311

MATHEMATICS

(Functions and Trigonometric functions, Calculus, Statistics and Probability)

2

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Foreword

Providing education to children is a fundamental right, and it's essential for the overalldevelopment of society. The government of Telangana plays a crucial role in ensuring thateducation is accessible to all, and they often establish institutions like the Telangana OpenSchool Society (TOSS) to cater to children who may be unable to access formal education due to various reasons.

To provide quality education to learners studying Intermediate Educationin Telangana Open School Society starting from the 2023 academic year, thetext books have been revised to align with the changing social situations and incorporate the fundamental principles of the National Education Policy 2020. The guidelines set forth in the policy aim to enhance theoverall learning experience and cater to the diverse needs of the learners. Earlier Textbooks were just guides with questions and answers. TOSS has designed the textbook with a student-centric approach, considering the different learning styles and needs of learners. Thisapproach encourages active engagement and participation in the learning process. The textbooks include supplementary teaching materials and resources to support educators in delivering effective and engaging lessons.

This textbook of Mathematics is broadly divided into six modules: Algebra, Coordinate Geometry, Three - dimensional Geometry, Trigonometry, Calculus, and Statistics. Book 2 contains three modules. In the module Functions students will learn about functions, and trigonometric functions. In the module on Calculus, students will be introduced to limits and continuity, derivatives, applications of derivatives and integration. The module Probability and statistics contains Measures of dispersion, random experiments, probability and distributions. Understanding all these chapters is essential for a comprehensive grasp of the subject.

We are indeed very grateful to the Government of Telangana and the Telangana State Board of Intermediate Education. Special thanks to the editor, co-coordinator, teachers, lecturers, and DTP operators who participated and contributed their services tirelessly to write this textbook.

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A Word WithYou

Dear Learner

Welcome to the senior secondary course. It gives me great pleasure to know that you have opted for mathematics as one of your subjects of study. Have you ever thought as to why we study mathematics? Can you think of a day when you have not counted something or used mathematics? Probably not.

Mathematics is the base of human civilization. From cutting vegetables to arranging books on the shelf, from tailoring clothe to motion of planets – mathematic applies everywhere. In fact, everything we do in our daily is governed by mathematics. Mathematics can be broadly defined as the scientific study of quantities, including their relationships, operations and their measurements, expressed by numbers and symbols. The mathematicians claim that the learning of mathematics can be real fun. It only requires complete concentration and love for mathematics.

The present curriculum has six modules, namely algebra, coordinate geometry, Three dimensional geometry, functions, calculus and statistics. There will be two books to cover the six modules.

VolumeII contains the three modules. In the module on functions, you will be introduced to functions, trigonometric functions,Inverse trigonometric functions and properties of triangles,

The fifth module on Calculus will introduce you to limits and continuity, differentiation of various functions, applications of derivatives, integration, definite integrals and solutions of differential equations.

The sixth; module is on Probability and statistics will introduce you to random experiments, probability, random variables and various probability distributions.

We would suggest to you that you go through all the solved examples given in the learning material and then try to solve independently all questions included in exercise and practice exercise given at the end of each lesson.

If you face any difficulty, please do write to us. Your suggestions are also welcome.

Yours,

Yours Academic Officer

(Mathematics)

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MATHEMATICS

Volume - I

MODULE - I

ALGEBRA

MODULE - II

CO-ORDINATE GEOMETRY

MODULE - III

THREE - DIMENSIONAL GEOMETRY AND VECTORS

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MATHEMATICS

Volume - II

MODULE - IV
FUNCTIONS AND TRIGONOMETRIC FUNCTIONS

MODULE - V
CALCULUS

MODULE - VI
STATISTICS AND PROBABILITY

CONTENTS

Chapter 16

SETS, RELATIONS AND FUNCTIONS

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- Define a set and represent the same in different forms;
- Define ordered pairs.
- Define Cartesian product of two sets;
- Define relation, function and cite examples thereof;
- Find domain and range of a function;
- Define and cite examples of diferent types of functions (one-one, many-one, onto, into and bijection);
- Determine wheather a function is one-one, many-one, onto or into;
- Draw the graph of functions;
- Define and cite examples of odd and even functions;
- Determine wheather a function is odd or even or neither;
- Define and cite examples of functions like $|x|$, $[x]$ the greatest integer function, polynomial functions, logarithmic and exponential functions;

- Define composition of two functions;
- Define the inverse of a function; and
- State the conditions for the inverse to exist.

PREREQUISITES

- Number systems, concept of ordered pairs.

Some standard notations to represent sets :

N : the set of natural numbers

W : the set of whole numbers

Z or I : the set of integers

Z+ : the set of positve integers

Z: the set of negative integers

Q : the set of rational numbers

R : the set of real numbers

C : the set of complex numbers

Other frequently used symbols are :

$\in$: 'belongs to'

$\notin$: 'does not belong to'

$\exists$: There exists,

$\nexists$: There does not exist.

If $a, b \in \mathbf{R}, a \leq b$ then

$(a, b) = \{x \in \mathbf{R} \mid a < x < b\}$

$[a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$

INTRODUCTION

A set is a collection of well defined objects. For a collection to be a set it is necessary that it should be well defined.

The word well defined was used by the German Mathematician George Cantor (1845- 1918A.D) to define a set. He is known as father of set theory. Now-a-days set theory has become basic to most of the concepts in Mathematics.

In our everyday life we come across different types of relations between the objects. The concept of relation has been developed in mathematical form.

All the scientists use mathematics essentially to study relationships. Physicists, chemists, Engineers, Biologists and social scientists, all seek to discern connection among the various elements of their chosen birds and so to arrive to a clear understanding of why these elements behave the way they do. A function is a special case of relation.

The word function was introduced by Leibnitz in 1694. Function is a special type of relation.

Each function is a relation but each ***relation is not a function***. In this lesson we shall discuss some basic definitions and operations involving sets, Cartesian product of two sets, relation between two sets, definition of function, different types of function and their properties. In order to have various important applications of functions later, it is essential to get a good grasp of the concepts in this chapter.

16.1 ORDERED PAIR

Let A and B sets. If $a \in A$ and $b \in B$ then (a, b) is an ordered pair a is called the first component and b is called the second component of the ordered pair

$(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$

16.2 CARTESIAN PRODUCT OF TWO SETS

Cartesion product of two sets : Let A and B be two sets. Then $\{(a, b) \,/\, a \in A \text{ and } b \in B\}$ is called cartesion product of A and B and is denoted by $A \times B$.

Consider two sets A and B where

$$A = \{1, 2\}, B = \{3, 4, 5\}.$$

Set of all ordered pairs of elements of A and B is

$$\{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$$

This set is denoted by $A \times B$ and is called the cartesian product of sets A and B.

i.e. $A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$

Cartesian product of B sets and A is denoted by $B \times A$.

In the present example, it is given by

$$B \times A = \{(3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)\}$$

Clearly $A \times B \neq B \times A$.

In the set builder form :

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

$$B \times A = \{(b, a) : b \in B \text{ and } a \in A\}$$

Note : If $A = \phi$ or $B = \phi$ or $A, B = \phi$

then $A \times B = B \times A = \phi$.

16.3 RELATIONS

If A and B are non empty sets, then any subset of $A \times B$ is called a relation from A to B $R \subseteq A \times B$.

If $A = \{1, 2, 3\}$ $\quad$ $B = \{2, 3, 4\}$

$A \times B = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$.

$f = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$ is a relation from A to B.

$g = \{(2, 2), (3, 3)\}$ is a relation from A to B.

If (i) $R = \phi$, R is called a void relation.

(ii) $R = A \times B$, R is called a universal relation.

(iii) If R is a relation defined from A to A, it is called a relation defined on A.

(iv) $R = \{(a, a) \ \forall \ a \in A\}$, is called the identity relation.

16.4 DOMAIN AND RANGE OF A RELATION

If R is a relation between two sets then the set of its first elements (components) of all the ordered pairs of R is called Domain and set of 2nd elements of all the ordered pairs of R is called range, of the given relation.

$f = \{(1, a), (2, b), (3, c)\}$

Domain = $\{1, 2, 3\}$

Range = $\{a, b, c\}$

Example 16.1 Given that $A = \{2, 4, 5, 6, 7\}$, $B = \{2, 3\}$.

R is a relation from A to B defined by

$R = \{(a, b) : a \in A, b \in B \text{ and } a \text{ is divisible by } b\}$

find (i) R in the roster form

(ii) Domain of R

(iii) Range of R

(iv) Repersent R diagramatically.

Solution : (i) $R = \{(2, 2), (4, 2), (6, 2), (6, 3)\}$

(ii) Domain of R = $\{2, 4, 6\}$

(iii) Range of R = $\{2, 3\}$

(iv)

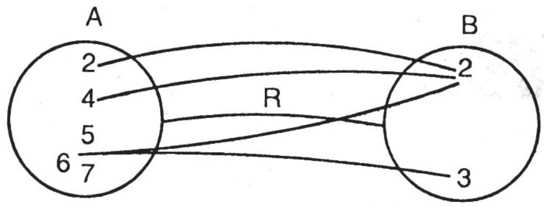


Fig. 16.1

Example 16.2 : If R is a relation 'is greater than' from A to B, where

A= {1, 2, 3, 4, 5} and B = {1,2,6}.

Find (i) R in the roster form. (ii) Domain of R (iii) Range of R.

Solution :

(i) R = {(3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)}

(ii) Domain of R = {3, 4, 5}

(iii) Range of R = {1, 2}

16.5 DEFINITION OF A FUNCTION

Consider the relation

$$f : \{(a, 1), (b, 2), (c, 3), (d, 5)\}$$

In this relation we see that each element of A has a unique image in *B*

This relation *f* from set *A* to *B* where every element of *A* has a unique image in *B* is defined as a function from *A* to *B*. So we observe that ***in a function no two ordered*** pairs have the same first element.

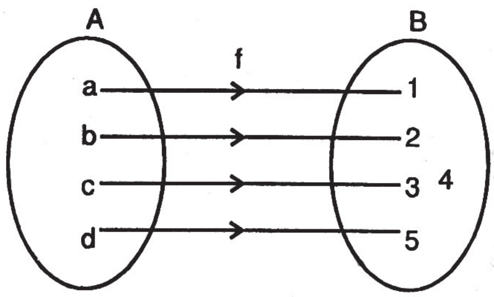


Fig.16.2

We also see that $\exists$ an element $\in$ B, i.e., 4 which does not have its preimage in A. Thus here:

(i) the set B will be termed as co-domain and

(ii) the set {1, 2, 3, 5} is called the range.

From the above we can conclude that ***range is a subset of co-domain.***

Symbolically, this function can be written as

$$f \; : \; f : A \rightarrow B \quad \text{or} \quad A \xrightarrow{f} B.$$

Example 16.3 : Which of the following relations are functions from A to B. Write their domain and range. If it is not a function give reason ?

(a) {(1, −2), (3, 7), (4, −6), (8, 1)}, A = {1, 3, 4, 8}, B = {−2, 7, −6, 1, 2}

(b) {(1, 0), (1, −1), (2, 3), (4, 10)}, A = {1, 2, 4}, B = {0, −1, 3, 10}

(c) {(a, b), (b, c), (c, b), (d, c)}, A = {a, b, c, d, e}, B = {b, c}

(d) {(2, 4), (3, 9), (4, 16), (5, 25), (6, 36)}

A = {2, 3, 4, 5, 6}, B = {4, 9, 16, 25, 36}

Solution :

(a) It is a function.

Domain= {1, 3, 4, 8} , Range {2,7, 6,1}

(b) It is not a function. Because Ist two ordered pairs have same first elements.

(c) It is not a function.

Domain= $\{a, b, c, d\} \neq$ A, Range = $\{b, c\}$

(d) It is a function.

Domain = {2,3,4,5,6} ,Range = {4, 9, 16, 25, 36}

First two ordered pairs have same first component and last two ordered pairs have also same first component.

Example 16.4 State whether each of the following relations represent a function or not.

(a)

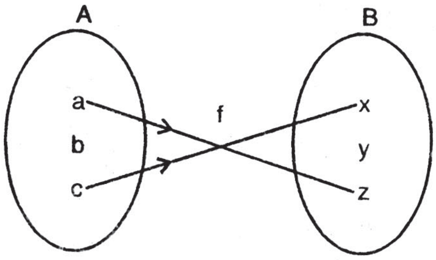


Fig.16.3

(b)

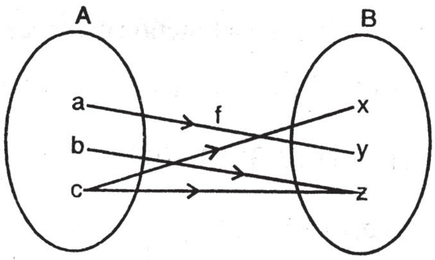


Fig.16.4

(c)

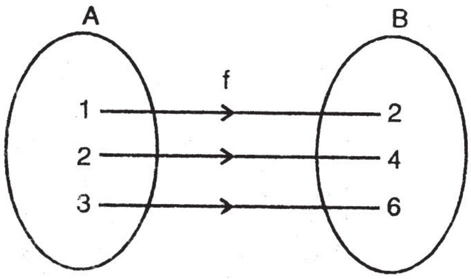


Fig. 16.5

(d)

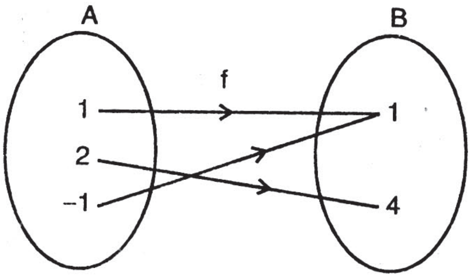


Fig.16.6

Solution :

(a) f is not a function because the element b of A does not have an image in B.

(b) f is not a function because the element c of A does not have a unique image in B.

(c) f is a function because every element of A has a unique image in B.

(d) f is a function because every element in A has a unique image in B.

EXERCISE 16.1

1. Which of the following relations are functions from A to B ?

   (a) {(1, –2), (3, 7), (4, –6), (8, 11)}, A = {1, 3, 4, 8},
   B = {–2, 7, –6, 11}

   (b) {(1, 0), (1, –1), (2, 3), (4, 10)}, A = {1, 2, 4},
   B = {1, 0, –1, 3, 10}

   (c) {(a, 2), (b, 3), (c, 2), (d, 3)}, A = {a, b, c, d}, B = {2, 3}

   (d) {(1, 1), (1, 2), (2, 3), (–3, 4)}, A = {1, 2, –3}, B = {1, 2, 3, 4}

   (e) $\left(2,\frac{1}{2}\right),\left(3,\frac{1}{3}\right),.....\left(10,\frac{1}{10}\right),$

   A = {1, 2, 3, 4}, B = $\left\{\frac{1}{2},\frac{1}{3},\frac{1}{4},....,\frac{1}{11}\right\}$

   (f) {(1, 1), (–1, 1), (2, 4), (–2, 4)}, A = {0, 1, –1, 2, –2},
   B = {1, 4}.

2. Which of the following relations represent a function ?

(a)

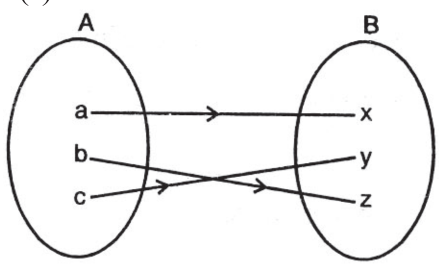


Fig. 16.7

(b)

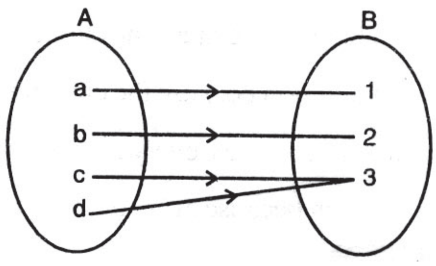


Fig. 16.8

(c)

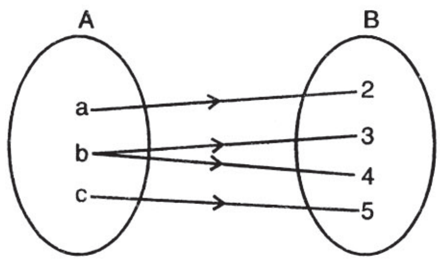


Fig. 16.9

(d)

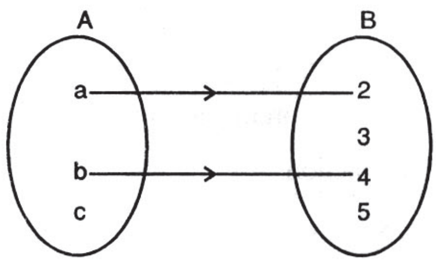


Fig. 16.10

3. Write domain and range for each of the following functions :

(a) $\left\{\left(\sqrt{2}, 2\right), \left(\sqrt{5}, -1\right), \left(\sqrt{3}, 5\right)\right\}$

(b) $\left\{\left(-3, \frac{1}{2}\right), \left(-2, \frac{1}{2}\right), \left(-1, \frac{1}{2}\right)\right\}$

(c) {(1, 1), (0, 0), (2, 2), (−1, −1)

(d) {(Deepak,16), (Sandeep, 28), (Rajan, 24)}

4. Write domain and range for each of the following mappings :

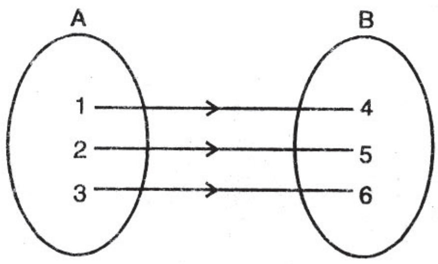


Fig. 16.10

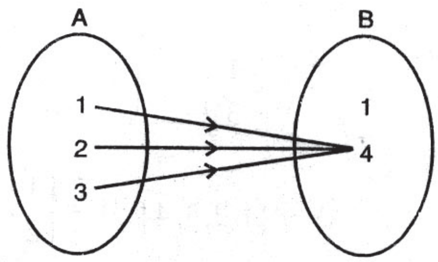


Fig. 16.11

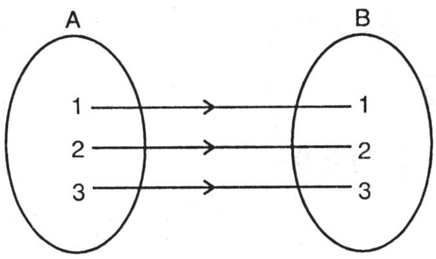


Fig. 16.12

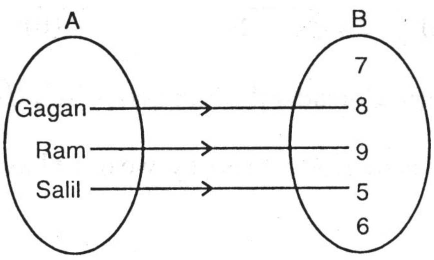


Fig. 16.13

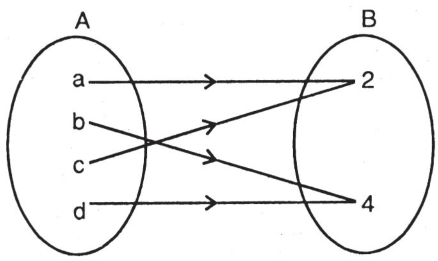


Fig. 16.14

16.5.1 Some More Examples on Domain and Range

Let us consider some functions which are only defined for a certain subset of the set of real numbers.

Example 16.5 Find the domain of each of the following functions :

(a) $y = \frac{1}{x}$ (ii) $y = \frac{1}{x-2}$ (iii) $y = \frac{1}{(x+2)(y-3)}$

Solution : The function $y = \frac{1}{x}$ can be described by the following set of ordered pairs.

$$\left\{........\left(-2, -\frac{1}{2}\right), (-1,-1), (1,1), \left(2, \frac{1}{2}\right)...\right\}$$

Here we can see that x can take all real values except 0 because the corresponding image, i.e., $\frac{1}{0}$ is not defined.

$\therefore$ Domain $= R - \{0\}$ [Set of all real numbers except 0]

***Note :* Here range = R − {0}**

(b) x can take all real values except 2 because the corresponding image, i.e., $\frac{1}{2-2}$ does not exist.

$\therefore$ Domain $= R - \{2\}$.

(c) Value of y does not exist for $x = -2$ and $x = 3$

$\therefore$ Domain $= R - \{-2, 3\}$.

Example 16.6 Find domain of each of the following functions :

(a) $y = +\sqrt{x-2}$ (b) $y = +\sqrt{(2-x)(4+x)}$

Solution : (a) Consider the function $y = \sqrt{(x-2)}$

In order to have real values of y, we must have $(x - 2) \geq 0$

i.e. $x \geq 2$

$\therefore$ Domain of the function will be all real numbers ≥ 2 .

(b) $y = +\sqrt{(2-x)(4+x)}$

In order to have real values of y, we must have $(2 - x)(4 + x) \geq 0$

We can achieve this in the following two cases.

Case I : $(2-x)\geq 0$ and $(4+x)\geq 0$

$\Rightarrow x\leq 2$ and $x\geq -4$

$\therefore$ Domain consists of all real values of x such that $-4\leq x\leq 2$

Case II : $2-x\geq 0$ and $4+x\leq 0$

$\Rightarrow 2\leq x$ and $x\leq -4$

But, x cannot take any real value which is greater than or equal to 2 and less than or equal to -4 .

$\therefore$ From both the cases, we have

Domain = $-4\leq x\leq 2 \quad \forall \in$ R.

Example 16.7 For the function

$f(x)=y=2x+1$, find the range when domain $\{-3, -2, -1, 0, 1, 2, 3\}$.

Solution : For the given values of x, we have

$f(-3)=2(-3)+1=-6+1=-5$

$f(-2)=2(-2)+1=-4+1=-3$

$f(-1)=2(-1)+1=-2+1=-1$

$f(0)=2(0)+1=0+1=1$

$f(1)=2(1)+1=2+1=3$

$f(2)=2(2)+1=4+1=5$

$f(3)=2(3)+1=6+1=7$

The given function can also be written as a set of ordered pairs.

i.e., $\{(-3,-5), (-2,-3), (-1,-1), (0,1)\ (1,3), (2,5)(3,7)\}$

$\therefore$ Range $\{-5, -3, -1, 1, 3, 5, 7\}$

Example 16.8 : If $f(x) = x^2$, $-3 \le x \le 3$, find its range.

Solution : Given $-3 \le x \le 3$

or $\quad 0 \le x^2 \le 9 \quad$ or $\quad 0 \le f(x) \le 9$

$\therefore$ Range $= \{ f(x) : 0 \le f(x) \le 9 \}$.

EXERCISE 16.2

1. Find the domain of each of the following functions x ∈ R :

   (a) (i) $y = 2x$ (ii) $y = 9x + 3$

   (iii) $y = x^2 + 5$

   (b) (i) $y = \dfrac{1}{3x-1}$ (ii) $y = \dfrac{1}{(4x+1)(x-5)}$

   (iii) $y = \dfrac{1}{(x-3)(x-5)}$ (iv) $y = \dfrac{1}{(3-x)(x-5)}$

   (c) (i) $y = \sqrt{6-x}$ (ii) $y = \sqrt{7+x}$

   (iii) $y = \sqrt{3x+5}$

   (d) (i) $y = \sqrt{(3-x)(x-5)}$ (ii) $y = \sqrt{(x-3)(x+5)}$

   (iii) $y = \dfrac{1}{\sqrt{(3+x)(7+x)}}$ (iv) $y = \dfrac{1}{\sqrt{(x-3)(7+x)}}$

2. Find the range of the function, given its domain in each of the following cases.

   (a) (i) $f(x) = 3x + 10, \quad x \in \{1, 5, 7, -1, -2\}$

   (ii) $f(x) = 2x^2 + 1, \quad x \in \{-3, 2, 4, 0\}$

   (iii) $f(x) = x^2 - x + 2, \quad x \in \{1, 2, 3, 4, 5\}$

   (b) (i) $f(x) = x - 2, \ 0 \le x \le 4$

   (ii) $f(x) = 3x + 4, \ -1 \le x \le 2$

(c) (i) $f(x) = x^2, -5 \le x \le 5$ (ii) $f(x) = 2x, \quad -3 \le x \le 3$

(iii) $f(x) = x^2+1, \quad -2 \le x \le 2$ (iv) $f(x) = \sqrt{x}, \quad 0 \le x \le 25$

(d) (i) $f(x) = x + 5, x \in R$ (ii) $f(x) = 2x - 3, x \in R$

(iii) $f(x) = x^3, \quad x \in R$ (iv) $f(x) = 1/x, \{x : x < 0\}$

(v) $f(x) = \dfrac{1}{x-2}, \quad \{x : x \le 1\}$ (vi) $f(x) = \dfrac{1}{3x-2}, \quad \{x : x < 0\}$

(vii) $f(x) = \dfrac{2}{x}, \quad \{x : x > 0\}$

(viii) $f(x) = \dfrac{x}{x+5}, \quad \{x : x \ne -5\}$

16.6 CLASSIFICATION OF FUNCTIONS

Image, Pre Image : If $f : A \to B$ is a function and if $f(a) = b$ then b is called the image of a under f and a is called a pre Image or inverse image of b under f.

$f : \{(1, a), (2, b), (3, c)\}$

$f(1) = a$. a is called the image of 1 amd 1 is called the pre image to a.

One - One function (Injection):

A function $f : A \to B$ is called an injection if distinct elements of A have distinct images in B. An Injection is called a One - One function.

$f : A \to B$ is an Injection $\Leftrightarrow a_1, a_2 \in A$ and $a_1 \ne a_2$ implies that $f(a_1) \ne f(a_2)$

$\Leftrightarrow a_1, a_2 \in A$ and $f(a_1) = f(a_2)$ implies that $a_1 = a_2$.

Ex:

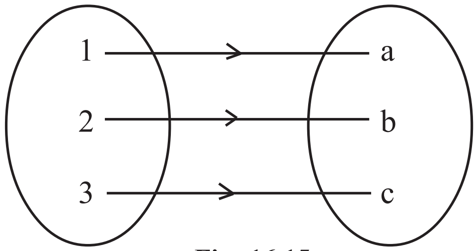


Fig. 16.15

f is an Injection.

Ex. Let $f : A \to B$ be defined by $f(x) = 2x + 1$. Then f is an Injection

Since for any $a_1, a_2 \in \mathbf{R}$ and $f(a_1) = f(a_2)$

$\Rightarrow 2a_1 + 1 = 2a_2 + 1$

$\Rightarrow a_1 = a_2$.

Onto function : (Surjection)

A function $f : A \to B$ is called a surjection if the range of f is equal to the codomain of f.

$f : A \to B$ is a surjection if for every $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$.

Ex.

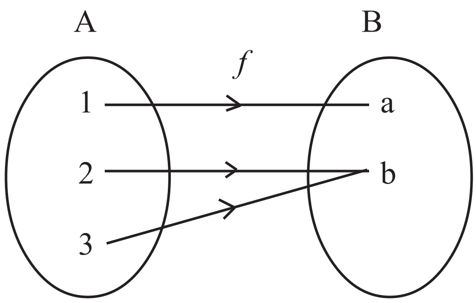


Fig. 16.16

f is a sursection.

Bijection

If $f : A \to B$ is both an Injection and a surjection then f is said to be a bijection form A to B.

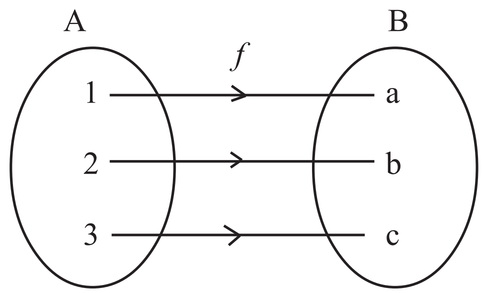


Fig. 16.17

f is bijection.

Identity Function :

Let A be a non empty set. Then the function $f : A \rightarrow A$ defined by $f(x) = x$, for all $x \in A$ is called the Identify function and is denoted by I_A.

If A = {1, 2, 3} then $I_A = \{(1, 1), (2, 2), (3, 3)\}$

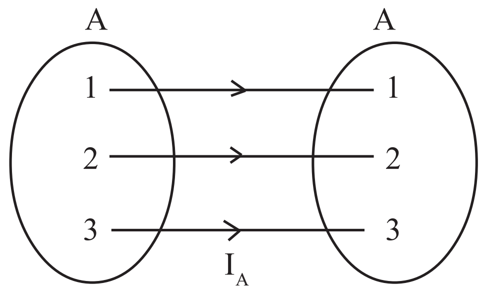


Fig. 16.18

Constant Function : A function $f : A \rightarrow B$ is said to be a constant function if the range of f contains only one element. i.e., $f(x) = c$ for all $x \in A$ for some fixed $c \in B$.

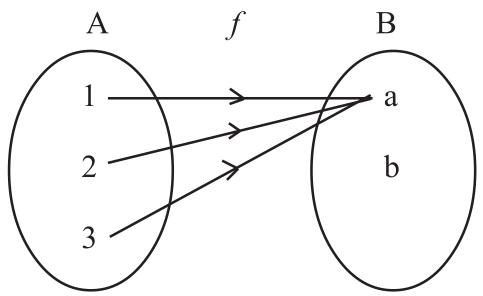


Fig. 16.19

f is constant function.

Example 16.8 Without using graph prove that the function

$f : R \rightarrow R$ defiend by $f(x) = 4 + 3x$ is ***one-to-one***.

Solution : For a function to be one-one function

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \;\; \forall \;\; x_1, x_2 \in \mathbf{R}$

$\therefore$ Now $f(x_1) = f(x_2)$ gives

$4 + 3x_1 = 4 + 3x_2$ or $x_1 = x_2$

$\therefore f$ is a ***one-one function***

Example 16.9 : Prove that

$f : R \rightarrow R$ defined by $f(x) = 4x^3 - 5$ is a bijection

Solution : Now $f(x_1) = f(x_2) \; \forall \; x_1 \; x_2 \in \mathbf{R}$

$\therefore \; 4x_1^3 - 5 = 4x_2^2 - 5$

$\Rightarrow x_1^3 = x_2^3 \Rightarrow x_1^3 - x_2^3 = 0 \Rightarrow (x_2 - x_1)\,(x_1^2 + x_1x_2 + x_2^2) = 0$

$\Rightarrow x_1 = x_2$ or

$x_1^2 + x_1x_2 + x_2^2 = 0$ (rejected). It has no real value of x_1 and x_2 .

$\therefore$ F is a ***one-one function***.

Again let $y = (x)$ where $y \in$ codomain, $x \in$ domain.

We have $y = 4x^3 - 5$ or $x = \left(\dfrac{y+5}{4}\right)^{1/3}$

$\therefore$ For each $y \in$ codomain $\exists \; x \in$ domain such that $f(x) = y$

Thus F is ***onto function***.

$\therefore$ F is a bijection.

Example 16.10 : Prove that $f : R \to R$ defined by $f(x) = x^3 + 3$ is neither ***one-one*** nor ***onto function***.

Solution : We have $f(x_1) = f(x_1)$ $\forall x_1, x_2 \in$ domain giving

$$x_1^2 + 3 = x_2^2 + 3 \Rightarrow x_1^2 = x_2^2$$

or $x_1^2 - x_2^2 = 0 \Rightarrow x_1 = x_2$ (or) $x_1 = -x_2$

or F is not ***one-one function***.

Again let $y = F(x)$ where $y \in$ codomain

$x \in$ domain.

$$\Rightarrow y = x^2 + 3 \Rightarrow x = \pm\sqrt{y-3}$$

$\Rightarrow$ $\forall y < 3$ no real value of x in the domain.

$\therefore$ F is not an ***onto finction***.

16.7 GRAPHICAL REPRESENTATION OF FUNCTIONS

Since any function can be represented by ordered pairs, therefore, a graphical representation of the function is always possible. For example, consider $y = x^2$.

$y = x^2$

x	0	1	–1	2	–2	3	–3	4	–4
y	0	1	1	4	4	9	9	16	16

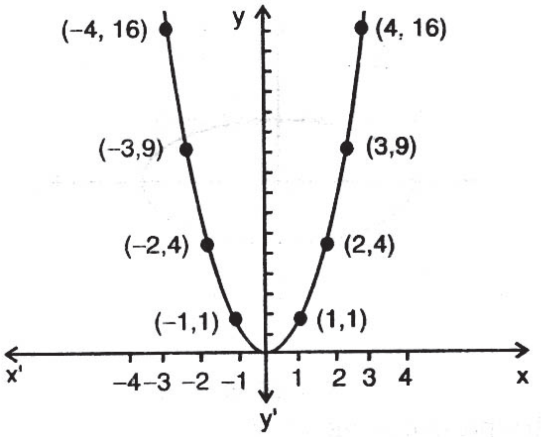


Fig. 16.20

Does this represent a function?

Yes, this represent a function because corresponding to each value of x $\exists$ a unique value of y.

Now consider the equation $x^2 + y^2 = 25$

$$x^2 + y^2 = 25$$

x	0	0	3	3	4	4	5	–5	–3	–3	–4	–4
y	5	–5	4	–4	3	–3	0	0	4	–4	3	–3

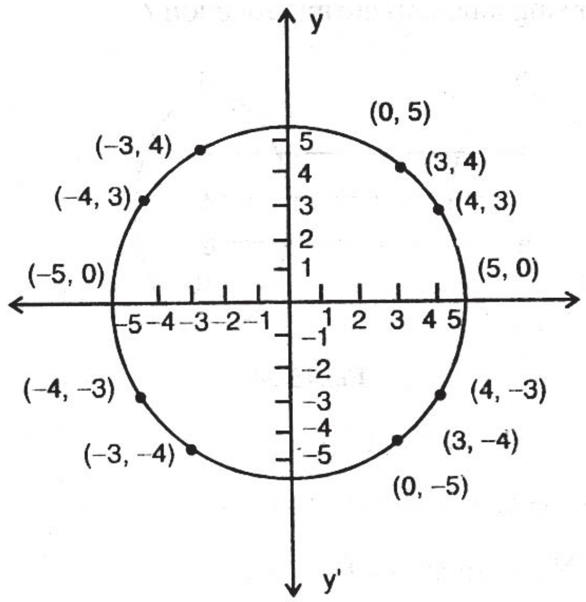


Fig. 16.21

This graph represents a circle.

Does it represent a function ?

No, this does not represent a function because corresponding to the same value of x, there does not exist a unique value of y.

EXERCISE 16.3

1. (i) Does the graph represent a function?

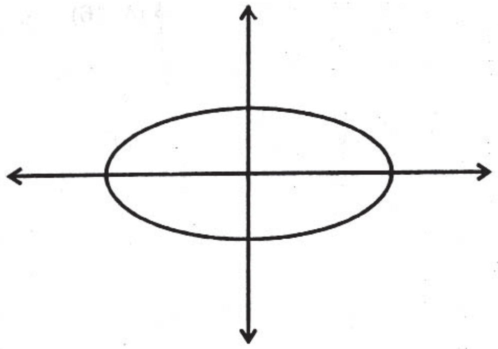


Fig. 16.22

(ii) Does the graph represent a function ?

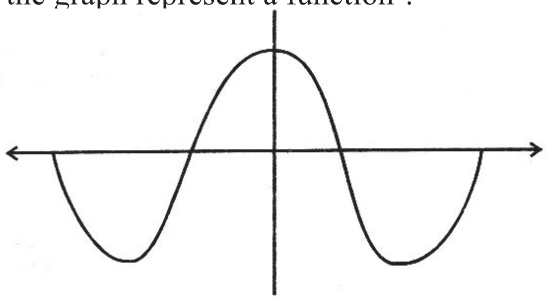


Fig. 16.23

2. Which of the following functions are into function ?

(a)

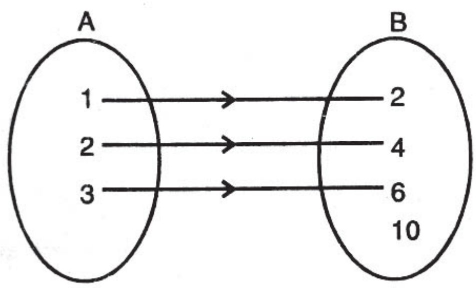


Fig. 16.24

(b) $f : \text{N} \to \text{N}$, defined as $f(x) = x^2$

Here N represents the set of natural numbers.

(c) $f : \text{N} \to \text{N}$, defined as $f(x) = x$

3. Which of the following functions are onto function if $f : R \to R$

   (a) $f(x) = 115x + 49$ (b) $f(x) = |x|$

4. Which of the following functions are one-to-one functions ?

   (a) $f : \{20, 21, 22\} \to \{40, 42, 44\}$ defined as $f(x) = 2x$

   (b) $f : \{7, 8, 9\} \to \{10\}$ defined as $f(x) = 10$

   (c) $f : R \to R$ defined as $f(x) = x^3$

   (d) $f : R \to R$ defined as $f(x) = 2 + x^4$

   (e) $f : N \to N$ defined as $f(x) = x^4 + 2x$

5. Which of the following graphs represents a function ?

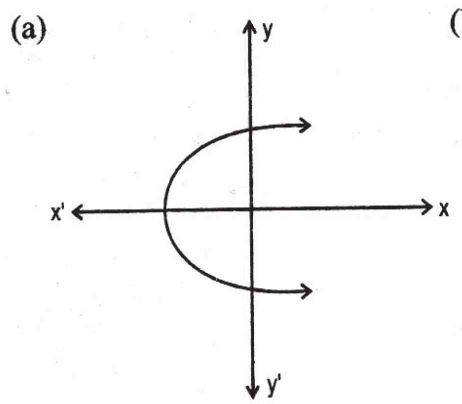


Fig. 16.26

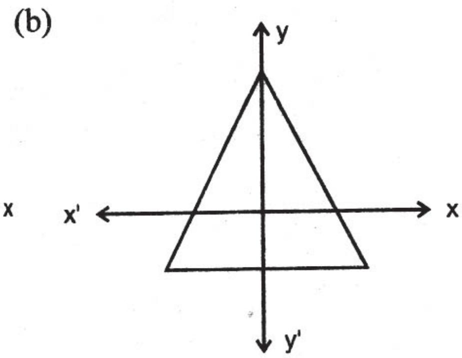


Fig. 16.27

Hint : If any line || to y-axis cuts the graph in more than one point, graph does not represent a function.

16.7.1 Even Function

A function is said to be an even function if for each x of domain

$F(-x) = F(x)$

For example, each of the following is an ***even function***.

(i) $F(x) = x^2$ then $F(-x) = (-x)^2 = x^2 = F(x)$

(ii) $F(x) = \cos x$ then $F(-x) = \cos(-x) = \cos x = F(x)$

(iii) $F(x) = (x)$ then $F(-x) = |-x| = |x| = F(x)$

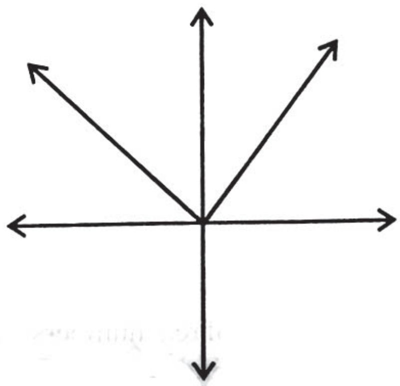


Fig. 16.28

The graph of this even function (modulus function) is shown in the figure above.

Observation

Graph is symmetrical about y-axis.

16.7.2 Odd Function

A function is said to be an odd function if for each x

$f(-x) = -f(x)$

For example

(i) If $f(x) = x^3$

$f(-x) = (-x)^3 = -x^3 = -f(x)$

(ii) If $f(x) = \sin x$

then $f(-x) = \sin(-x) = -\sin x = -f(x)$

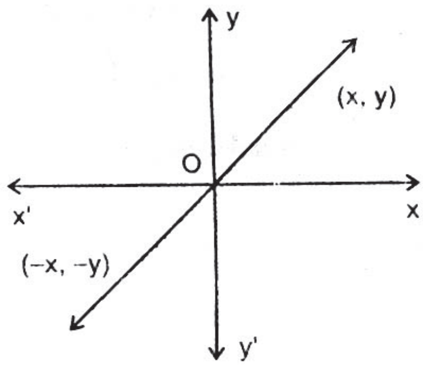


Fig. 16.29

Graph of the odd function y = x is given in Fig.16.29

Observation

Graph is symmetrical about origin.

16.17.3 Greatest Integer Function (Step Function)

$f(x) = [x]$ which is the greatest integer less than or equal to x.

$f(x)$ is called Greatest Integer Function or Step Function. Its graph is in the form of steps, as shown in Fig. 16.47.

Let us draw the graph of $y = [x],\ x \in R$

$[x] = 1,\ 1 \le x \le 2$

$[x] = 2,\ 2 \le x \le 3$

$[x] = 3,\ 3 \le x \le 4$

$[x] = 0,\ 0 \le x \le 1$

$[x] = -1,\ -1 \le x \le 0$

$[x] = -2,\ -2 \le x \le -1$

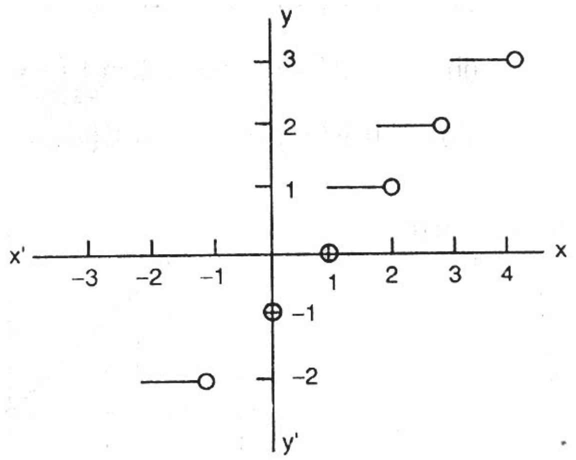


Fig. 16.29

- Domain of the step function is the set of real numbers.
- Range of the step function is the set of integers.

16.7.4 Polynomial Function

Any function defined in the form of a polynomial is called a polynomial function.

For example,

(i) $f(x) = 3x^2 - 4x - 2$

(ii) $f(x) = x^3 - 5x^2 - x + 5$

(iii) $f(x) = 3$

are all polynomial functions

Note : Functions of the type $f(x) = k$, where k is a constant is also called a constant function

16.7.5 Rational Function

Function of the type $f(x) = \dfrac{g(x)}{h(x)}$. Where $h(x) \neq 0$ and $g(x)$ and $h(x)$are polynomial functions are called rational functions.

For example: $f(x) = \dfrac{x^2 - 4}{x+1}, \; x \neq -1$

is a rational function.

16.7.6 Reciprocal Function

Functions of the type $y = \dfrac{1}{x}, \; x \neq 0$ is called a reciprocal function.

16.7.7 Exponential Functions

A swiss mathematician Leonhard Euler introduced a number e in the form of an infinite series. In fact

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + ... + \frac{1}{\lfloor n} + ... \qquad ...(1)$$

It is well known that the sum of its infinite series tends to a finite limit (i.e., this series is convergent) and hence it is a positive real number denoted by e. This number e is a transcendental irrational number and its value lies between 2 and 3.

Consider now the infinite series

$$1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + ... + \frac{x^n}{\lfloor n} + ...$$

It can be shown that the sum of its infinite series also tends to a finite limit, which we denote by e^x. Thus,

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + ... + \frac{x^n}{\lfloor n} + ... \qquad ...(2)$$

This is called the **Exponential Theorem** and the infinite series is called the **exponential series**. We easily see that we would get (1) by putting x = 1 in (2).

The function $f(x) = e^x$, where x is any real number is called an **Exponential Function**.

The graph of the exponential function

$$y = e^x$$

is obtained by considering the following important facts :

(i) As x increases, the y values are increasing very rapidly, whereas as x decreases, the y values are getting closer and closer to zero.

(ii) There is no x-intercept, since $e^x \neq 0$ for any value of x.

(iii) The y intercept is 1, since $e^0 = 0$ and $e \neq 0$.

(iv) The specific points given in the table will serve as guidelines to sketch the graph of e^x (Fig. 15.48).

x	−3	−2	−1	0	1	2	3
$y = e^x$	0.04	0.13	0.36	1.00	2.71	7.38	20.08

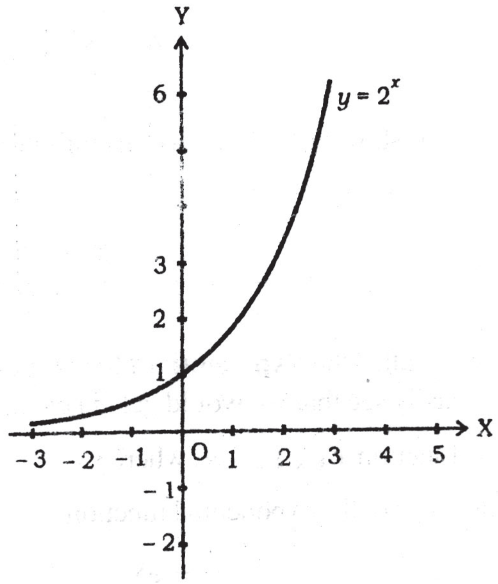


Fig. 16.30

If we take the base different from e, say a, we would get exponential function

$f(x) = a^x$, provbided $a > 0$, $a \neq 1$.

For example, we may take a = 2 or a = 3 and get the graphs of the functions

$y = 2x$ (see Fig. 16.30)

and $y = 3^x$ (see Fig. 16.31)

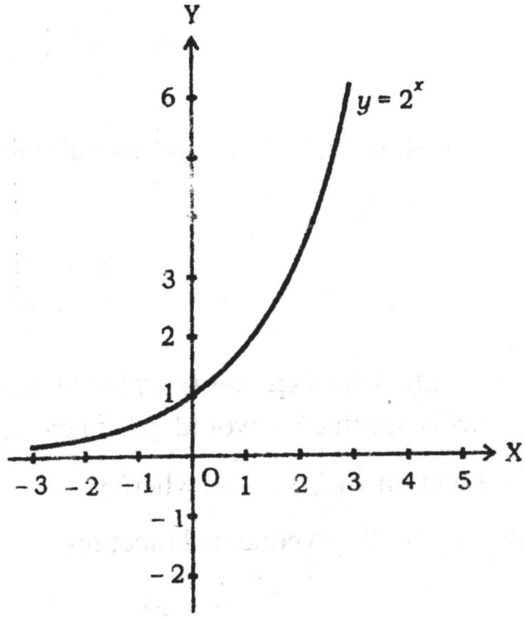


Fig. 16.31

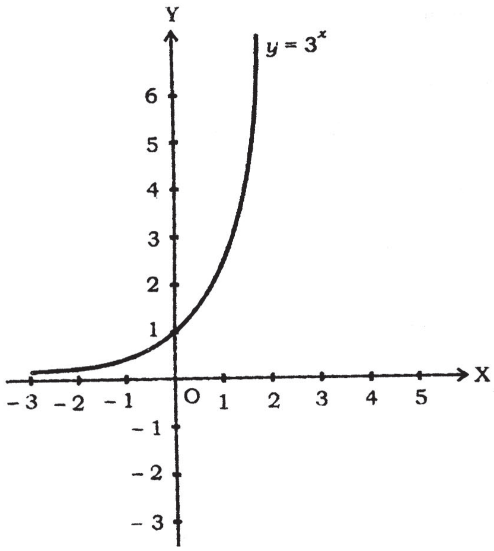


Fig. 16.32

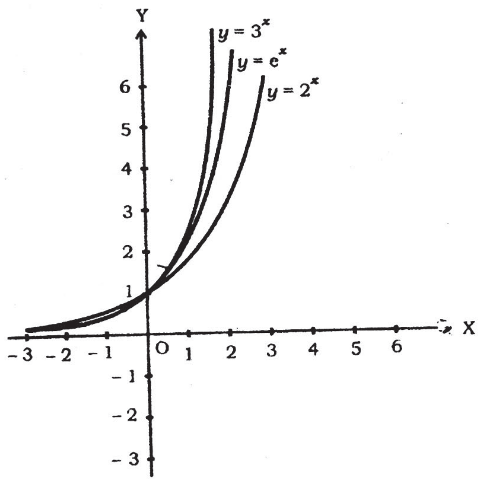


Fig. 16.33

16.7.8 Logarithmic Functions

Consider now the function

$$y = e^x \qquad .....(3)$$

We write it equivalently as

$$x = \log_e y$$

Thus, $\quad y = \log_e x$ .....(4)

is the inverse function of $y = e^x$

The base of the logarithm is not written if it is e and so $\log_e x$ is usually written as $\log x$.

As $y = e^x$ and $y = \log x$ are inverse functions, their graphs are also symmetric w.r.t. the line

$$y = x$$

The graph of the function $y = \log x$ can be obtained from that of $y = e^x$ ex by reflecting it in the line $y = x$.

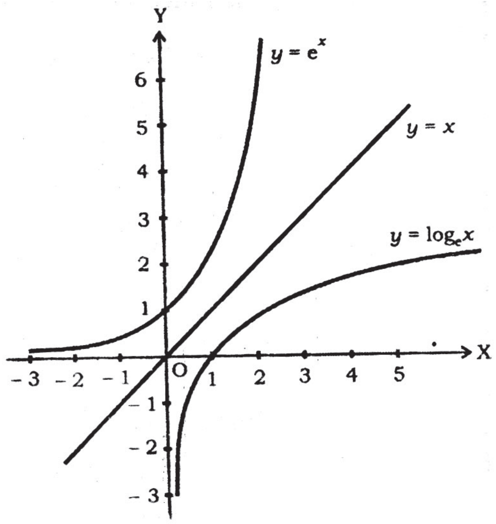


Fig. 16.34

Note

(i) The learner may recall the laws of indices which you have already studied in the Secondary Mathematics :

If $a > 0$, and m and n are any rational numbers, then

$$a^m \cdot a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$\left(a^m\right)^n = a^{mn}$$

$$a^0 = 1.$$

(ii) The corresponding laws of logarithms are

$$\log_a(mn) = \log_a m + \log_a n$$

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a(m)^n = n\log_a m$$

$$\log_b m = \frac{\log_a m}{\log_a b} \quad \ldots$$

or $\log_b m = \log_a m \, . \log_b a$

Here $\log a, b > 0 \ \ a \neq 1, \ b \neq 1.$

EXERCISE 16.4

I. Tick mark the correct statement.

(i) Function $f(x) = 2x^4 + 7x^2 + 9x$ is an even function.

(ii) Odd function is symmetrical about y-axis.

(iii) $f(x) = x^{1/2} - x^3 + x^5$ is a polynomial function.

(iv) $f(x) = \dfrac{x-3}{3+x}$ is a rational function for all $x \in R$.

(v) $f(x) = \dfrac{\sqrt{5}}{3}$ is a constant function.

(vi) $f(x) = \dfrac{1}{x}$

Domain of the function is the set of real numbers except 0.

(vii) Greatest integer function is neither even nor odd.

2. Specify the following functions as polynomial function, rational function, reciprocal function or constant function.

(a) $y = 3x^8 - 5x^7 + 8x^5$

(b) $y = \dfrac{x^2 + 2x}{x^3 - 2x + 3}, \ x^3 - 2x + 3 \neq 3$

(c) $y = \dfrac{3}{x^2}, \ x \neq 0$

(d) $y = 3 + \dfrac{2x+1}{x}, \ x \neq 0$

(e) $y = 1 - \dfrac{1}{x}, \ x \neq 0$

(f) $y = \dfrac{x^2 - 5x + 6}{x - 2}, \ x \neq 2$

(g) $y = \dfrac{1}{9}$

16.8 COMPOSITION OF FUNCTIONS

Consider the two functions given below:

$$y = 2x + 1, \qquad x \in \{1, 2 \;\; 3\}$$

$$z = y + 1, \qquad y \in \{3, 5 \;\; 7\}$$

Then z is the composition of two functions x and y because z is defined in terms of y and y in terms of x.

Graphically one can represent this as given below :

$$f(x) = y = 2x + 1 \qquad g(y) = z = y + 1$$

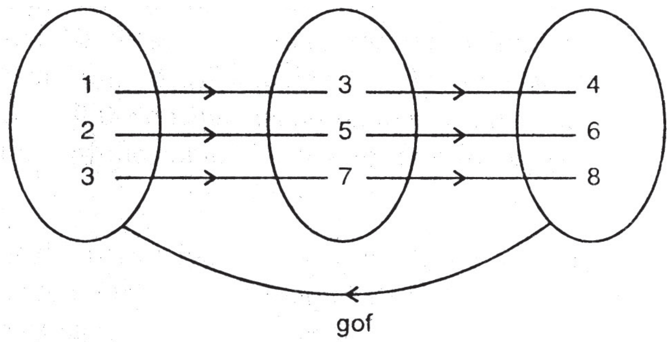


Fig. 16.35

The composition, say, gof of function g and f is defined as function g of function f.

If $f : A \to B$ And $g : B \to C$ then $gof : A \to C$

$$gof(x) = g[\,f(x)\,]$$

Let $f(x) : 3x + 1$ and $g(x) : x^2 + 2$

Then $fog\ (x) = f\,[g(x)]$

$$= f(x^2 + 2)$$

$$= 3[x^2 + 2] + 1 = 3x^2 + 7$$

and $(gof\,)\ (x) = g\,[\,f(x)]$

$$= g(3x + 1)$$

$$= (3x + 2)^2 + 2 = 9x^2 + 6x + 3$$

Check from (i) and (ii), if

$$fog = gof$$

Evidently, $fog \neq gof$

Similarly, $(fof)(x) = f[f(x)] = f(3x+1)$ [Read as function of function f].

$$= 3(3x+1)+1$$

$$= 9x + 3 + 1 = 9x + 4$$

$(gog)(x) = g[g(x)] = g[x^2+2]$ (Read as function of function g)

$$= (x^2+2)^2 + 2$$

$$= x^4 + 4x^2 + 4 + 2$$

$$= x^4 + 4x^2 + 6.$$

Example 16.11 If $f(x) = \sqrt{x+1}$ and $g(x) = x^2 + 2$ calculate fog and gof.

Solution : $fog(x) = f[g(x)]$

$$= f(x^2+2)$$

$$= \sqrt{x^2+2+1} = \sqrt{x^2+3}$$

$$(gof)(x) = g[f(x)]$$

$$= g\left(\sqrt{x+1}\right)$$

$$= \left(\sqrt{x+1}\right)^2 + 2$$

$$= x + 1 + 2 = x + 3$$

Here again, we see that $(fog) \neq (gof)$

Example 16.12 : If $f(x) = x^3$, $f: \mathrm{R} \rightarrow \mathrm{R}$

$$g(x) = \frac{1}{x}, \quad g: \mathrm{R}\text{-}\{0\} \rightarrow \mathrm{R} \rightarrow \{0\}$$

Find fog and gof.

Solution : $(fog)\,(x) = f\,[\,g\,(x)]$

$$= f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 = \frac{1}{x^3}$$

$$(gof\,)\,(x) = g\,[\,f\,(x)] = g(x^3) = \frac{1}{x^3}$$

Here we see that $fog = gof$

Note : We observe from Example 16.12 and Example 16.13 that *fog* and *gof* may or may not be equal.

EXERCISE 16.5

1. For each of the following functions write *fog, gof, fof* and *gog*.

   (a) $f(x) = x^2 - 4,\quad g(x) = 2x + 5$

   (b) $f(x) = x^2,\quad g(x) = 3$

   (c) $f(x) = 3x - 7,\quad g(x) = \frac{2}{x},\ x \neq 0$

2. Let $f(x) = x^2 + 3,\quad g(x) = x - 2$

   Prove that $fog \neq gof$ and

$$f\left[f\left(\frac{3}{2}\right)\right] = g\left[f\left(\frac{3}{2}\right)\right]$$

3. If $f(x) = x^2,\quad g(x) = \sqrt{x}$ Show that $fog = gof$.

16.9 INVERSE OF A FUNCTION

Let A, B are two sets, $a \in$ A, $b \in$ B, f : A $\rightarrow$ B is abijection then f^{-1} : B $\rightarrow$ A is called Inverse function of f.

(A) Consider the relation

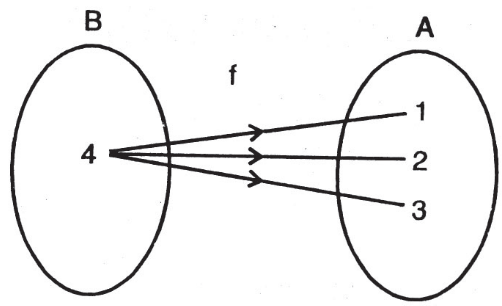


Fig. 16.36

This is a many-to-one function. Now let us find the inverse of this relation.

Pictorially, it can be represented as

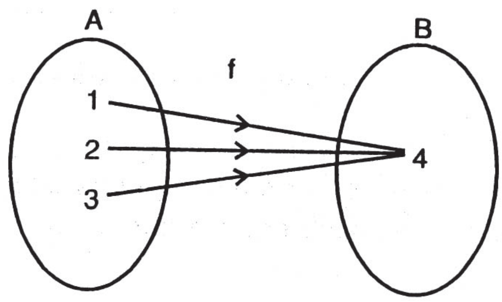


Fig 16.37

Clearly this relation does not represent a function. (Why ?)

(B) Now take another relation

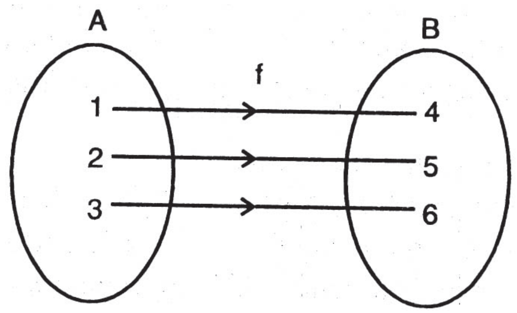


Fig.16.38

It represents one-to-one onto function. Now let us find the inverse of this relation, which is represented pictorially as

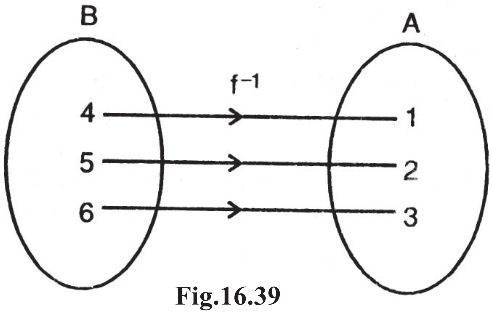


Fig.16.39

This represents a function.

(C) Consider the relation

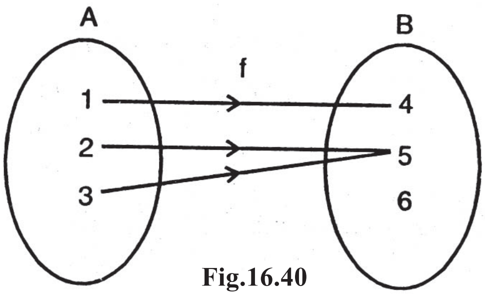


Fig.16.40

Ir represents many-to-one function. Now find the inverse of the relation.

Pictorially it is represented as

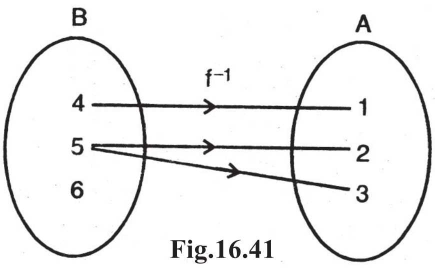


Fig.16.41

This does not represent a function, because element 6 of set B is not associated with any element of A. Also note that the elements of B does not have a unique image.

(D) Let us take the following relation

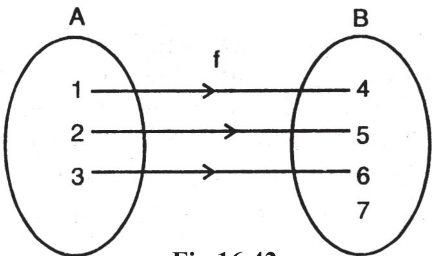


Fig.16.42

It represent one-to-one into function.

Find the inverse of the relation.

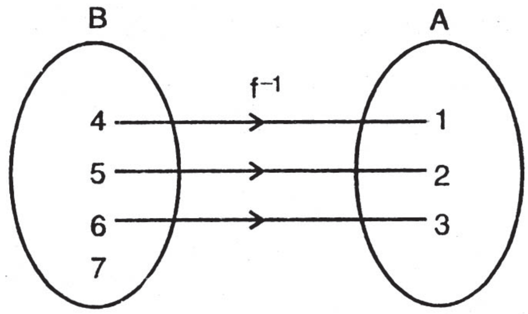


Fig.16.43

It does not represent a function because the element 7 of B is not associated with any element of A. From the above relations we see that we may or may not get a relation as a function when we find the inverse of a relation (function).

We see that the inverse of a function exists only if the function is one-to-one onto function i.e. only if it is a bijective function.

EXERCISE 16.6

1. Show that the inverse of the function

$$y = 4x - 7 \quad \text{exists.}$$

(ii) Let f be a one-to-one and onto function with domain A and range B. Write the domain and range of its inverse function.

2. Find the inverse of each of the following functions (if it exists) :

(a) $f(x) = x + 3 \;\forall\; x \in R$

(b) $f(x) = 1 - 3x \;\forall\; x \in R$

(c) $f(x) = x^2 \;\forall\; x \in R$

(d) $f(x) = \dfrac{x+1}{x},\; x \neq 0,\; x \in R$

16.10 REAL VARIABLE FUNCTION

1. A function $f : A \to B$ is called a real variable function if $A \subseteq R$.
2. A function $f : A \to B$ is called real valued function if $B \subseteq R$.
3. A function $f : A \to B$ is called real function if $A \subseteq R$, $B \subseteq R$.

Example: a^x $(a > 0)$, $\sin x$, $\log x$ these are all real functions.

16.10.1 n^{th} root of a non-negative real number

Let x be a non-negative real number and n be a positive integer then there exists a unique non-negative real number y such that n such that $y^n = x$. This number y is called the n^{th} root of x and is denoted as $x^{\frac{1}{n}}$ (or) $\sqrt[n]{x}$

Example 16.13: The domain of the real valued function

$f(x) = \sqrt{a^2 - x^2}$ $(a > 0)$ is $[-a, a]$.

[since $\sqrt{a^2 - x^2} \in R \quad (a > 0)$

$\Leftrightarrow a^2 - x^2 \geq 0 \Leftrightarrow x^2 \leq a^2$

$\Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$]

16.10.2 Algebra of real valued functions

If f and g are real valued functions with domains A and B respectively, then both f and g are defined on $A \cap B$ when $A \cap B = \phi$.

i) $(f \pm g)(x) = f(x) \pm g(x)$ defined on $A \cap B$

ii) $(f g)(x) = f(x)\, g(x)$ defined on $A \cap B$

iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $\forall x \in E$, $E = \{x \in A \cap B / g(x) \neq 0\}$

iv) Let $f : A \to R$ and $n \in N$, we define $|f|$ and f^n on A by $|f|(x) = |f(x)|$ $f^n(x) = (f(x))^n \; \forall\, x \in A$.

v) If E = $\{x \in A / f(x) \geq 0\} \neq \phi$, then we define $\sqrt{f}$ on E by $\sqrt{f}(x) = \sqrt{f(x)}$, for all $x \in E$.

Find the domains of the following real valued functions.

Example 16.14: $f(x) = \dfrac{1}{6x - x^2 - 5}$

Solution: $f(x) = \dfrac{1}{6x - x^2 - 5} = \dfrac{1}{(x-1)(5-x)} \in R$

$\Leftrightarrow (x - 1)(x - 5) \neq 0$

$\Leftrightarrow x \neq 1, 5$

$\therefore$ Domain of f is R − {1, 5}.

Example 16.15 : $f(x) = \sqrt{(x+2)(x-3)}$

Solution : $\sqrt{(x+2)(x-3)} \in R$

$\Leftrightarrow (x + 2)(x - 3) \geq 0$

$\Leftrightarrow x \leq -2$ or $x \geq 3$

$\Leftrightarrow x \in (-\infty, -2] \cup [3, \infty)$

$= R - (-2, 3)$.

Example 16.16: $f(x) = \sqrt{x^2 - 1} + \dfrac{1}{\sqrt{x^2 - 3x + 2}}$

Sol : $f(x) = \sqrt{x^2 - 1} + \dfrac{1}{\sqrt{x^2 - 3x + 2}} \in R$

$\Leftrightarrow x^2 - 1 \geq 0,\; x^2 - 3x + 2 > 0$

$\Leftrightarrow$ $(x + 1)(x - 1) \geq 0$ and $(x - 1)(x - 2) > 0$

$\Leftrightarrow x \in (-\infty, -1) \cup [1, \infty]$ and $x \in (-\infty, 1) \cup (2, \infty)$

$\Leftrightarrow$ $x \in R - (-1, 1) \cap R - (1, 2)$

$\Leftrightarrow x \in R - \{(-1, 1) \cup (1, 2)\}$, $\Leftrightarrow x + R - [-1, 2] = (-\infty, -1)] \cup (2, \infty)$

Domain of f is « $(-\infty, 1) \cup (2, \infty) = R - (-1, 2)$.

Example 16.17 : (i) : $f = \{(4, 5) (5, 6), (6, -4)\}$ and $g = \{(4, -4), (6, 5), (8, 5)\}$ find (i) $(f - g)$

Sol : $f - g = \{(4, 5 + 4), (6, -4 -5)\}$

$= \{(4, 9), (6, -9)\}$.

Example 16.18: : Find $2f + 4g$

Sol : Domain of $2f = \{4, 5, 6\}$ Domain of $4g = \{4, 6, 8\}$

But $2f = \{(4, 10) (5, 12) (6, -8)\}$

$4g = \{(4, -16) (6, 20) (8, 20)\}$

Domain of $2f + 4g$ « $\{4, 6\}$

$2f + 4g = \{(4, 10 - 16), (6, -8 + 20)\}$

$= \{(4, -6), (6, 12)\}$.

Example 16.19 : Find fg

Sol : Domain of fg $A \cap B = \{4, 6\}$

$fg = \{(4, (5) (-4)), (6, (-4)(-5)\}$

$= \{(4, -20), (6, 20)\}$.

Example 16.20 : Find $\frac{f}{g}$

Sol : Domain of $\frac{f}{g} = \{4, 6\}$

$$\therefore \frac{f}{g} = \left\{\left(4, \frac{-5}{4}\right), \left(6, \frac{-4}{5}\right)\right\}$$

Example 16.21: Find f^2

Sol : Domain of f^2 = A = {4, 5, 6}

$$f^2 = \{(4, 25), (5, 36), (6, 16)\}.$$

Find the Domains and ranges of the following real valued functions.

Example 16.22 : $f(x) = \frac{2+x}{2-x}$

Sol : $f(x) = \frac{2+x}{2-x} \in R \Leftrightarrow 2-x \neq 0 \Leftrightarrow x \neq 2$

$\Leftrightarrow x \in R - \{2\}$

Domain of f is R − {2}

Let $f(x) = y$

$$\Rightarrow \frac{2+x}{2-x} = y \Rightarrow x = \frac{2(y-1)}{y+1}$$

x is not defined for $y + 1 = 0$ i.e., when $y = -1$

$\therefore$ range of f « R − { −1}.

Example 16.23: $f(x) = \sqrt{9-x^2}$

Sol : $\sqrt{9-x^2} \in R \Leftrightarrow 9-x^2 \geq 0$

$\Leftrightarrow (3+x)(3-x) \geq 0$

$\Leftrightarrow x \in [-3, 3]$

$\therefore$ Domain of $f = [-3, 3]$

$y = f(x) \Rightarrow x = \sqrt{9-y^2} \in R$

$\Leftrightarrow 9-y^2 \geq 0 \Leftrightarrow (3+y)(3-y) \geq 0$

$\therefore -3 \leq y \leq 3$ but $f(x)$ attains only non negative values.

$\therefore$ range of $f = [0, 3]$.

16.11 TYPES OF FUNCTIONS

16.11.1 Implicit function :

y is said to be an implicit function of x if it is given in the form of

$f(x, y) = 0$.

Ex: $x^2 + xy + y^2 - 2 = 0$.

EXERCISE 16.7

1. Find the domain of real valued function $f(x) = \dfrac{2x^2 - 5x + 7}{(x-1)(x-2)(x-3)}$.

2. Find the domain of $f(x) = \sqrt{4x - x^2}$.

3. Find the domain of $f(x) = \sqrt{x^2 - 25}$.

4. Find the range of the following functions.

   i) $\dfrac{\sin \pi[x]}{1+[x]^2}$ ii) $\dfrac{x^2-4}{x-2}$

5. If $f(x) = 2x - 1$, $g(x) = x^2$ then find the following.

i) $\left[\frac{\sqrt{f}}{g}\right](x)$ ii) $(f + g + 2)(x)$

6. If $f = \{(1, 2)\ (2, -3), (3, -1)\}$ then find

i) $2f$ ii) f^2 iii) $\sqrt{f}$

KEY WORDS

- Set is a well defined collection of objects
- To represent a set in Roster form all elements are to be written but in set builder form a set is represented by the common property.
- If the elements of a set can be counted then it is called a finite set and if the elements cannot be counted, it is infinite.
- Cartesian product of two sets A and B is the set of all ordered pairs of the elements of A and B. It is denoted by $A \times B$. i.e.

  i.e., $A \times B = \{(a, b) : a \in A, \text{ and } b \in B\}$.
- Relation is a sub set of $A \times B$ where A and B are sets.

  i.e., $R \subseteq A \times B\ \{(a, b) : a \in A \text{ and } b \in B \text{ and } a\,R\,b\}$
- Function is a special type of relation.
- Functions $f : A \to B$ is a rule of correspondence from A to B such that to every element of A $\exists$ a unique element in B.
- Functions can be described as a set of ordered pairs.
- Let f be a function from A to B.

  Domain : Set of all first elements of ordered pairs belonging to f.

  Range : Set of all second elements of ordered pairs belonging to f.

- Functions can be written in the form of equations such as $y = f(x)$ where x is independent variable, y is dependent variable.

  Domain : Set of independent variable.

  Range : Set of dependent variable.

  Every equation does not represent a function.

- **Vertical line test :** To check whether a graph is a function or not, we draw a line parallel to y-axis. If this line cuts the graph in more than one point, we say that graph does not represent a function.

- Let f be a function from a set A to a set B. Symbolically we can write it as $f : A \to B$

  (i) If every element of B is not an image of some element of A then f is said to be into function. In this case, range $\subset$ co-domain.

  (ii) If range = co-domain, then f is said to be onto function.

  (iii) If distinct elements of set A have distinct images in set B then f is called one-to-one function.

  (iv) If many elements in the domain of a function have the same image element in the range, then the function is called many-to-one function.

- **Horizontal Line Test :** To check the one-to-oneness of a function, draw a line parallel, to x-axis. If it cuts the graph at one point, we say that it is one-to-one function.

- A function is said to be monotonic on an interval if it is either increasing or decreasing on that interval.

- A function is called even function if $f(x) = f(-x)$, and odd function if $f(-x) = -f(x),\ x,\ -x \in D_f$

- Inverse of a function exists if it is one-to-one and onto.

- A function $f : A \to B$ is a real valued function if $A \subseteq R$, $B \subseteq R$.
- If $y^n = x$ then y is n^{th} root of x. This can be written as $x^{\frac{1}{n}}$ or $\sqrt[n]{x}$
- If f, g are real functions with domains A, B two sets then both f, g are defined on $A \cap B$.
- $(f \pm g)x = f(x) \pm g(x)$ domain $= A \cap B$
- $(f\,g)x = f(x)\ g(x)$ domain $= A \cap B$
- $\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}$ Đ Domain = $\{x/x \in A \cap B, g(x) \neq 0\}$
- Let A be a non empty subset of R such that $-x \in A\ \forall x \in A$ and $f : A \to R$.

  (i) If $f(-x) = f(x)\ \forall x \in A$ then f is called even.

  (ii) If $f(-x) = -f(x)\ \forall x \in A$ then f is called odd.
- If $f(a) = a^x$ $(a > 0)$ then $f(x)$ is Exponential function.
- If $f(x) = \log_a x$ $(a \neq 1,\ x > 0)$ then $f(x)$ is logarithmic function.

SUPPORTIVE WEBSITES

- *http://www.wikipedia.org*
- *http://mathworld.walfram.com*

PRACTICE EXERCISE

1. Which of the following statements are true or false :

   (i) $\{1, 2, 3\} = \{1, \{2\}, 3\}$ (ii) $\{1, 2, 3\} = \{3, 1, 2\}$

   (iii) $\{a, e, o\} = \{a, b, c\}$ (iv) $\{\phi\} = \{\ \}$

2. Write domain and range of the following functions :

   f_1 : {(0, 1), (2, 3), (4, 5), (6, 7), ... (100, 101)}

f_2 : {(–2, 4), (–4, 16), (–6, 36), ...}

f_3 : {(1, 1), (1/2, –1), (1/3, 1), (1/4,–1) .....}

f_4 : {.....(3, 0), (–1, 2), (4, –1)}

f_5 : {.....(–3, 3), (–2, 2), (–1, 1), (0, 0), (1, 1), (2, 2), ...}

3. Write domain of the following functions:

(a) $f(x) = x^3$ (b) $f(x) = \dfrac{1}{x^2 - 1}$

(c) $f(x) = \sqrt{3x+1}$ (d) $f(x) = \dfrac{1}{\sqrt{(x+1)(x+3)}}$

(e) $f(x) = \dfrac{1}{\sqrt{(x-1)(2x-5)}}$

4. Which of the following graphs represent a function ?

(a)

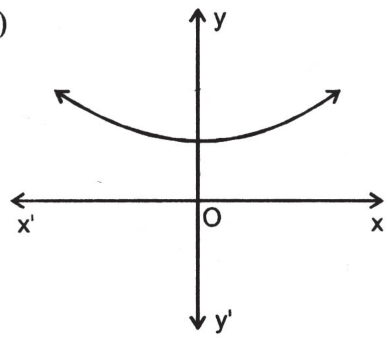


Fig. 16.44

(b)

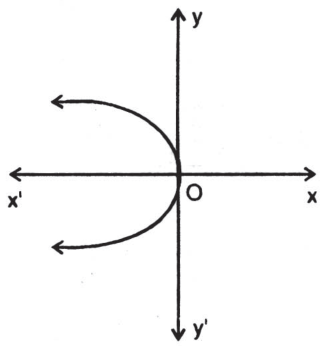


Fig. 16.45

(c)

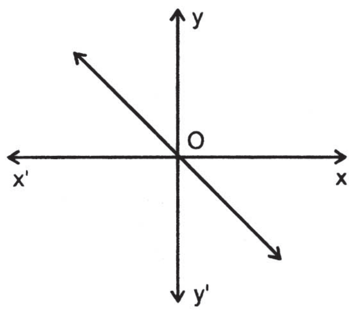


Fig. 16.46

(d)

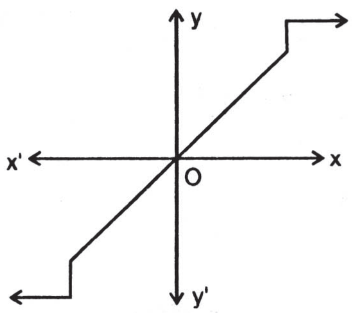


Fig. 16.47

(e)

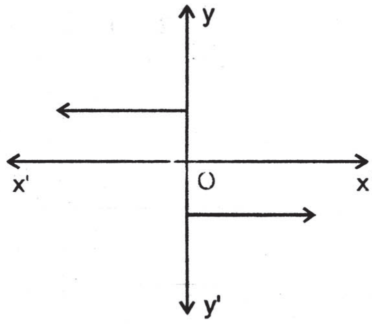


Fig. 16.48

(f)

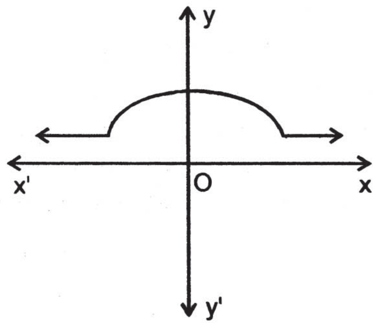


Fig. 16.49

(g)

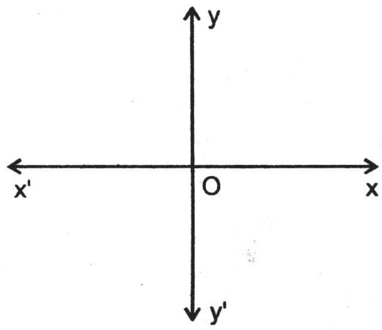


Fig. 16.50

(h)

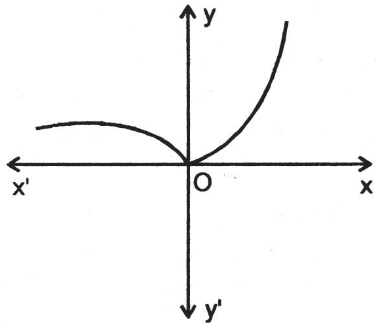


Fig. 16.51

5. Which of the following functions are one-to-one or many-to-one function?

   (a) $y = 3x + 5,\ x \in R$

   (b) $y = 1 - 2x^2,\ x \in R$

   (c) $f: \{(1, 2), (2, 2), (3, 2), (4, 2)\}$

   (d) $f: \{1, 2, 3\} \rightarrow \{4, 5, 6, 8\}$ such that $f(1) = 4,\ f(2) = 6, f(3) = 8$.

6. Which of the following functions are rational functions ?

   (a) $f(x) = \dfrac{2x-3}{x+2},\ x \in R - \{-2\}$ (b) $f(x) = \dfrac{x}{\sqrt{x}},\ x \in R^{-1}$

   (c) $f(x) = \dfrac{x+2}{x^2+4x+4},\ x \in R - \{-2\}$ (d) $y = x,\ x \in R$

7. Which of the following functions are polynomial functions ?

   (a) $f(x) = x^2 + \sqrt{3}x + 2$

(b) $f(x) = (x + 2)^2$

(c) $f(x) = 3 - x + 2x^3 - x^4$

(d) $f(x) = \sqrt{x} + x - 5,\ x \geq 0.$

(e) $f(x) = \sqrt{x^2 - 4},\ x \notin (-2, 2)$

8. Which of the following functions are even or odd functions ?

(a) $f(x) = \sqrt{9 - x^2},\ x \in [-3, 3]$

(b) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

(c) $f(x) = |x|$

(d) $f(x) = x - x^5$

9. Write for each of the following functions *fog, gof, fof, gogo*

(a) $f(x) = x^3$ $g(x) = 4x - 1$

(b) $f(x) = \dfrac{1}{x^2},\ x \neq 0$ $g\,|\,x\,| = x^2 - 2x + 3$

(c) $f(x) = \sqrt{x - 4},\ x \geq 4$ $g(x) = x - 4$

(d) $f(x) = x^2 - 1$ $g(x) = x^2 + 1$

10. (a) Let $f(x) = |x|$, $g(x) = 1/x$, $x \neq 0$, $h(x) = x^{1/3}$ Find *fogoh*

(b) $f(x) = x^2 + 3$, $g(x) = 2x^2 + 1$ Find *fog* (3) and *gof* (3)

ANSWERS

EXERCISE 16.1

1. (a) Function (b) Not a function

(c) Function (d) Not a function

(e) Not a function (f) Not a function

2. (a) Domain = $\{\sqrt{2}, \sqrt{5}, \sqrt{3}\}$, range = $\{-1, 2, 5\}$

(b) Domain = {–3, –2, –1}, range = $\left\{\frac{1}{2}\right\}$

(c) Domain = {–1, 0, 1, 2}, range = {–1, 0, 1, 2}

(d) Domain = {Deepak, Sandeep, Rajan}, range = {16, 24, 28}

4. (a) Domain = {1, 2, 3}, range = {4, 5, 6}

(b) Domain = {1, 2, 3}, range = {4}

(c) Domain = {1, 2, 3}, range = {1, 2, 3}

(d) Domain = {Gagon, Ram, Shyam}, range = {5, 8, 9}

(e) Domain = {a, b, c, d}, range = {2, 4}

EXERCISE 16.2

1. (a) (i) Domain = Set of real numbers.

(ii) Domain = Set of real numbers

(iii) Domain = Set of a real numbers.

(b) (i) Domain = $R - \left\{\frac{1}{3}\right\}$ (ii) Domain = $R - \left\{-\frac{1}{4}, 5\right\}$

(iii) Domain = R – {3, 5} (iv) Domain = R – {3, 5}

(c) (i) Domain = $\{x \in R;\ x \le 6\}$ (ii) Domain = $\{x \in R;\ x \ge -7\}$

(iii) Domain = $\{x : x \in R;\ x \ge -5/3\}$

(d) (i) Domain = $\{x : x \in R \text{ and } 3 \le x \le 5\}$

(ii) Domain = $\{x : x \in R,\ x \ge 3, x \le -5\}$

(iii) Domain = $\{x : x \in R,\ x \ge -3, x \le -7\}$

(iv) Domain = $\{x : x \in R,\ x \ge 3, x \le -7\}$

2. (a) (i) {4, 7, 13, 25} (ii) {1, 9, 18, 33} (iii) {2, 4, 8, 14, 22}

(b) (i) (i) [–2, 2] (ii) [1, 0]

(c) (i) [−25, 25] (ii) [−6, 6] (iii) [1, 5] (iv) [0, 5]

3 (a) (i) Range = {13, 25, 31, 7, 4}

(ii) Range = {19, 9, 33, 1}

(b) (i) Range = $\{ f(x) : -2 \le f(x) \le 2\}$

(ii) Range = $\{ f(x) : 1 \le f(x) \le 10\}$

(c) (i) Range = $\{ f(x) : 1 \le f(x) \le 25\}$

(ii) Range = $\{ f(x) : -6 \le f(x) \le 6\}$

(iii) Range = $\{ f(x) : 1 \le f(x) \le 5\}$

(iv) Range = $\{ f(x) : 0 \le f(x) \le 5\}$

(d) (i) Range = R (ii) Range = R

(iii) Range = R

(iv) Range = $\{ f(x) : f(x) < 0\}$

(v) Range = $\{ f(x) : -1 \le f(x) \le 0\}$

(vi) Range = $\{ f(x) : 0.5 \le f(x) < 0\}$

(vii) Range = $\{ f(x) : f(x) > 0\}$

(viii) Range: All values of f(x) except values at x = −5.

EXERCISE 16.3

1. (i) No. (ii) Yes
2. (a), (b)
3. (a)
4. (a), (c), (e)
5. (a)

EXERCISE 16.4

1. v, vi, vii are true statements.

   (i), (ii), (iii), (iv) are incorrect statement.

2. (a) Polynomial function (b) Rational function.

   (c) Rational function. (d) Rational function.

   (e) Rational function. (f) Rational function.

   (g) Constant function

EXERCISE 16.5

1. (a) $fog = 4x^2 + 20x + 21,\quad gof = 2x^2 - 3,$

   $fof = x^4 - 8x^2 + 12$

   $gog = 4x + 15$

   (b) $fog = 9,\quad gof = 3,\quad fof = x^4,\quad gog = 3.$

   (c) $fog = \frac{6-7x}{x},\quad gof = \frac{2}{3x-7},\quad fof = 9x - 28,\quad gog = x.$

EXERCISE 16.6

1. (ii) Domain is B. Range is A.

2. (a) $f^{-1} = x - 3$

   (b) $f^{-1}(x) = \frac{1-x}{3}$

   (c) Inverse does not exist.

   (d) $f^{-1}(x) = \frac{1}{x-1}$

EXERCISE 16.7

1. R – {1, 2, 3}

2. [0, 4]

3. R – (–5, 5)

4. i) {0} ii) R – {4}

5. i) $\frac{\sqrt{2x-1}}{x^2}$ ii) $(x + 1)^2$

6. i) {(1, 4), (2, –6), (3, –2)}

   ii) {(1, 4) (2, 9), (3, 1)}

   iii) $\left\{(1, \sqrt{2})\right\}$

PRACTICE EXERCISE

1. (i) False (ii) True

   (iii) False (iv) False

2. f_1 - Domain = {0, 2, 4, 6, ... 100}

   Range = {1, 3, 5, 7, ... 101}

   f_2 - Domain ={–2, –4, –6, ... }

   Range ={4, 16, 3, 6, ...}

   f_3 - Domain = {1, 1/2, 1/3, 1/4, ...}, Range ={1, –1}

   f_4 - Domain ={3, –1, 4}, Range ={0, 1, 2, 3, ...}

   f_5 - Domain = {.....–3, –2, –1, 0, 1, 2, 3, ...}

   Range = {0, 1, 2, 3, ...}

3. (a) R (b) R = {–1, 1} (c) $x \geq -\frac{1}{3}$ (d) $[1, \infty)$

(e) $x > \frac{5}{2}$

4. (a) Function (b) Not function (c) Function

(d) Not function (e) Not function (f) Not function

(g) function (h) function

5. (a) One - One (b) Not One - One

(c) Not One - One (d) One - One

6. a, c 7. a, b, c

8. (a) Even (b) Even (c) Even (d) odd

9. (a) $fog = (4x - 1)^3$, $gof = 4x^3 - 1$, $fof = x^9$, $gog = 16x - 5$

(b) $fog = \frac{1}{(x^2-2x+3)^2}$, $gof = \frac{3x^4-2x^2+1}{x^4}$

$fog = x^4$, $gog = x^4 - 4x^3 + 4x^2$

(c) $fog = \sqrt{x-8}$ $gof = \sqrt{x-4}-4$

$fof = \sqrt{\sqrt{x-4}-4}$, $gog = x - 8$

(d) $fog = x^4 + 2x^2$, $gof = x^4 - 2x^2 + 2$,

$fof = x^4 - 2x^2$, $gog = x^4 + 2x^2 + 2$

10. (a) $\left|\frac{1}{x^{\frac{1}{3}}}\right|$ (b) (fog) (3) = 364, (gof) (3) = 289.

TRIGONOMETRIC FUNCTIONS

Chapter **17**

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- define positive and negative angles;
- define degree and radian as a measure of an angle;
- convert measure of an angle from degrees to radians and vice-versa;
- state the formula $l = r\,\theta$ where r and θ have their usual meanings;
- define trigonometric functions of a real number;
- draw the graphs of trigonometric functions;
- establish the addition and subtraction formulae for :

  $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

  $\sin(A \pm B) = \sin A \sin B \pm \cos A \cos B$ and $\tan(A \pm B) = \dfrac{\tan A + \tan B}{1 \mp \tan A \tan B}$
- solve problems using the addition and subtraction formulae
- state the formulae for the multiples and sub-multiples of angles such as $\cos 2A$, $\sin 2A$, $\tan 2A$, $\cos 3A$, $\sin 3A$, $\tan 3A$, $\sin \dfrac{A}{2}$, $\cos \dfrac{A}{2}$ and $\tan \dfrac{A}{2}$; and

- sum and product of trigonometric ratios

  $\sin C \pm \sin D$, $\cos C \pm \cos D$

- solve simple trigonometric equations of the type :

  $\sin x = 0$, $\cos x = 0$, $\tan x = 0$

  $\sin x = \sin \alpha$, $\cos x = \cos \alpha$, $\tan x = \tan \alpha$

PREREQUISITES

- Definition of an angle.
- Definition of trigonometric functions.
- Values of trigonometric ratios.
- Trigonometric functions of complementary and supplementary angles.
- Trigonometric identities.

INTRODUCTION

The word 'trigonometry' can be read as trigon-o-metry. This word id derived form two Greek words (i) trigonon (ii) metron.

The word 'trigonon' means a triangle and the word 'metron' means a measure. Thus trigonometry is the science deals with measurements of triangles. Trigonometry has great use in measurement of areas, heights, distances etc. It has many applications in almost all branches of science in general and in physics and Engineering in particular.

We have read about trigonometric ratios in our earlier classes. Recall that we defined the ratios of the sides of a right triangle as follows:

$$\sin\theta = \frac{c}{b},\ \cos\theta = \frac{a}{b},\ \tan\theta = \frac{c}{a}$$

and $\text{cosec}\,\theta = \dfrac{b}{c},\ \sec\theta = \dfrac{b}{a},\ \cot\theta = \dfrac{a}{c}$

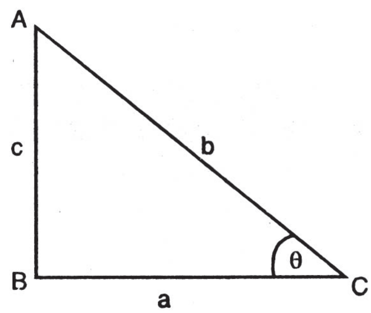


Fig.17.1

We also developed relationships between these trigonometric ratios as

$\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, $1 + \cot^2\theta = \text{cosec}^2\theta$

We shall try to describe this knowledge gained so far in terms of functions, and try to develop this lesson using functional approach.

In this lesson, we shall develop the science of trigonometry using functional approach. We shall develop the concept of trigonometric functions using a unit circle. We shall discuss the radian measure of an angle and also define trigonometric functions of the type

$y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \cot x$, $y = \sec x$,

$y = \text{cosec } x$, $y = a \sin x$, $y = b \cos x$

etc., where x, y are real numbers.

We shall draw the graphs of functions of the type

$y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \cot x$, $y = \sec x$, and

$y = \text{cosec } x$,

In this lesson we will establish addition and subtraction formulae for $\cos(A \pm B)$, $\sin(A \pm B)$ and $\tan(A \pm B)$. We will also state the formulae for the multiple and sub multiples of angles and solve examples thereof. The general solutions of simple trigonometric functions also discussed in the lesson.

17.1 CIRCULAR MEASURE OF ANGLE

An angle is a union of two rays with the common end point. An angle is formed by the rotation of a ray as well. Negative and positive angles are formed according as the rotation is clockwise or anticlock-wise.

17.1.1 A Unit Circle

It can be seen easily that when a line segment makes one complete rotation, its end point describes a circle. In case the length of the rotating line be one unit then the circle described will be a circle of unit radius. Such a circle is termed as ***unit circle.***

17.1.2 A Radian

A radian is another unit of measurement of an angle other than degree.

A radian is the measure of an angle subtended at the centre of a circle by an arc equal in length to the radius (r) of the circle. In a unit circle one radian will be the angle subtended at the centre of the circle by an arc of unit length.

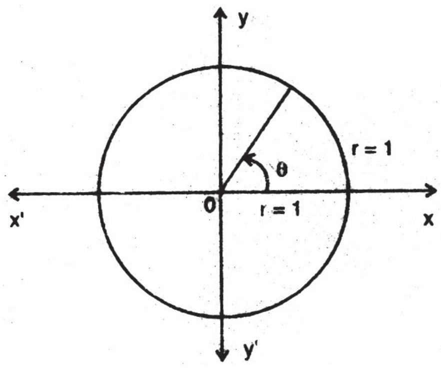


Fig. 17.2

Note: A radian is a constant angle; implying that the measure of the angle subtended by an are of a circle, with length equal to the radius is always the same irrespective of the radius of the circle.

17.1.3 Relation between Degree and Radian

An arc of unit length subtends an angle of 1 radian. The circumference 2π $(r = 1)$ subtend an angle of 2π radian.

Hence $2\pi = 360^0$

π radians $= 180^0$

$\frac{\pi}{2}$ radians $= 90^0$

$\frac{\pi}{4}$ radians $= 45^0$

$$1 \text{ radians} = \left(\frac{360}{2\pi}\right)^0 = \left(\frac{180}{\pi}\right)^0$$

$$\text{or } 1^0 = \frac{2\pi}{360} \text{ radians} = \frac{\pi}{180} = \text{radians}$$

Example 17.1 Convert

(i) 90^0 into radians (ii) 15^0 into radians

(iii) $\frac{\pi}{6}$ radians into degrees. (iv) $\frac{\pi}{10}$ radians into degrees.

Solution: $1^0 = \frac{2\pi}{360}$ radians

$$\Rightarrow 90^0 = \frac{2\pi}{360} \times 90 \text{ radians} \quad \text{or} \quad 90^0 = \frac{\pi}{2} \text{ radians}$$

(ii) $15^0 = \frac{2\pi}{360} \times 15$ radians or $15^0 = \frac{\pi}{12}$ radians

(iii) 1 radian $= \left(\frac{360}{2\pi}\right)^0$

$$\frac{\pi}{6} \text{ radians} = \left(\frac{360}{2\pi} \times \frac{\pi}{6}\right)^0$$

$$\frac{\pi}{6} \text{ radians} = 30^0$$

(iv) $\frac{\pi}{10}$ raidans $= \left(\frac{360}{2\pi} \times \frac{\pi}{10}\right)^0$

$$\frac{\pi}{10} \text{ raidans} = 18^0.$$

EXERCISE 17.1

1. Convert the following angles (in degrees) into radians :

   (i) 60^0 (ii) 15^0 (iii) 75^0

   (iv) 105^0 (v) 270^0.

2. Convert the following angles into degrees:

   (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{12}$ (iii) $\frac{\pi}{20}$

   (iv) $\frac{\pi}{60}$ (v) $\frac{2\pi}{3}$

3. The angles of a triangle are 45^0, 65^0 and 70^0. Express these angles in radians

4. The three angles of a quadrilateral are $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}$ Find the fourth angle in radians.

5. Find the angle complementary to $\frac{\pi}{6}$.

17.1.4 Relation Between Length of an Arc and Radius of the Circle

An angle of 1 radian is subtended by an arc whose length is equal to the radius of the circle. An angle of 2 radians will be substened if arc is double the radius.

An angle of $2\frac{1}{2}$ radians willbe subtended if arc is $2\frac{1}{2}$ times the radius.

All this can be read from the following table :

Length of the arc (l)	Angle subtended at the centre of the circle θ (in radians)
r	1
$2r$	2
$\left(2\frac{1}{2}\right)r$	$2\frac{1}{2}$
$4r$	4

Therefore, $\theta = \frac{l}{r}$ or $l = r\,\theta$.

where r = radius of the circle,

θ = angle substended at the centre in radians

and l = length of the arc.

The angle subtended by an arc of a circle at the centre of the circle is given by the ratio of the length of the arc and the radius of the circle.

Note : In arriving at the above relation, we have used the radian measure of the angle and not the degree measure. Thus the relation $\theta = \frac{l}{r}$ is valid only when the angle is measured in radians.

Example 17.2 : Find the angle in radians subtended by an arc of length 10 cm at the centre of a circle of radius 35 cm.

Solution : l = 10cm and r = 35cm.

$$\theta = \frac{l}{r} \text{ radians} \qquad \text{or} \qquad \theta = \frac{10}{35} \text{ radians}$$

$\because$

or $\theta = \frac{2}{7}$ radians

Example 17.3 : If D and C represent the number of degrees and radians in an angle prove that

$$\frac{D}{180} = \frac{C}{\pi}$$

Solution: $\left(\frac{360}{2\pi}\right)^0$ or $\left(\frac{180}{\pi}\right)^0$

$$\therefore \quad C \text{ radians} = \left(C \times \frac{180}{\pi}\right)^0$$

Since D is the degree measure of the same angle, therefore,

$$D = C \times \frac{180}{\pi}$$

which implies $\frac{D}{180} = \frac{C}{\pi}$.

Example 17.4 : A railroad curve is to be laid out on a circle. What should be the radius of a circular track if the railroad is to turn through an angle of 45° in a distance of 500m?

Solutin : Angle θ is given in degrees. To apply the formula $l = r\theta$, θ must be changed to radians.

$$\theta = 45^0 = 45 \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{\pi}{4} \text{ radians} \qquad ...(1)$$

$l = 500$ m

$l = r\,\theta$ gives $r = \dfrac{l}{\theta}$

$\therefore\ r = \dfrac{500}{\pi/4} m$ [using (1) and (2)]

$= 500 \times \dfrac{4}{\pi}$ m

$= 2000 \times 0.32m \quad \left[\dfrac{1}{\pi} = 0.32\right]$

$\therefore\ r = 640$ m.

Example 17.5 : A train is travelling at the rate of 60 km per hour on a circular track. Through what angle will it turn in 15 seconds if the radius of the track is $\dfrac{5}{6}$ km.

Solution : The speed of the train is 60 km per hour. In 15 seconds, it will cover

$$\frac{60 \times 15}{60 \times 60} \text{ km}$$

$$= \frac{1}{4} \text{km}$$

$\therefore$ We have, $l = \dfrac{1}{4}$km and $r = \dfrac{5}{6}$km

$$\theta = \frac{l}{r} = \frac{\frac{1}{4}}{\frac{5}{6}} \text{ radians}$$

$$= \frac{3}{10} \text{ radians.}$$

EXERCISE 17.2

1. Express the following angles in radians :

(a) 30^0 (b) 60^0 (c) 150^0

2. Express the following angles in degrees :

   (a) $\frac{\pi}{5}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{9}$

3. Find the angle in radians and in degrees subtended by an arc of length 2.5 cm at the centre of a circle of radius 15 cm.

4. A train is travelling at the rate of 20 km per hour on a circular track. Through what angle will it turn in 3 seconds if the radius of the track is $\frac{1}{12}$ of a km?

5. A railroad curve is to be laid out on a circle. What should be the radius of the circular track if the railroad is to turn through an angle of 60° in a distance of 100 m?

6. Complete the following table for l, r, θ having their usual meanings.

	l	r	θ
a)	1.25m	......	135^0
b)	30 cm	.....	$\pi/4$
c)	0.5 cm	2.5m	.....
d)	.....	6m	120^0
e)	....	150 cm	$\pi/15$
f)	150 cm	40 m	.....
g)	.....	12m	$\pi/6$
h)	1.5m	0.75m	.....
i)	25m	.....	75^0

17.2 TRIGONOMETRIC FUNCTIONS

While considering, a unit circle you must have noticed that for every real number between 0 and 2π , there exists a ordered pair of numbers x and y. This ordered pair (x, y) represents the coordinates of the point P.

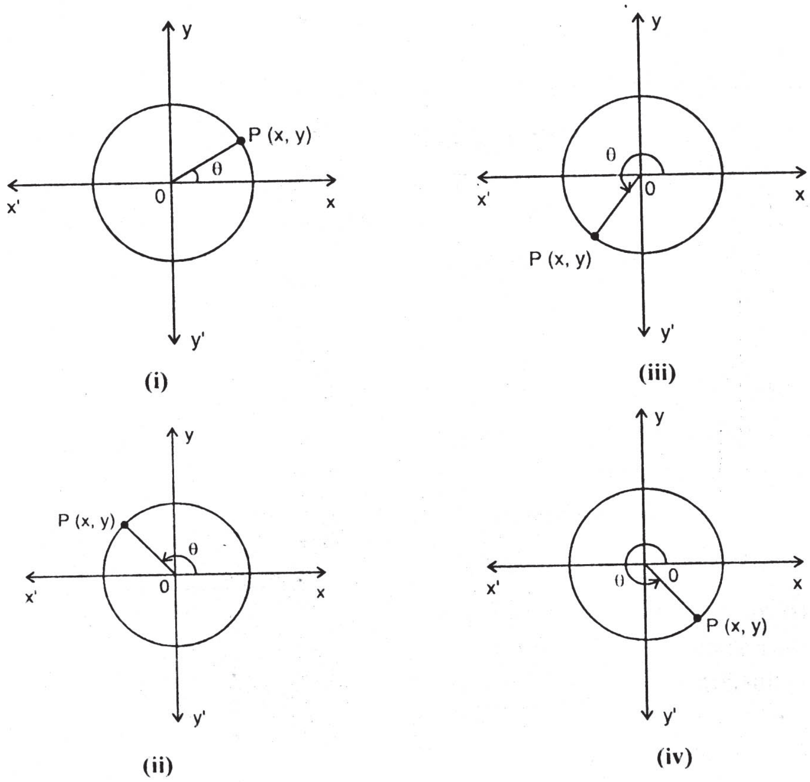


Fig. 17.3

If we consider $\theta = 0$ on the unit circle, we will have a point whose coordinates are (1,0).

If $\theta = \dfrac{\pi}{2}$ then the corresponding point on the unit circle will have its coordinates (0,1).

In the above figures you can easily observe that no matter what the position of the point, corresponding to every real number θ we have a unique set of coordinates (x, y). The values of x and y will be negative or positive depending on the quadrant in which we are considering the point.

Considering a point P (on the unit circle) and the corresponding coordinates (x, y), we define trigonometric functions as :

$$\sin\theta = y, \qquad \cos\theta = x, \qquad \tan\theta = \frac{y}{x} \text{ (for } x \neq 0)$$

$$\cot\theta = \frac{x}{y} \text{ (for } y \neq 0), \quad \sec\theta = \frac{1}{x} \ (x \neq 0), \quad \operatorname{cosec}\theta = \frac{1}{y} \ (y \neq 0)$$

Now let the point P move from its original position in anti-clockwise direction. For various positions of this point in the four quadrants, various real numbers θ will be generated. We summarise, the above discussion as follows. For values of θ in the :

I quadrant, both x and y are positve.

II quadrant, x will be negative and y will be positive.

III quadrant, x as well as y will be negative.

IV quadrant, x will be positive and y will be negative.

or	I quadrant	II quadrant	III quadrant	IV quadrant
	All positive	sin positive	tan positive	cos positive
		cosec positive	cot positive	sec positive

Where what is positive can be rememebred by :

	All	sin	tan	cos
Quardrant	I	II	III	IV

If (x, y) are the coordinates of a point P on a unit circle and T , the real number generated by the position of the point, then $\sin\theta = y$ and $\cos\ \theta = x$. This means the coordinates of the point P can also be written as $(\cos\theta\ , \sin\ \theta\ )$

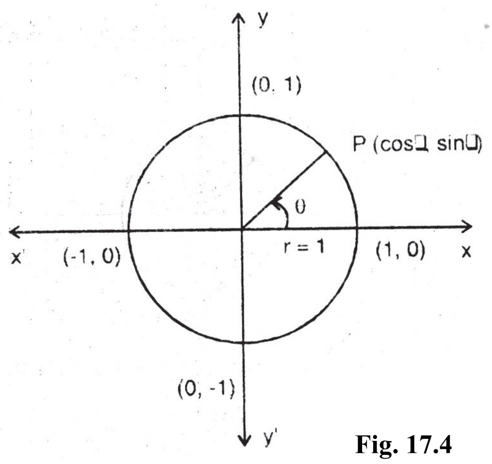


Fig. 17.4

From Fig. 17.5, you can easily see that the values of x will be between 1 and + 1 as P moves on the unit circle. Same will be true for y also.

Thus, for all P on the unit circle

$$-1 \le x \le 1 \quad \text{and} \quad -1 \le y \le 1$$

Thereby, we conclude that for all real numbers θ

$-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$

In other words, sin θ and cos θ can not be numerically greater than 1.

Quadrant Angles: The Angles $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ have their terminal side along either X-axis or Y-axis. Hence these angles are called Quadrant angles.

Negative Angle :

If the angle $\theta(0 \le \theta \le 2\pi)$ is measured in anti clock wise direction (starting from the initial side OX), it is defined as positive angle and if the same angle θ is measured in clock wise direction, it is defined as negative angle and it is identified with - θ are defined as follows.

$$\sin(-\theta) = \sin(2\pi - \theta) = \frac{-y}{r} = -\sin\theta$$

$$\cos(-\theta) = \cos(2\pi - \theta) = \frac{x}{r} = \cos\theta$$

$$\tan(-\theta) = -\tan\theta,\ \theta \notin \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$$

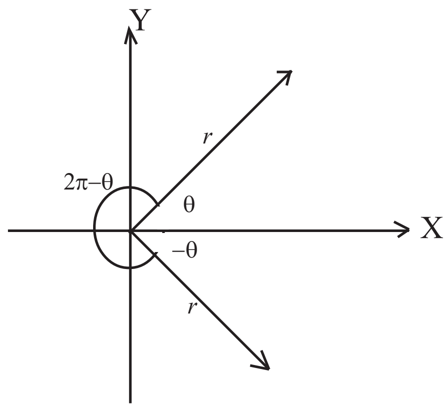


Fig. 17.5

Example 17.6 : What will be sign of the following ?

(i) $\sin\frac{7\pi}{18}$ (ii) $\cos\frac{4\pi}{9}$ (iii) $\tan\frac{5\pi}{9}$

Solution : (i) Since $\frac{7\pi}{18}$ lies in the first quadrant, the sign of $\sin\frac{7\pi}{18}$ will be posilive.

(ii) Since $\frac{4\pi}{9}$ lies in the first quadrant, the sign of $\cos\frac{4\pi}{9}$ will be positive.

(iii) Since $\frac{5\pi}{9}$ lies in the second quadrant, the sign of $\tan\frac{5\pi}{9}$ will be negative.

Example 17.7 : Write the values of (i) $\sin\frac{\pi}{2}$ (ii) $\cos 0$ (iii) $\tan\frac{\pi}{2}$

Solution :

(i) From Fig. 17.5, we can see that the coordinates of the point *A* are (0,1)

$\therefore \sin\frac{\pi}{2} = 1$, as $\sin\theta = y$

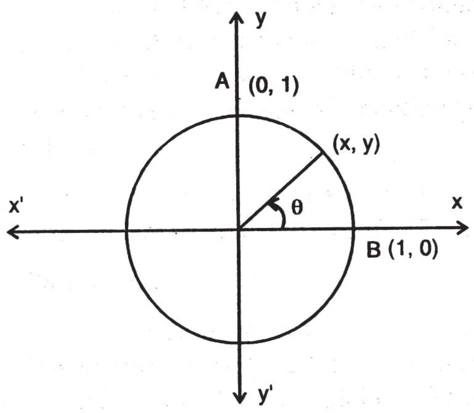


Fig.17.6

(ii) Coordinates of the point B are (1, 0)

$\therefore \cos 0 \quad 1$, as $\cos\theta = x$

(iii) $\tan\frac{\pi}{2} = \dfrac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = \frac{1}{0}$ which is not defined

Thus $\tan\frac{\pi}{2}$ is not defined.

Example 17.8 : Write the minimum and maximum values of $\cos\theta$.

Solution : $-1 \le \cos\theta \le 1$

$\therefore$ The maximum value of $\cos\theta$ is 1 and the minimum value of $\cos\theta$ is 1.

EXERCISE 17.3

1. What will be the sign of the following ?

(i) $\cos^2\frac{\pi}{3}$ (ii) $\tan\frac{5\pi}{6}$ (iii) $\sec\frac{2\pi}{3}$

(iv) $\sec\frac{35\pi}{18}$ (v) $\tan\frac{25}{18}$ (vi) $\cot\frac{3\pi}{4}$

(vii) $\operatorname{cosec}\frac{8\pi}{3}$ (viii) $\cot\frac{7\pi}{8}$

2. Write the value of each of the following :

(i) $\cos\frac{\pi}{2}$ (ii) $\sin 0$ (iii) $\cos\frac{2\pi}{3}$

(iv) $\tan\frac{3\pi}{4}$ (v) $\sec 0$ (vi) $\tan\frac{\pi}{2}$

(vii) $\tan\frac{3\pi}{2}$ (viii) $\cos 2\pi$

17.2.1 Relation Between Trigonometric Functions

By definition $x = \cos\theta$

$y = \sin\theta$

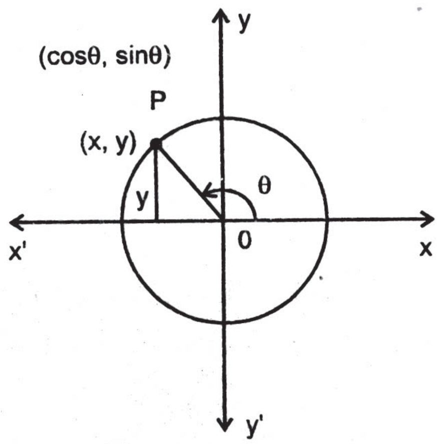


Fig. 17.7

As $\tan\theta = \frac{y}{x}, \ (x \neq 0)$

$= \frac{\sin\theta}{\cos\theta}, \ \theta \neq \frac{n\pi}{2}$

and $\cot\theta = \frac{x}{y}, \ (y \neq 0)$

i.e., $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} \ (\theta \neq n\pi)$

Similarly, $\sec\theta = \frac{1}{\cos\theta} \ \left(\theta \neq \frac{n\pi}{2}\right)$

and $\operatorname{cosec}\theta = \frac{1}{\sin\theta} \ (\theta \neq n\pi)$

Using Pythagoras theorem we have, $x^2 + y^2 = 1$

i.e., $(\cos\theta)^2 + (\sin\theta)^2$

or $\cos^2\theta + \sin^2\theta = 1$

Note: $(\cos\theta)^2$ is written as $\cos^2\theta$ and $(\sin\theta)^2$ as $\sin^2\theta$

Again $x^2 + y^2 = 1$

or $1+\left(\frac{y}{x}\right)^2 = \left(\frac{1}{x}\right)^2$, for $x \neq 0$ or $1 + (\tan\theta)^2 = (\sec\theta)^2$

i.e., $\sec^2\theta = 1 + \tan^2\theta$

Similarly, $\text{cosec}^2\theta = 1 + \cot^2\theta$.

Example 17.9 : Prove that $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$.

Solution : LHS $= \sin^4\theta + \cos^4\theta$

$$= \sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta$$

$$= (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta$$

$$= 1 - 2\sin^2\theta\cos^2\theta \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= \text{RHS}$$

Example 17.10 : Prove that $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$.

Solution : LHS $= \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$

$$= \sqrt{\frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta - \tan\theta = \text{R.H.S}.$$

Example 17.11 : If $\sin\theta = \frac{21}{29}$, prove that $\sec\theta + \tan\theta = 2\frac{1}{2}$ given that θ lies in the first quadrant.

Solution: $\sec\theta = \frac{21}{29}$

Also $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \cos^2\theta = 1 - \sin^2\theta = 1 - \frac{441}{841} = \frac{400}{841} = \left(\frac{20}{29}\right)^2$$

$$\Rightarrow \cos\theta = \frac{20}{29}$$ (cos θ is positive as θ lies in the first quardrant)

$$\therefore \tan\theta = \frac{21}{29}$$

$$\therefore \sec\theta + \tan\theta = \frac{29}{20} + \frac{21}{20} = \frac{50}{20}$$

$$= \frac{5}{2} = 2\frac{1}{2} = \text{R.H.S}$$

EXERCISE 17.4

1. Prove that $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$.
2. If $\tan\theta = \frac{1}{2}$, find the other five trigonometric functions.
3. If $\operatorname{cosec}\theta = \frac{b}{a}$, find the other five trigonometric functions, if θ lies in the first quardrant.
4. Prove that $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$.
5. If $\cot\theta + \operatorname{cosec}\theta = 1.5$, show that $\cos\theta = \frac{5}{13}$.
6. If $\tan\theta + \sec\theta = m$, find the value of $\cos\theta$.
7. Prove that $(\tan A + 2)(2\tan A + 1) = 5\tan A + \sec^2 A$.

8. Prove that $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$.

9. Prove that $\dfrac{\cos\theta}{1-\tan\theta} + \dfrac{\sin\theta}{1-\cot\theta} = \cos\theta + \sin\theta$.

10. Prove that $\dfrac{\tan\theta}{1+\cos\theta} + \dfrac{\sin\theta}{1-\cos\theta} = \cot\theta + \operatorname{cosec}\theta.\sec\theta$.

17.3 TRIGONOMETRIC FUNCTIONS OF SOME SPECIFIC REAL NUMBERS

The values of the trigonometric functions of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ and $\frac{\pi}{2}$ are summarised below in the form of a table :

Real Numbers → / Function ↓	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

As an aid to memory, we may think of the following pattern for above mentioned values of sin function :

$$\sqrt{\frac{0}{4}}, \sqrt{\frac{1}{4}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{4}}$$

On simplification, we get the values as given in the table. The values for cosines occur in the reverse order.

Complement, Supplement Angles : If θ is any angle then $\frac{\pi}{2} - \theta$ is called its complement and $\pi - \theta$ is called its supplement.

In other words two angles θ, ϕ are said to be complementary angles it $\theta+\phi=\frac{\pi}{2}$ and supplementary angles if $\theta+\phi=\pi$.

$\frac{\pi}{6},\frac{\pi}{3}$ are complementary angles

$\frac{\pi}{6},\frac{5\pi}{6}$ are supplementary angles.

Example 17.12 : Find the value of the following :

(a) $\sin\frac{\pi}{4}.\sin\frac{\pi}{3}-\cos\frac{\pi}{4}.\cos\frac{\pi}{3}$

(b) $4\tan^2\frac{\pi}{4}-\text{cosec}^2\frac{\pi}{4}-\cos^2\frac{\pi}{3}$

Solution : (a) $\sin\frac{\pi}{4}.\sin\frac{\pi}{3}-\cos\frac{\pi}{4}.\cos\frac{\pi}{3}$

$$=\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

(b) $4\tan^2\frac{\pi}{4}-\text{cosec}^2\frac{\pi}{4}-\cos^2\frac{\pi}{3}$

$$=4(1)^2-(2)^2-\left(\frac{1}{2}\right)^2$$

$$=4-4-\frac{1}{4}=-\frac{1}{4}.$$

Example 17.13 : $A=\frac{\pi}{3}$ and $B=\frac{\pi}{6}$, verify that

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

Solution : LHS $=\cos(A+B)$

$$=\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\cos\frac{\pi}{2}=0.$$

R.H.S. $= \cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\,.\,\sin\frac{\pi}{6}$

$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2}$

$= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0$

L.H.S. = 0 = R.H.S.

$\therefore$ cos (A + B) = cos A cos B − sinA sin B.

EXERCISE 17.5

1. Find the value of

(i) $\sin^2\frac{\pi}{6} + \tan^2\frac{\pi}{4} + \tan^2\frac{\pi}{3}$

(ii) $\sin^2\frac{\pi}{3} + \text{cosec}^2\frac{\pi}{6} + \sec^2\frac{\pi}{4} - \cos^2\frac{\pi}{3}$

(iii) $\cos\frac{2\pi}{3}\cos\frac{\pi}{3} - \sin\frac{2\pi}{3}\sin\frac{\pi}{3}$

(iv) $4\cot^2\frac{\pi}{3} + \text{cosec}^2\frac{\pi}{4} + \sec^2\frac{\pi}{3}.\tan^2\frac{\pi}{4}$

(v) $\left(\sin\frac{\pi}{6} + \sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{3} - \cos\frac{\pi}{4}\right) + \frac{1}{4}$

2. Show that

$\left(1 + \tan\frac{\pi}{6}.\tan\frac{\pi}{3}\right) + \left(\tan\frac{\pi}{6} - \tan\frac{\pi}{3}\right) = \sec^2\frac{\pi}{6}\,.\,\sec^2\frac{\pi}{3}.$

3. Taking $A = \frac{\pi}{3}$, $B = \frac{\pi}{6}$, verify that

(i) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\ \tan B}$

(ii) cos(A + B) = cos A cos B − sin A sinB

4. If $\theta = \frac{\pi}{4}$, verify the following :

(i) $\sin 2\theta = 2 \sin \theta \cos \theta$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 2 \cos^2\theta - 1$

$= 1 - 2 \sin^2\theta$

5. If $A = \frac{\pi}{6}$ verify that

(i) $\cos 2A = 2\cos^2 A - 1$ (ii) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

(iii) $\sin 2A = 2 \sin A \cos A$

17.4 GRAPHS OF TRIGONOMETRIC FUNCTIONS

Given any function, a pictorial or a graphical representation makes a lasting impression on the minds of learners and viewers. The importance of the graph of functions stems from the fact that this is a convenient way of presenting many properties of the functions. By observing the graph we can examine several characteristic properties of the functions such as (i) periodicity, (ii) intervals in which the function is increasing or decreasing (iii) symmetry about axes, (iv) maximum and minimum points of the graph in the given interval. It also helps to compute the areas enclosed by the curves of the graph.

17.4.1 Variations of sin θ as θ Varies From 0 to 2π

Let *X'OX* and *Y'OY* be the axes of coordinates.With centre *O* and radius *OP* = unity, draw a circle. Let *OP* starting from *OX* and moving in anticlockwise direction make an angle Twith the x-axis, i.e. $\angle XOP = \theta$. Draw $PM \perp X'OX$, then $\sin\theta = MP$ as $OP=1$.

∴ The variations of sin θ are the same as those of MP.

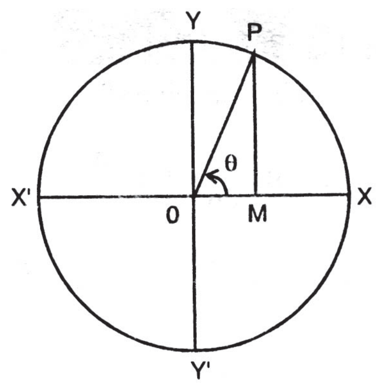


Fig. 17.8

I. Quadrant :

As θ increases continuously from 0 to $\frac{\pi}{2}$ PM is positive and increases from 0 to 1.

$\therefore$ sin θ is positive.

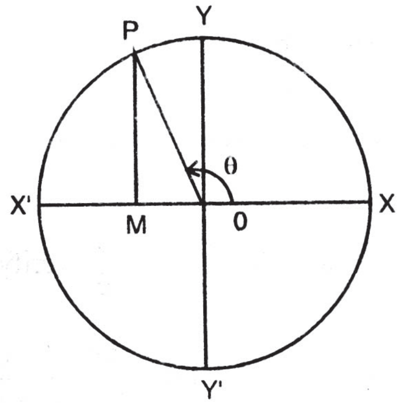


Fig. 17.9

II Quadrant : $\left[\frac{\pi}{2}, \pi\right]$

In this interval, θ lies in the second quadrant.

Therefore, point *P* is in the second quadrant. Here PM = y is positive, but decreases from 1 to 0 as θ varies from $\frac{\pi}{2}$ to π. Thus sin θ is positive.

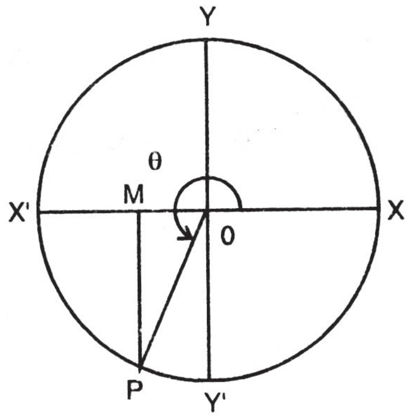


Fig. 17.10

III Quadrant : $\left[\pi, \frac{3\pi}{2}\right]$

In this interval, θ lies in the third quandrant. Therefore, point *P* can move in the third quadrant only. Hence *PM = y* is negative and decreases from 0 to −1 as θ varies from π to $\frac{3\pi}{2}$. In this interval sinθ decreases from 0 to 1. In this interval sin θ is negative.

IV Quadrant : $\left[\frac{3\pi}{2}, 2\pi\right]$

In this interval, θ lies in the fourth quadrant. Therefore, point P can move in the fourth quadrant only. Here again PM = y is negative but increases from −1 to 0 as θ varies from $\frac{3\pi}{2}$ to 2π. Thus sin θ is negative in this interval.

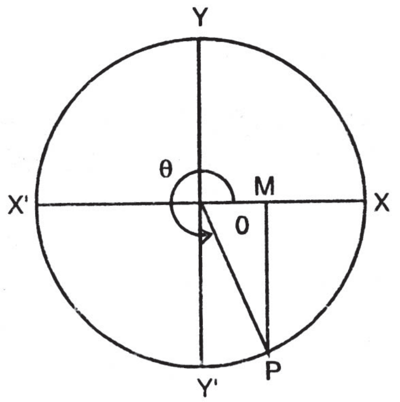


Fig. 17.11

17.4.2 Graph of sin θ as θ varies from 0 to 2π.

Let $X'OX$ and $Y'OY$ be the two coordinate axes of reference. The values of θ are to be measured along x-axis and the values of sine θ are to be measured along y-axis.

(Approximate value of $\sqrt{2} = 1.41$, $\frac{1}{\sqrt{2}} = 0.707$, $\frac{\sqrt{3}}{2} = 0.87$)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sin θ	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-0.1	-0.87	-0.5	0

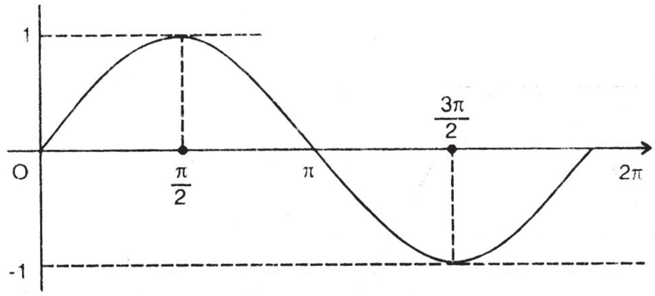


Fig. 17.12

Some Observations

(i) Maximum value of sin θ is 1.

(ii) Minimum value of sin θ is –1.

(iii) It is continuous everywhere.

(iv) It is increasing from 0 to $\frac{\pi}{2}$ and from $\frac{3\pi}{2}$ to 2π. It is decreasing from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. With the help of the graph drawn in Fig. 16.12 we can always draw another graph. $y = \sin\theta$ in the interval of [2π, 4π] (see Fig. 16.11)

What do you observe ?

The graph of y = sin θ in the interval [2π, 4π] is the same as that in 0 to2π. Therefore, this graph can be drawn by using the property sin (2π + θ) = sin θ. Thus, sin θ repeats itself when θ is increased by 2π. This is known as the periodicity of sin θ .

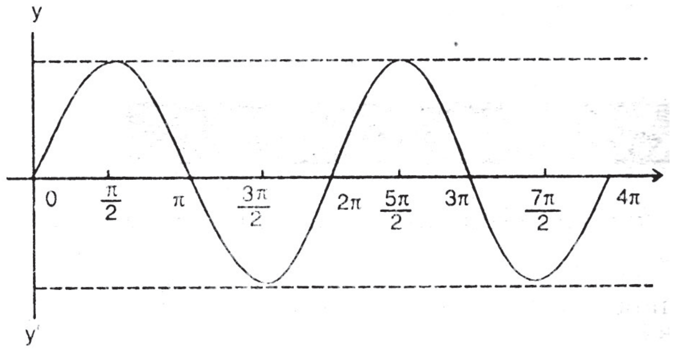


Fig. 17.12

We shall discuss in details the periodicity later in this lesson.

Example 17.14 : Draw the graph of $y = \sin 2\theta$

Solution :

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
2θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\sin 2\theta$	0	0.87	1	0.87	0	–0.87	–1	–0.87	0

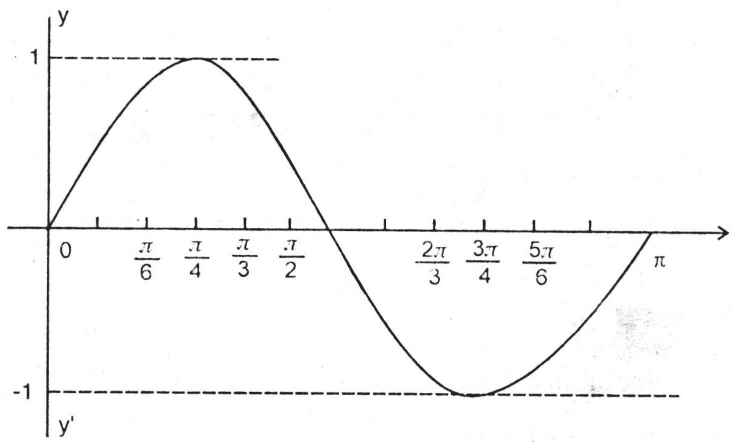


Fig. 17.13

The graph is similar to that of $y = \sin \theta$

Some Observations

1. The other graphs of $\sin \theta$, like $a \sin \theta$, $3 \sin 2\theta$ can be drawn applying the same method.

2. Graph of sin θ, in other intervals namely [4π, 6π] , [−2π, 0], [−4π, − 2π] , can also be drawn easily. This can be done with the help of properties of allied angles: sin (θ + 2θ) sin θ, sin (θ − 2π) = sin θ. i.e.,θ repeats itself when increased or decreased by 2π.

EXERCISE 17.6

1. What are the maximum and minimum values of sin θ in $[0, 2\pi]$.
2. Explain the symmetry in the graph of sin θ in $[0, 2\pi]$
3. Sketch the graph of $y = 2\sin\theta$, in the interval $[0, 2\pi]$
4. For what values of θ in $[\pi, 2\pi]$, sin θ becomes

   (a) $-\frac{1}{2}$ (b) $-\frac{\sqrt{3}}{2}$

5. Sketch the graph of y = sin x in the interval of [−π, π]

17.4.3 Graph of cos θ as θ Varies From 0 to 2π

As in the case of sin θ, we shall also discuss the changes in the values of cos θ when θ assumes values in the intervals $\left[0, \frac{\pi}{2}\right], \left[\frac{\pi}{2}, \pi\right], \left[\pi, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$.

I Quadrant : In the interval $\left[0, \frac{\pi}{2}\right]$, point *P* lies in the first quadrant, therefore, *OM* = *x* is positive but decreases from 1 to 0 as θ increases from 0 to $\frac{\pi}{2}$. Thus in this interval cos θ decreases from 1 to 0.

∴ cos θ is positive in this quadrant.

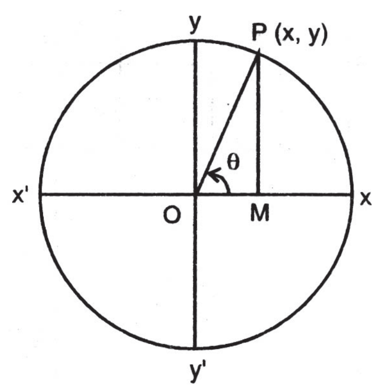


Fig.17.14

II Quadrant: In the interval $\left[\frac{\pi}{2}, \pi\right]$, point P lies in the second quadrant and therefore point M lies on the negative side of x-axis. So in this case $OM = x$ is negative and decreases from 0 to -1 as θ increases $\frac{\pi}{2}$ to π. Hence in this inverval $\cos\theta$ decreases from 0 to −1.

$\therefore$ $\cos\theta$ is negative.

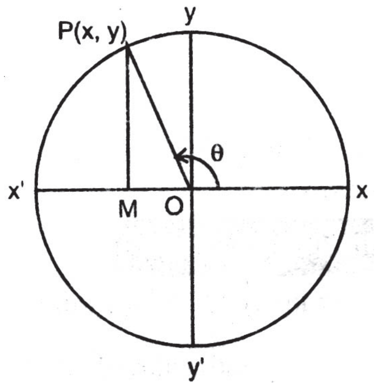


Fig.17.15

III Quadrant: In the interval $\left[\pi, \frac{3\pi}{2}\right]$ point P lies in the third quadrant and therefore, $OM = x$ remains negative as it is on the negative side of x-axis. Therefore $OM = x$ is negative but increases from −1 to 0 as θ increases from π to $\frac{3\pi}{2}$. Hence in this interval $\cos\theta$ increases from −1 to 0.

$\therefore$ $\cos\theta$ is negative.

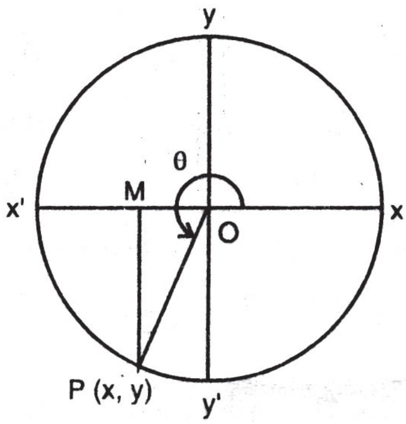


Fig.17.16

IV Quadrant: In the interval $\left[\frac{3\pi}{2}, 2\pi\right]$, point P lies in the fourth quadrant and M moves on the positive side of x-axis. Therefore $OM = x$ is positive. Also it increases from 0 to 1 as θ increases from $\frac{3\pi}{2}$ to 2π.

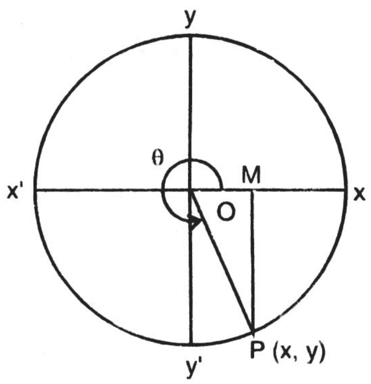


Fig.17.17

Thus in this interval $\cos\theta$ increases from 0 to 1.

$\therefore$ $\cos\theta$ is positive.

Let us tabulate the values of cosines of some suitable values of θ.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos\theta$	1	0.87	0.5	0	0.5	–0.87	–1	–0.87	–0.5	0	0.5	0.87	1

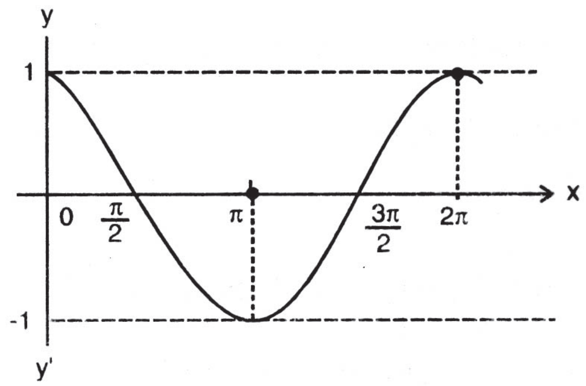


Fig. 17.18

Let *X'OX* and *Y'OY* be the axes. Values of θ are measured along x-axis and those of $\cos\theta$ along y-axis.

Some observations

(i) Maximum value of $\cos\theta = 1$

(ii) Minimum value of $\cos\theta = -1$.

(iii) It is continuous everywhere.

(iv) $\cos(\theta + 2\pi) = \cos\theta$, Also $\cos(\theta - 2\pi) = \cos\theta$ repeats itself when θ is increased or decreased by 2π It is called periodicity of $\cos\theta$. We shall discuss in details about this in the later part of this lesson.

(v) Graph of $\cos\theta$ in the intervals $[2\pi, 4\pi]$, $[4\pi, 6\pi]$, $[-2\pi, 0]$ will be the same as in $(\theta, 2\pi)$.

Example 17.15 Draw the graph of $\cos\theta$ as θ varies from $-\pi$ to π. From the graph read the values of θ when $\cos\theta = \pm 0.5$.

Solution:

θ	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
cosθ	–1.0	–0.87	–0.5	0	0.50	–0.87	1.0	0.87	0.5	0	–0.5	–0.87	–1

$\cos\theta = 0.5$

when $\theta = \frac{\pi}{3}, \frac{-\pi}{3}$

$\cos\theta = -0.5$

when $\theta = \frac{2\pi}{3}, \frac{-2\pi}{3}$

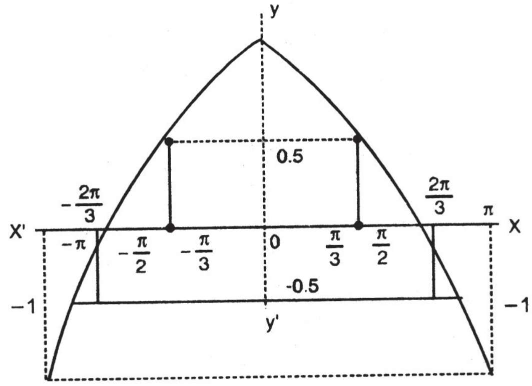


Fig. 17.18

Example 17.16: Draw the graph of $\cos 2\theta$ in the interval 0 to π.

Solution :

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
2θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
cos2θ	1	0.5	0	–0.5	–1	–0.5	0	0.5	1

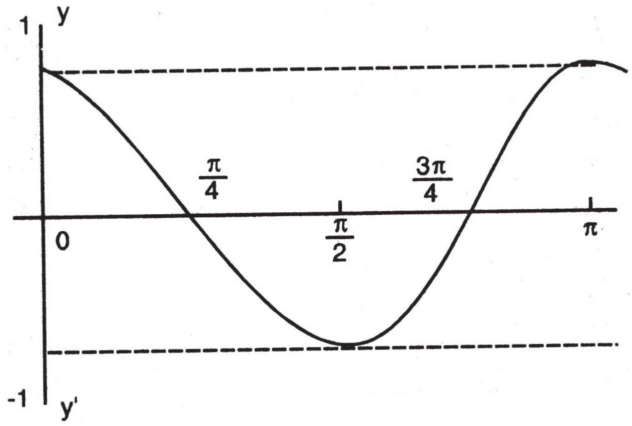


Fig. 17.19

EXERCISE 17.7

I. (a) Sketch the graph of y = cos θ as θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

(b) Draw the graph of y = 3 cos θ as θ varies from 0 to 2π .

(c) Draw the graph of y = cos 3θ from −π to π and read the values of θ when cos θ = 0.87 and cos θ = –0.87.

(d) Does the graph of y = cos θ in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ lie above x-axis or below x-axis?

(e) Draw the graph of y = cos θ in [2π, 4 π]

17.4.4 Graph of tan θ as θ Varies from 0 to 2π

In I Quadrant : tan θ can be written as $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Behaviour of tan θ depends upon the behaviour of sin θ and $\frac{1}{\cos\theta}$

In I quadrant, sin θ increases from 0 to 1, cos θ decreases from 1 to 0

But x $\frac{1}{\cos\theta}$ increases from 1 indefintely (and write it as increasses from 1 to ∞) tan θ > 0

$\therefore$ tan θ increases from 0 to ∞. (See the table and graph of tan ∞).

In II Quadrant : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

sin θ decreases from 1 to 0.

cos θ decreases from 0 to 1.

tan θ is negative and increases from $-\infty$ to 0

In III Quadrant : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

sin θ decreases from 0 to –1.

cos θ decreases from –1 to 0.

$\therefore$ tan θ is negative and increases from 0 to ∞

In IV Quadrant : $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

sin θ increases from –1 to 0

cos θ increases from 0 to 1.

tan θ is negative and increases from $-\infty$ to 0

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}+0^0$	$\frac{\pi}{2}+0^0$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0^0$	$\frac{3\pi}{2}+0^0$	$\frac{5\pi}{6}$	$\frac{11\pi}{6}$	2π
tan θ	0	.58	1.73	$-\infty$	–1.73	–.58	0	.58	1.73	$+\infty$	$-\infty$	–1.73	–.58	0	0

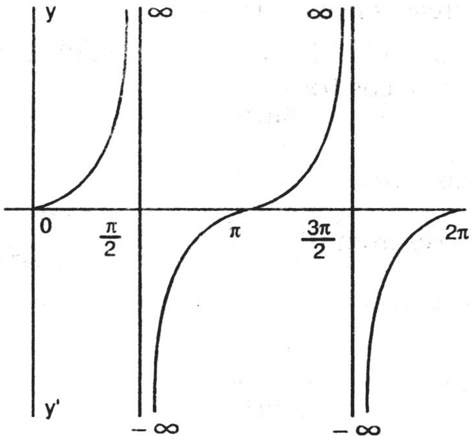


Fig. 17.20

Observations

(i) tan $(180^0 + \theta)$ = tan θ Therefore, the complete graph of tan θ consists of infinitely many repetitions of the same to the left as well as to the right.

(ii) Since tan (–θ) = – tan θ, therefore, if (θ, tan θ)is any point on the graph then (–θ, –tan θ) will also be a point on the graph.

(iii) By above results, it can be said that the graph of y = tan θ is symmetrical in opposite quadrants.

(iv) tan θ may have any numerical value, positve or negative.

(v) The graph of tan θ is discontinuous (has a break) at the points $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

(vi) As θ passes through these values, tan T suddenly changes from $+\infty$ to $-\infty$.

17.4.5 Graph of cot θ as θ Varies From 0 to 2π

The behaviour of cot θ depends upon the behaviour of cos θ and $\frac{1}{\sin\theta}$ as

$\cot\theta = \cos\theta\,\frac{1}{\sin\theta}$.

We discuss it in each quadrant.

I Quadrant : $\cot\theta = \cos\theta \times \frac{1}{\sin\theta}$

cos θ decreases from 1 to 0.

sin θ increases from 0 to 1.

$\therefore$ cot θ also decreases from $-\infty$ to 0 but cot θ > 0.

II Quadrant : $\cot\theta = \cos\theta \times \frac{1}{\sin\theta}$

cos θ decreases from 0 to −1.

sin θ decreases from 1 to 0.

$\Rightarrow$ cot θ < 0 or cot θ decreases from 0 to $-\infty$.

III Quadrant : $\cot\theta = \cos\theta \times \frac{1}{\sin\theta}$

cos θ increases from −1 to 0.

sin θ decreases from 0 to −1.

$\therefore$ cot θ decreases from $+\infty$ to 0.

IV Quadrant : $\cot\theta = \cos\theta \times \frac{1}{\sin\theta}$

cos θ increases from 0 to 1.

sin θ increases from −1 to 0.

$\therefore$ cot θ < 0.

cos θ decreases from 0 to $-\infty$.

Graph of cos θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi-0$	$\pi+0$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	11π	2π
$\cot\theta$	∞	1.73	0.58	0	–0.58	–1.73	$-\infty$	$-\infty$	1.73	0.58	0	–0.58	–1.73	$-\infty$

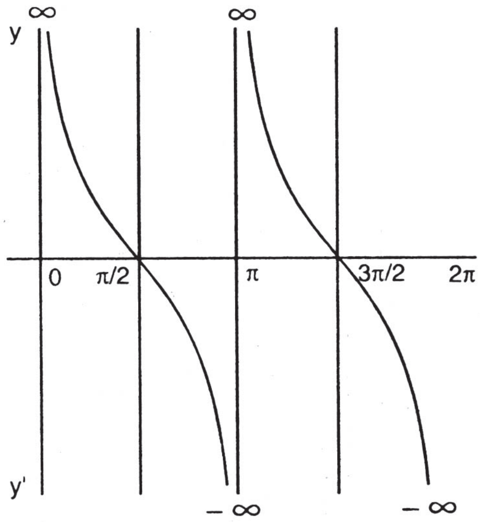


Fig 17.21

Observations

(i) Since $\cos(\pi+\theta) = \cot\theta$, the complete graph of $\cot\theta$ consists of the portion from $\theta = 0$ to $\theta = \pi$ or $\theta = \frac{\pi}{2}$ τo• $\theta = \frac{3\pi}{2}$.

(ii) $\cot\theta$ can have any numerical value - positive or negative.

(iii) The graph of $\cot\theta$ is discontinuous, i.e. it breaks at $0, \pi, 2\pi$.

(iv) As θ takes values $0, \pi, 2\pi$, $\cot\theta$ suddently changes from $-\infty$ to $+\infty$.

EXERCISE 17.8

1. (a) What is the maximum value of $\tan\theta$?

   (b) What changes do you observe in $\tan\theta$ at $\frac{\pi}{2}, \frac{3\pi}{2}$?

(c) Draw the graph of $y = \tan\theta$ from $-\pi$ to π. Find from the graph the value of θ for which $\tan\theta = 1.7$.

2. (a) What is the maximum value of $\cot\theta$?

(b) Find the value of θ when $\cot\theta = -1$, from the graph.

17.4.6 To Find the Variations And Draw The Graph of sec θ As Varies From 0 to 2π.

Let *X'OX* and *Y'OY* be the axes of coordinates. With centre *O*, draw a circle of unit radius.

Let *P* be any point on the circle. Join *OP* and draw

PM $\perp$ X'OX

$$\sec\theta = \frac{OP}{OM} = \frac{1}{OM}$$

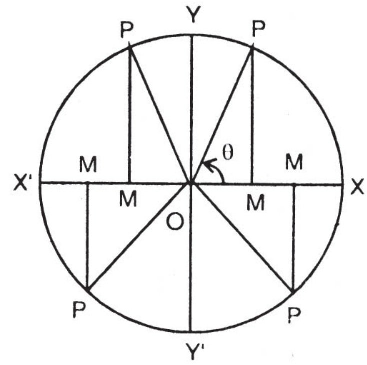


Fig.17.22

$\therefore$ Variations will depend upon OM

I Quadrant: $\sec\theta$ is positive as *OM* is positive.

Also $\sec\theta = 1$ and $\sec\frac{\pi}{2} = \infty$ when we approach $\frac{\pi}{2}$ from the right.

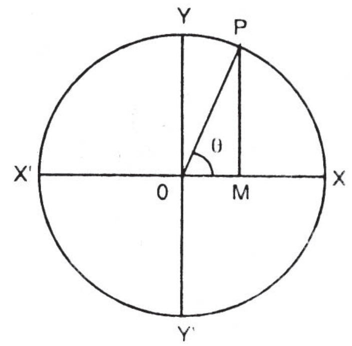


Fig.17.23

$\therefore$ As θ varies from 0 to $\frac{\pi}{2}$, $\sec\theta$ increases from 1 to ∞.

II Quadrant : $\sec\theta$ is negative as *OM* is negative. $\sec\frac{\pi}{2} = -\infty$ when we approach $\frac{\pi}{2}$ from the left. Also $\sec\pi = -1$.

$\therefore$ As θ varies from $\frac{\pi}{2}$ to π, $\sec\theta$ changes from $-\infty$ to -1.

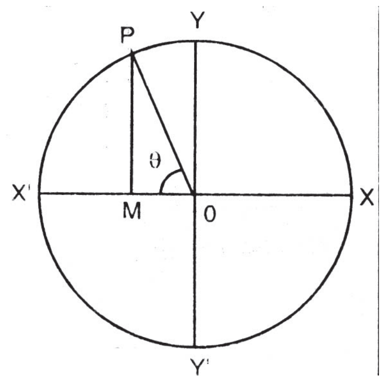


Fig.17.24

It is observed that as θ passes through $\frac{\pi}{2}$, sec θ changes from $+\infty$ to $-\infty$.

III Quadrant : sec θ is negative as *OM* is negative. sec π = −1 and $\sec\frac{3\pi}{2} = -\infty$ when the angle approaches $\frac{3\pi}{2}$ in the counter clockwise direction. As θ varies from π to $\frac{3\pi}{2}$, secθ decreases from −1 to $-\infty$.

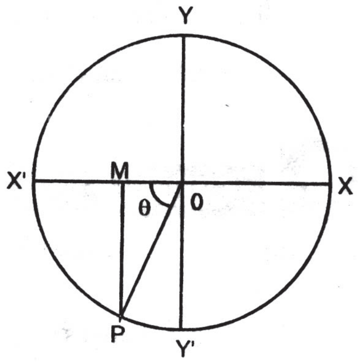


Fig.17.25

IV Quadrant : sec θ is positive as OM is positive. when θ is slightly greater than $\frac{3\pi}{2}$, sec θ is positive and very large.

Also sec 2π = 1. Hence sec θ decreases from ∞ to 1 as θ varies from $\frac{3\pi}{2}$ to 2π.

It may be observed that as θ passes through $\frac{3\pi}{2}$, sec θ changes from $-\infty$ to $+\infty$.

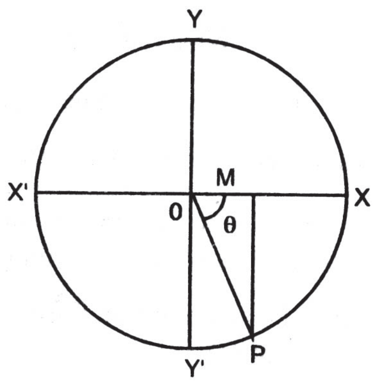


Fig.17.26

Graph of sec θ as θ varies from 0 to 2π

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}-0$	$\frac{\pi}{2}+0$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0$	$\frac{3\pi}{2}+0$	$\frac{5\pi}{6}$	$\frac{11\pi}{6}$	2π
cot θ	0	1.15	2	$+\infty$	$-\infty$	−2	−1.15	−1	−1.15	−2	$-\infty$	$+\infty$	2	1.15	

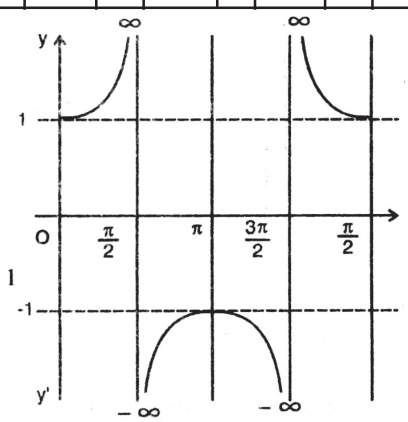


Fig.17.27

Observations

a) $\sec\theta$ cannot be numerically less than 1.

b) Graph of $\sec\theta$ is discontinuous, discontinuties (breaks) occuring at $(\frac{\pi}{2}$ and $\frac{3\pi}{2})$

c) As θ passes through $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, $\sec\theta$ changes abruptly from $+\infty$ to $-\infty$ and then from $-\infty$ to $+\infty$ respectively.

17.4.7 Graph of cosec θ as θ Varies From 0 to 2π

Let *X'OX* and *Y'OY* be the axes of coordinates. With centre *O* draw a circle of unit radius. Let *P* be any point on the circle. Join *OP* and draw *PM* perpendicular to *X'OX*.

$$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{1}{MP}$$

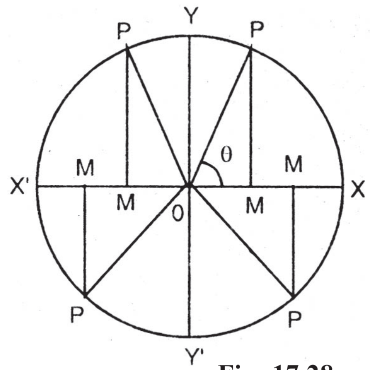


Fig. 17.28

$\therefore$ The variation of $\operatorname{cosec}\theta$ will depend upon *MP*.

I Quadrant: $\operatorname{cosec}\theta$ is positive as *MP* is positive

$\operatorname{cosec}\frac{\pi}{2} = 1$ when θ is very small, *MP* is also small and therefore, the value of $\operatorname{cosec}\theta$ is very large.

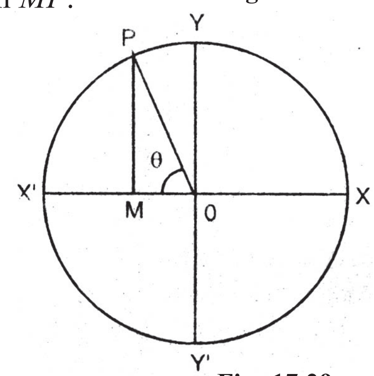


Fig. 17.29

$\therefore$ As θ varies from 0 to $\frac{\pi}{2}$, $\operatorname{cosec}\theta$ decreases from ∞ to 1.

II Quadrant : PM is positive. Therefore, $\operatorname{cosec}\theta$ is positive. $\operatorname{cosec}\frac{\pi}{2} = 1$ and $\operatorname{cosec}\pi = \infty$ when the revolving line approaches π in the counter clockwise direction.

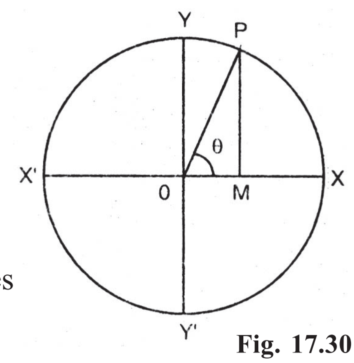


Fig. 17.30

$\therefore$ As θ varies from $\frac{\pi}{2}$ to π, $\operatorname{cosec}\theta$ increases from 1 to ∞.

III Quadrant : PM is negative

$\therefore$ cosec θ is negative. When θ is slightly greater than π,

cosec is very large and negative.

$$\text{Also } \operatorname{cosec}\frac{3\pi}{2} = -1$$

$\therefore$ As varies from π to $\frac{3\pi}{2}$, cosec θ changes from $-\infty$ to -1.

It may be observed that as θ passes through π, cosec θ changes from $+\infty$ to $-\infty$.

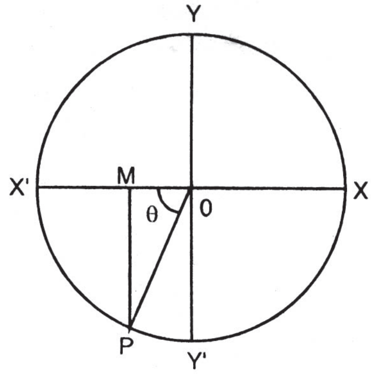


Fig. 17.31

IV Quadrant : PM is negative

Therefore, cosec $\theta = -\infty$, θ, 2π.

$\therefore$ As θ varies from $\frac{3\pi}{2}$ to 2π, cosec θ varies from 1 to $-\infty$.

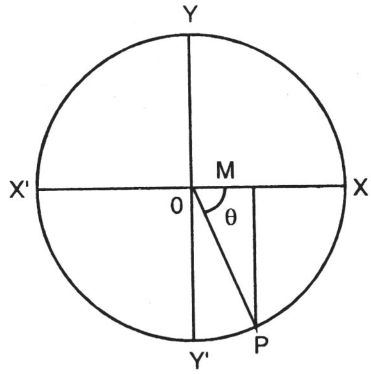


Fig. 17.32

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi-0$	$\pi+0$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{6}$	$\frac{11\pi}{6}$	2π
cosec θ	∞	2	1.15	1	1.15	2	$+\infty$	$-\infty$	–2	–1.15	–1	–1.15	–2	$-\infty$

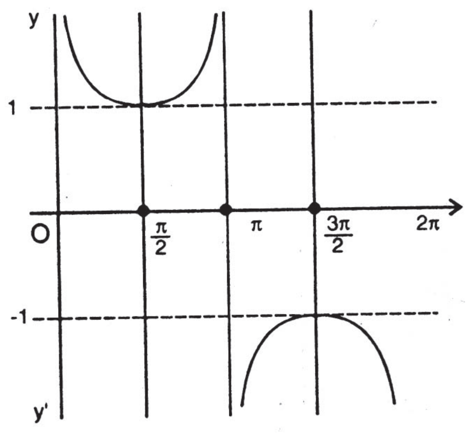


Fig. 17.33

Observations

(a) cosec θ cannot be numerically less than 1.

(b) Graph of cosec θ is discountinous and it has breaks at $\theta = 0, \pi, 2\pi$.

(c) As θ passes through π, cosec θ changes from $0 +\infty$ to $-\infty$. The values at 0 and 2π are $+\infty$ and $-\infty$ respectively.

Example 16.17 : Trace the changes in the values of sec θ as θ lies in $-\pi$ to π.

Solution:

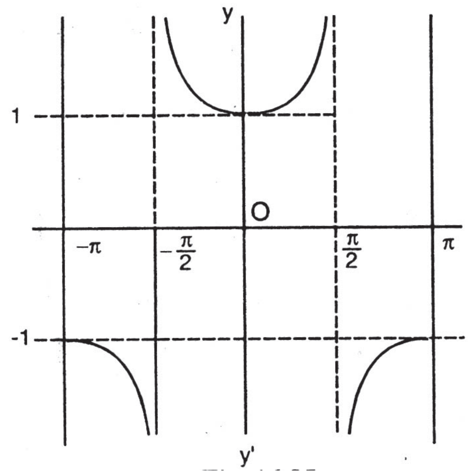


Fig. 17.34

EXERCISE 17.9

I. (a) Trace the changes in the values of sec θ when θ lies between 2π and 2π and draw the graph between these limits.

(b) Trace the graph of cosec θ,when θ lies between -2π and 2π.

17.5 PERIODICITY OF THE TRIGONOMETRIC FUNCTIONS

From your daily experience you must have observed things repeating themselves after regular intervals of time. For example, days of a week are repeated regularly after 7 days and months of a year are repeated regularly after 12 months. Position of a particle on a moving wheel is another example of the type. The property of repeated occurence of things over regular intervals is known as ***periodicity.***

Definition : A function *f (x)* is said to be periodic if its value is unchanged when the value of the variable in increased by a constant, that is if $f(x + p) = f(x)$ for all x.

If p is smallest positive constant of this type, then p is called the period of the function $f(x)$.

If $f(x)$ is a periodic function with period p, then $\frac{1}{f(x)}$ is also a periodic function with period p.

17.5.1 Periods of Trigonometric Functions

(i) $\sin x = \sin(x + 2n\pi)$; $n = 0, \pm 1, \pm 2, ...$

(ii) $\cos x = \cos(x + 2n\pi)$; $n = 0, \pm 1, \pm 2, ...$

Also there is no p, lying in 0 to 2π, for which

$\sin x = \sin(x + p)$

$\cos x = \cos(x + p)$, for all x

$\therefore$ 2π is the smallest positive value for which

$\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$

$\Rightarrow$ $\sin x$ and $\cos x$ each have the period 2π.

(iii) The period of cosec x is also 2π because $\operatorname{cosec} x = \frac{1}{\sin x}$

(iv) The period of sec x is also 2π as $\sec x = \frac{1}{\cos x}$.

(v) Also $\tan(x + \pi) = \tan x$.

Suppose p $(0 < p < \pi)$ is the period of tan x, then

$\tan(x + p) \quad \tan x$, for all x.

Put $x = 0$, then $\tan p = 0$, i.e., $p = 0$ or π .

$\Rightarrow$ the period of tan x is π .

$\therefore$ p can not values between 0 and π for which tan x = tan(x + p).

$\therefore$ The period of tan x is π

(vi) Since $\cot x = \dfrac{1}{\tan x}$ therefore, the period of cot x is also π.

Example 17.18 : Find the period of each the following functions :

(a) $y = 3 \sin 2x$ (b) $y = \cos \dfrac{x}{2}$ (c) $y = \tan \dfrac{x}{4}$

Solution :

(a) Period is $\dfrac{2\pi}{2}$, i.e., π

(b) $y = \cos \dfrac{1}{2}x$, therefore period $\dfrac{2\pi}{1/2} = 4\pi$

(c) Period of $y = \tan \dfrac{x}{4} = \dfrac{\pi}{1/4} = 4\pi$

EXERCISE 17.10

1. Find the period of each of the following functions :

   (a) $y = 2 \sin 3x$ (b) $y = 3 \cos 2x$

   (c) $y = \tan 3x$ (d) $y = \sin^2 2x$

17.6 ADDITION AND MULTIPLICATION OF TRIGONOMETRIC FUNCTIONS

In earlier sections we have learnt about circular measure of angles, trigonometric functions, values of trigonometric functions of specific numbers and of allied numbers.

You may now be interested to know whether with the given values of trigonometric functions of any two numbers A and B, it is possible to find trigonometric functions of sums or differences.

You will see how trigonometric functions of sum or difference of numbers are connected with those of individual numbers. This will help you, for instance, to find the value of trigonometric functions of $\frac{\pi}{12}$ and $\frac{5\pi}{12}$ etc.

$\frac{\pi}{12}$ can be expressed as $\frac{\pi}{4}-\frac{\pi}{6}$

$\frac{5\pi}{12}$ can be expressed as $\frac{\pi}{4}+\frac{\pi}{6}$

How can we express $\frac{7\pi}{12}$ in the form of addition or subtraction?

In this section we propose to study such type of trigonometric functions.

17.6.1 Addition Formulae

For any two numbers A and B,

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

In given figure trace out

$\angle SOP = A$

$\angle POQ = B$

$\angle SOR = -B$

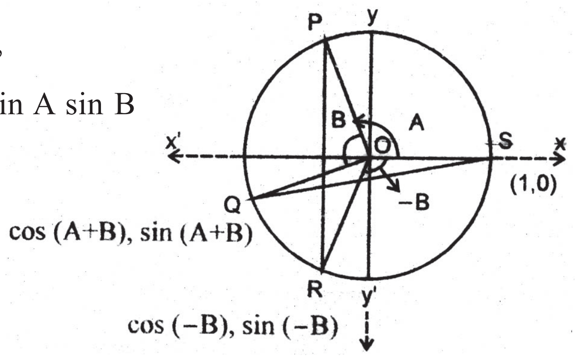


Fig. 17.44

where points P, Q, R, S lie on the unit circle.

Coordinates of P, Q, R, S will be $(\cos A, \sin A)$, $[\cos(A+B), \sin(A+B)]$, $[\cos(-B), \sin(-B)]$ and $(1, 0)$

From the given figure, we have

side OP = side OQ

$\angle POR = \angle QOS$

(each angle $= \angle B + \angle QOR$)

side OR = side OS

$\Delta POR \cong \Delta QOS$ (by SAS)

$\therefore \quad PR = QS$

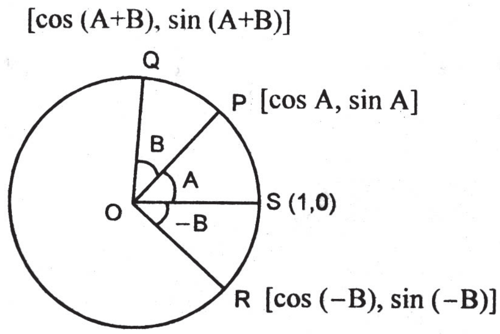


Fig. 17.2

$$PR = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin(-B))^2}$$

$$QS = \sqrt{(\cos(A+B) - 1)^2 + (\sin(A+B) - 0)^2}$$

Since $PR^2 = QS^2$

$\therefore \quad \cos^2 A + \cos^2 B - 2\cos A \cos B + \sin^2 A + \sin^2 B + 2 \sin A \sin B$

$= \cos^2(A + B) + 1 - 2\cos(A + B) + \sin^2(A + B)$

$\Rightarrow 1 + 1 - 2(\cos A \cos B - \sin A \sin B) = 1 + 1 - 2\cos(A + B)$

$\Rightarrow \cos A \cos B - \sin A \sin B = \cos(A + B)$ ...(I)

Extreme values of trignometric functions

For any $\theta \in \mathbf{R}$, $-1 \le \sin\theta \le 1$. Hence the minimum and maximum values of sin θ are −1 and 1 respectively as $\theta \in \mathbf{R}$. Each of them is called an extreme value of sin θ. Similarly the minimum and maximum values of cos θ are −1 and 1.

If $a, b, c \in \mathbf{R}$ such that $a^2 + b^2 \pm 0$, then the maximum and minimum values of $a \sin x + b \cos x + c$ are respectively

$C + \sqrt{a^2 + b^2}$ and $C - \sqrt{a^2 + b^2}$ over **R**.

$f(x) = a \sin x + b \cos x + c$ for all $x \in \mathbf{R}$

Put $a = r\cos\theta$, $b = r\sin\theta$ where $r = \sqrt{a^2 + b^2}$

Then $f(x) = r\cos\theta \sin x + r\sin\theta \cos x + c$

$= r[\cos\theta \sin\theta + \sin x \cos x] + c$

$= r\sin(\theta + x) + c$

$-1 \le \sin(\theta + x) \le 1$, so that

$-r \le r\sin(\theta + x) \le r$

$c - r \le \{c + r\sin(\theta + x)\} \le c + r$

Hence the maximum and minimum vales of f over **R** are respectively $c + \sqrt{a^2 + b^2}$ and $c - \sqrt{a^2 + b^2}$.

Example 17.19: Find Extreme values of $7\cos x - 24\sin x + 5$.

Let $a = 7$, $b = -24$, $c = 5$

$$\sqrt{a^2+b^2} = \sqrt{7^2+(-24)^2}$$

$$= \sqrt{49+576}$$

$$= \sqrt{625}$$

$$= 25.$$

Maximum value = $c+\sqrt{a^2+b^2} = 5 + 25 = 30$

Minimum value = $c-\sqrt{a^2+b^2} = 5 - 25 = -20$.

Corollary 1

For any two numbers A and B, cos (A – B) = cos A cos B + sin A sin B

Proof: Replace B by– B in (I)

cos (A – B) = cos A cos B + sin A sin B

[$\because$ cos (–B) = cosB and sin(–B) = –sinB]

Corollary 2

For any two numbers A and B

sin(A + B) = sin A cos B + cos A sin B

Proof: We know that $\cos\left(\frac{\pi}{2}-A\right)=\sin A$

and $\sin\left(\frac{\pi}{2}-A\right)=\cos A$

$$\therefore \sin(A+B) = \cos\left[\left(\frac{\pi}{2}-(A+B)\right)\right]$$

$$= \cos\left[\left(\frac{\pi}{2}-A\right)-B\right]$$

$$= \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$$

or $\quad \sin(A + B) = \sin A \cos B + \cos A \sin B \quad$ ...(II)

Corollary 3

For any two numbers A and B

$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

Proof : Replacing B by –B in (2), we have

$$\sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B)$$

or $\sin(A - B) = \sin A \cos B - \cos A \sin B.$

Example 17.20

(a) Find the value of each of the following :

(i) $\sin\frac{5\pi}{12}$ (ii) $\cos\frac{\pi}{12}$ (iii) $\cos\frac{7\pi}{12}$

(b) If $\sin A = \frac{1}{\sqrt{10}}$, $\sin B = \frac{1}{\sqrt{5}}$ show that $A + B = \frac{\pi}{4}$

Solution :

(a) (i) $\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \sin\frac{\pi}{4}.\cos\frac{\pi}{6} + \cos\frac{\pi}{4}.\sin\frac{\pi}{6}.$

$$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\therefore \sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

(ii) $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$$= \cos\frac{\pi}{4} . \cos\frac{\pi}{6} + \sin\frac{\pi}{4}.\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\therefore \quad \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Observe that $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12}$

(iii) $\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$

$$= \cos\frac{\pi}{3} . \cos\frac{\pi}{4} - \sin\frac{\pi}{3} . \sin\frac{\pi}{4}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$\therefore \cos\left(\frac{7\pi}{12}\right) = \frac{1-\sqrt{3}}{2\sqrt{2}}$$

(b) sin (A + B) = sin A cos B + cos A sin B

$\cos A = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}}$ and $\cos B = \sqrt{1-\frac{1}{5}} = \frac{2}{\sqrt{5}}$

Substituting all these values in (II), we get

$$\sin(A+B) = \frac{1}{\sqrt{10}} \cdot \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{5}}$$

$$= \frac{5}{\sqrt{10}\sqrt{5}} + \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}.$$

or $\quad A+B = \frac{\pi}{4}$

EXERCISE 17.11

1. (a) Find the values of each of the following :

(i) $\sin\frac{\pi}{12}$ (ii) $\sin\frac{\pi}{9} . \cos\frac{2\pi}{9} + \cos\frac{\pi}{9}\sin\frac{2\pi}{9}$

(b) Prove the following :

(i) $\sin\left(\frac{\pi}{6}+A\right) = \frac{1}{2}(\cos A+\sqrt{3}\sin A)$

(ii) $\sin\left(\frac{\pi}{4}-A\right) = \frac{1}{\sqrt{2}}(\cos A-\sin A)$

(c) If $\sin A = \frac{8}{17}$ and $\sin B = \frac{5}{13}$, find $\sin(A - B)$.

2. (a) Find the value of $\cos\frac{5\pi}{12}$.

(b) Prove the following :

(i) $\cos\theta+\sin\theta = \sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)$

(ii) $\sqrt{3}\sin\theta-\cos\theta = 2\sin\left(\theta-\frac{\pi}{6}\right)$

(iii) $\cos(n+1)A\cos(n-1)A+\sin(n+1)A\sin(n-1)A=\cos 2A$

(iv) $\cos\left(\frac{\pi}{4}+A\right)\cos\left(\frac{\pi}{4}-B\right)+\sin\left(\frac{\pi}{4}+A\right)\sin\left(\frac{\pi}{4}-B\right) = \cos(A+B)$

Corollary 4 : $\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}$

Proof : $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

Dividing by cos A, cos B, we have

$$\tan(A+B) = \frac{\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}}$$

or $$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B} \quad ...(III)$$

Corollary 5 : $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

Proof : Replacing B by $-B$ in (III), we get the required result.

Corollary 6 : $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$

Proof : $\cot(A+B)=\dfrac{\cos(A+B)}{\sin(A+B)}=\dfrac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$

Dividing by sin A sin B, we have

$$\cot(A+B)=\frac{\cot A\cot B-1}{\cot B+\cot A}.$$

Corollary 7 : $\tan\left(\dfrac{\pi}{4}+A\right)=\dfrac{1+\tan A}{1-\tan A}$

Proof : $\tan\left(\dfrac{\pi}{4}+A\right)=\dfrac{\tan\dfrac{\pi}{4}+\tan A}{1-\tan\dfrac{\pi}{4}.\tan A}$

$$=\frac{1+\tan A}{1-\tan A}\quad\text{as}\quad\tan\frac{\pi}{4}=1,$$

Similarly, it can be proved that

$$\tan\left(\frac{\pi}{4}-A\right)=\frac{1-\tan A}{1+\tan A}.$$

Example 17.21: Find $\tan\dfrac{\pi}{12}$

Solution : $\tan\dfrac{\pi}{12}=\tan\left(\dfrac{\pi}{4}-\dfrac{\pi}{6}\right)=\dfrac{\tan\dfrac{\pi}{4}-\tan\dfrac{\pi}{6}}{1+\tan\dfrac{\pi}{4}.\tan\dfrac{\pi}{6}}$

$$=\frac{1-\dfrac{1}{\sqrt{3}}}{1+1.\dfrac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}+1)} = \frac{4-2\sqrt{3}}{2}$$

$$= 2-\sqrt{3}$$

$$\therefore \tan\frac{\pi}{12} = 2-\sqrt{3}.$$

Example 17.22: Prove the following :

(a) $$\frac{\cos\frac{7\pi}{36}+\sin\frac{7\pi}{36}}{\cos\frac{7\pi}{36}-\sin\frac{7\pi}{36}} = \tan\frac{4\pi}{9}.$$

(b) $\tan 7A - \tan 4A - \tan 3A = \tan 7A \tan 4A . \tan 3A$

(c) $$\tan\frac{7\pi}{18} = \tan\frac{\pi}{9} + 2\tan\frac{5\pi}{18}$$

Solution: (a) Dividing numerator and denominator by $\cos\frac{7\pi}{36}$, we get

$$\text{LHS} = \frac{\cos\frac{7\pi}{36}+\sin\frac{7\pi}{36}}{\cos\frac{7\pi}{36}-\sin\frac{7\pi}{36}} = \frac{1+\tan\frac{7\pi}{36}}{1-\tan\frac{7\pi}{36}}$$

$$= \frac{\tan\frac{\pi}{4}+\tan\frac{7\pi}{36}}{1-\tan\frac{\pi}{4}.\tan\frac{7\pi}{36}}$$

$$= \tan\left(\frac{\pi}{4}+\frac{7\pi}{36}\right)$$

$$= \tan\frac{16\pi}{36} = \tan\frac{4\pi}{9} = \text{R.H.S.}$$

(b) $$\tan 7A = \tan(4A+3A) = \frac{\tan 4A + \tan 3A}{1-\tan 4A \tan 3A}$$

or $\tan 7A - \tan 7A \tan 4A \tan 3A = \tan 4A + \tan 3A$

or $\tan 7A - \tan 4A - \tan 3A = \tan 7A \tan 4A \tan 3A.$

(c) $\tan\frac{7\pi}{18} = \tan\left(\frac{5\pi}{18}+\frac{2\pi}{18}\right) = \dfrac{\tan\frac{5\pi}{18}+\tan\frac{2\pi}{18}}{1-\tan\frac{5\pi}{18}.\tan\frac{2\pi}{18}}$

$$\tan\frac{7\pi}{18} - \tan\frac{7\pi}{18}\tan\frac{5\pi}{18}\tan\frac{2\pi}{18} = \tan\frac{5\pi}{18}+\tan\frac{2\pi}{18} \qquad ...(1)$$

$$\tan\frac{7\pi}{18} = \tan\left(\frac{\pi}{2}-\frac{\pi}{9}\right) = \cot\frac{\pi}{9} = \cot\frac{2\pi}{18}.$$

$\therefore$ (1) can be written as

$$\tan\frac{7\pi}{18} - \cot\frac{2\pi}{18}\ \tan\frac{5\pi}{18}\tan\frac{2\pi}{18} = \tan\frac{5\pi}{18}+\tan\frac{\pi}{9}$$

$$\therefore \tan\frac{7\pi}{18} = \tan\frac{\pi}{9} + 2\tan\frac{5\pi}{18}.$$

EXERCISE 17.12

1. Fill in the blanks :

(i) $\sin\left(\frac{\pi}{4}+A\right)\sin\left(\frac{\pi}{4}-A\right) = ...$

(ii) $\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = ...$

2. (a) Prove the following :

(i) $\tan\left(\frac{\pi}{4}+\theta\right)\tan\left(\frac{\pi}{4}-\theta\right) = 1$

(ii) $\cot(A-B) = \dfrac{\cot A\cot B+1}{\cot B-\cot A}$

(iii) $\tan\frac{\pi}{12}+\tan\frac{\pi}{6}+\tan\frac{\pi}{12}.\tan\frac{\pi}{6} = 1.$

(b) If $\tan A = \frac{a}{b}$, $\tan B = \frac{c}{d}$, Prove that

$$\tan(A+B) = \frac{ad+bc}{bd-ac}.$$

(c) Find the value of $\frac{11\pi}{12}$.

3. (a) Prove the following :

(i) $\tan\left(\frac{\pi}{4}+A\right)\tan\left(\frac{3\pi}{4}+A\right)=-1$

(ii) $\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}=\tan\left(\frac{\pi}{4}+\theta\right)$

(iii) $\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\tan\left(\frac{\pi}{4}-\theta\right)$

17.7 TRANSFORMATION OF PRODUCTS INTO SUMS AND VICE VERSA

17.7.1 Transformation of Products into Sums or Differences

We know that

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$\sin (A - B) = \sin A \cos B - \cos A \sin B$

$\cos (A + B) = \cos A \cos B - \sin A \sin B$

$\cos (A - B) = \cos A \cos B + \sin A \sin B$

By adding and subtracting the first two formulae, we get respectively

$2 \sin A \cos B = \sin (A + B) + \sin(A - B)$ ...(1)

and $2\cos A \sin B = \sin(A + B) - \sin (A - B)$ ...(2)

Similarly, by adding and subtracting the other two formulae, we get

$2 \cos A \cos B = \cos (A + B) + \cos(A - B)$ ...(3)

and $2\cos A \sin B = \cos(A - B) - \cos (A + B)$ ...(4)

We can also quote these as

2 sin A cos B = sin (sum) + sin (difference)

2 cos A sin B = sin (sum) – sin(difference)

2 cos A cos B = cos (sum) + cos (difference)

2 sin A sin B = cos (difference) – cos (sum).

17.7.2 Transformation of Sums or Differences into Products

In the above results put

$$A + B = C$$

and $$A - B = D$$

Then $A = \frac{C+D}{2}$ and $B = \frac{C-D}{2}$ and (1), (2), (3) and (4) become

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$$

17.7.3 Further Applications of Addition and Subtraction Formulae

We shall prove that

(i) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$.

(ii) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$ (or) $\cos^2 B - \sin^2 A$

Proof : (i) $\sin(A+B)\sin(A-B)$

$= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$

$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$

$= \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 B$.

$1 - \cos^2 A - (1 - \cos^2 B) = \cos^2 B - \cos^2 A$.

(ii) $\cos(A + B)\cos(A - B)$

$= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$

$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$

$= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B$

$= \cos^2 A - \sin^2 B.$

$= (1 - \sin^2 A) - (1 - \cos^2 B)$

$= \cos^2 A - \sin^2 B.$

Example 17.23 : Express the following products as a sum or difference

(i) $2\sin 3\theta \cos 2\theta$ (ii) $\cos 6\theta \cos \theta$ (iii) $\sin\frac{5\pi}{12}\sin\frac{\pi}{12}$

Solution : (i) $2\sin 3\theta\cos 2\theta = \sin(3\theta + 2\theta) + \sin(3\theta - 2\theta)$

$= \sin 5\theta + \sin\theta$

(ii) $\cos 6\theta \cos\theta = \frac{1}{2}(2\cos 6\theta\cos\theta)$

$= \frac{1}{2}[\cos(6\theta + \theta) + \cos(6\theta - \theta)]$

$= \frac{1}{2}[\cos 7\theta + \cos 5\theta]$

(iii) $\sin\frac{5\pi}{12}\sin\frac{\pi}{12} = \frac{1}{2}\left[2\sin\frac{5\pi}{12}\sin\frac{\pi}{12}\right]$

$= \frac{1}{2}\left[\cos\left(\frac{5\pi - \pi}{12}\right) - \cos\left(\frac{5\pi + \pi}{12}\right)\right]$

$= \frac{1}{2}\left[\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\right]$

Example 17.24 : Express the following sums as products.

(i) $\cos\frac{5\pi}{9} + \cos\frac{7\pi}{9}$ (ii) $\sin\frac{5\pi}{36} + \cos\frac{7\pi}{36}$

Solution:

(i) $\cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = 2\cos\frac{5\pi + 7\pi}{9\times 2}\cos\frac{5\pi - 7\pi}{9\times 2}$

$= 2\cos\frac{2\pi}{3}\cos\frac{\pi}{9}$ $\left[\because \cos\left(\frac{-\pi}{9}\right) = \cos\frac{\pi}{9}\right]$

$$= 2\cos\left(\pi - \frac{\pi}{3}\right)\cos\frac{\pi}{9}$$

$$= -2\cos\frac{\pi}{3}\ \cos\frac{\pi}{9}$$

$$= -\cos\frac{\pi}{9} \qquad \left[\because\ \cos\frac{\pi}{3} = \frac{1}{2}\right]$$

(ii) $$\sin\frac{5\pi}{36} + \cos\frac{7\pi}{36} = \sin\left(\frac{\pi}{2} - \frac{13\pi}{36}\right) + \cos\frac{7\pi}{36}$$

$$= \cos\frac{13\pi}{36} + \cos\frac{7\pi}{36}$$

$$= 2\cos\frac{13\pi + 7\pi}{36\times 2}\cos\frac{13\pi - 7\pi}{36\times 2}$$

$$= 2\cos\frac{5\pi}{18}\ \cos\frac{\pi}{12}.$$

Example 17.25 : Prove that $\dfrac{\cos 7A - \cos 9A}{\sin 9A - \sin 7A} = \tan 8A$

Solution:

$$\text{L.H.S.} = \frac{2\sin\frac{7A+9A}{2}\sin\frac{9A-7A}{2}}{2\cos\frac{9A+7A}{2}\sin\frac{9A-7A}{2}}$$

$$= \frac{\sin 8A\ \sin A}{\cos 8A\ \sin A} = \frac{\sin 8A}{\cos 8A} = \tan 8A = \text{RHS}$$

Example 17.26 : Prove the following :

(i) $$\cos^2\left(\frac{\pi}{2} - A\right) - \sin^2\left(\frac{\pi}{4} - B\right) = \sin(A+B)\cos(A-B)$$

(ii) $$\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \frac{1}{\sqrt{2}}\sin A.$$

Solution: (i) $\cos^2 A - \sin^2 B = \cos(A + B)\cos(A - B)$, we have

$$\text{LHS} = \cos\left[\frac{\pi}{4} - A + \frac{\pi}{4} - B\right]\cos\left[\frac{\pi}{4} - A - \frac{\pi}{4} + B\right]$$

$$= \cos\left[\frac{\pi}{2} - (A+B)\right] \cos[-(A-B)]$$

$$= \sin(A+B)\cos(A-B) = \text{RHS}$$

(ii) Applying the formula

$\sin^2 A - \sin^2 B = \cos(A+B)\cos(A-B)$

$$\text{LHS} = \sin\left[\frac{\pi}{8} + \frac{A}{2} + \frac{\pi}{8} - \frac{A}{2}\right] \sin\left[\frac{\pi}{8} + \frac{A}{2} - \frac{\pi}{8} + \frac{A}{2}\right]$$

$$= \sin\frac{\pi}{4}\sin A$$

$$= \frac{1}{\sqrt{2}}\sin A = \text{RHS.}$$

Example 17.27: Prove that

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{\pi}{3}\cos\frac{4\pi}{9} = \frac{1}{16}.$$

Solution : $\text{LHS} = \cos\frac{\pi}{3}\left[\cos\frac{2\pi}{9}\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[2\cos\frac{2\pi}{9}\cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9} \qquad \left[\because \cos\frac{\pi}{3} = \frac{1}{2}\right]$$

$$= \frac{1}{4}\left[\cos\frac{\pi}{3} + \cos\frac{\pi}{9}\right]\cos\frac{4\pi}{9}$$

$$= \frac{1}{8}\cos\frac{4\pi}{9} + \frac{1}{8}\left[2\cos\frac{4\pi}{9}\cos\frac{\pi}{9}\right]$$

$$= \frac{1}{8}\cos\frac{4\pi}{9} + \frac{1}{8}\left[\cos\frac{5\pi}{9} + \cos\frac{\pi}{3}\right]$$

$$= \frac{1}{8}\cos\frac{4\pi}{9} + \frac{1}{8}\cos\frac{5\pi}{9} + \frac{1}{16} \qquad \text{...(1)}$$

Now $\cos\frac{5\pi}{9} = \cos\left[\pi - \frac{4\pi}{9}\right] = -\cos\left(\frac{4\pi}{9}\right) \qquad \text{...(2)}$

From (1) and (2), we get

$$\text{LHS} = \frac{1}{16} = \text{RHS}.$$

EXERCISE 17.13

1. Express each of the following as sums or differences :

   (a) $2\cos 3\theta \sin 2\theta$ (b) $2\sin 4\theta \sin\theta$

   (c) $2\cos\frac{\pi}{4}\cos\frac{\pi}{12}$ (d) $2\sin\frac{\pi}{3}\cos\frac{\pi}{6}$

2. Express each of the following as a product :

   (a) $\sin 6\theta + \sin 4\theta$ (b) $\sin 7\theta - \sin 3\theta$

   (c) $\cos 2\theta - \cos 4\theta$ (d) $\cos 7\theta + \cos 5\theta$

3. Prove the following :

   (a) $\sin\frac{5\pi}{18} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$ (b) $\dfrac{\cos\frac{\pi}{9} - \cos\frac{7\pi}{18}}{\sin\frac{7\pi}{18} - \sin\frac{\pi}{9}} = 1$

   (c) $\sin\frac{5\pi}{18} - \sin\frac{7\pi}{18} + \sin\frac{\pi}{18} = 0$ (d) $\cos\frac{\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = 0$

4. Prove the following :

   (a) $\sin^2(n+1)\theta - \sin^2 n\theta = \sin(2n+1)\theta.\ \sin\theta.$

   (b) $\cos\beta\cos(2\alpha - \beta) = \cos 2\alpha - \sin^2(\alpha - \beta)$

   (c) $\cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{12} = \frac{\sqrt{3}}{4}.$

5. $\cos^2\left(\frac{\pi}{4} + \theta\right) - \sin^2\left(\frac{\pi}{4} - \theta\right)$

6. Show that

   (a) $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \tan 4\theta$

   (b) $\sin\frac{\pi}{18}\sin\frac{5\pi}{6}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18} = \frac{1}{16}$

   (c) $(\cos\theta + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\frac{\alpha - \beta}{2}.$

17.8 TRIGONOMETRIC FUNCTIONS OF MULTIPLES OF ANGLES

(a) To express sin 2A in terms of sin A, cos A and tan A.

We know that

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

By putting B = A, we get

$$\sin 2A = \sin A \cos A + \cos A \sin A$$

$$= 2 \sin A \cos A.$$

$\therefore$ sin 2A can also be written as

$$\sin 2A = \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A} \qquad (\because\ 1 = \cos^2 A + \sin^2 A)$$

Dividing numerator and denomunator by $\cos^2 A$, we get

$$\sin 2A = \frac{2\left(\frac{\sin A \cos A}{\cos^2 A}\right)}{\frac{\cos^2 A}{\cos^2 A} + \frac{\sin^2 A}{\cos^2 A}} = \frac{2 \tan A}{1 + \tan^2 A}$$

(b) To express cos 2A in terms of sin A, cos A and tan A.

We know that

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

Putting B = A, we have

$$\cos 2A = \cos A \cos A - \sin A \sin A$$

(or) $\cos 2A = \cos^2 A - \sin^2 A.$

Also $\cos 2A = \cos^2 A - (1 - \cos^2 A)$

$$= \cos^2 A - 1 + \cos^2 A$$

i.e., $\cos {}^2A = 2\cos^2 A - 1 \Rightarrow \cos^2 A = \dfrac{1 + \cos 2A}{2}$

Also $\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - \sin^2 A - \sin^2 A$

i.e., $\cos 2A = 1 - 2\sin^2 A$ $\quad\Rightarrow \sin^2 A = \dfrac{1+\cos 2A}{2}$

$$\therefore \cos 2A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

•Dividing the numerator and denominator of R.H.S. by $\cos^2 A$, we have

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

(c) To express tan 2A in terms of tan A.

$$\tan 2A = \tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$= \frac{2\tan A}{1-\tan^2 A}.$$

Thus we have derived the following formulae :

$$\sin 2A = 2\sin A \cos A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$$

$$= 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\cos^2 A = \frac{1+\cos 2A}{A}, \qquad \sin^2 A = \frac{1-\cos 2A}{2}.$$

Example 17.28: If $A = \dfrac{\pi}{6}$, verify the following :

(i) $\sin 2A = 2\sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Solution: (i) $\sin 2A = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$.

$$2\sin A\cos A = 2\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 2\times\frac{1}{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$\frac{2\tan A}{1+\tan^2 A} = \frac{2\tan\frac{\pi}{6}}{1+\tan^2\frac{\pi}{6}} = \frac{\left(2\times\frac{1}{\sqrt{3}}\right)}{1+\frac{1}{3}} = \frac{2}{\sqrt{3}}\times\frac{3}{4} = \frac{\sqrt{3}}{2}.$$

Thus, it is verified that

$$\therefore \sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$$

(ii) $\cos 2A = \cos\dfrac{\pi}{3} = \dfrac{1}{2}$

$$\cos^2 A - \sin^2 A = \cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

$$2\cos^2 A - 1 = 2\cos^2\frac{\pi}{6} - 1 = 2\left(\frac{\sqrt{3}}{2}\right)^2 - 1$$

$$= 2\times\frac{3}{4} - 1 = \frac{1}{2}.$$

$$1 - 2\sin^2 A = 1 - 2\sin\frac{\pi}{6} = 1 - 2\left(\frac{1}{2}\right)^2 = 1 - 2\times\frac{1}{4} = \frac{1}{2}.$$

$$\frac{1-\tan^2 A}{1+\tan^2 A} = \frac{1-\tan^2\frac{\pi}{6}}{1+\tan^2\frac{\pi}{6}} = \frac{1-\left(\frac{1}{\sqrt{3}}\right)^2}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{1-\frac{1}{3}}{1+\frac{1}{3}} = \frac{2}{3}\times\frac{3}{4} = \frac{1}{2}.$$

Thus, it is verified that

$$\therefore \cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

(iii) $\tan 2A = \tan\frac{\pi}{3} = \sqrt{3}$

$$\frac{2\tan A}{1-\tan^2 A} = \frac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}} = \frac{2\times\frac{1}{\sqrt{3}}}{1-\frac{1}{3}} = \frac{2}{\sqrt{3}}\times\frac{3}{2} = \sqrt{3}$$

Thus, it is verified that $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Example 17.29 : Prove that $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$

Solution: $\dfrac{\sin 2A}{1+\cos 2A} = \dfrac{2\sin A\cos A}{2\cos^2 A}$

$$= \frac{\sin A}{\cos A} = \tan A.$$

Example 17.30 : cos A – tan A = 2 cot 2A Jx x~¡¶„²OKÇO_•.

Solution: $\cot A - \tan A = \dfrac{1}{\tan A} - \tan A$

$$= \frac{1-\tan^2 A}{\tan A}$$

$$= \frac{2(1-\tan^2 A)}{2\tan A}$$

$$= \frac{2}{\left(\dfrac{2\tan A}{1-\tan^2 A}\right)} = \frac{2}{\tan 2A}$$

$$= 2\cot 2A\ .$$

Example 17.31: Evaluate $\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}$.

Solution: $\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8} = \dfrac{1+\cos\frac{\pi}{4}}{2}+\dfrac{1+\cos\frac{3\pi}{4}}{2}$

$$=\frac{1+\frac{1}{\sqrt{2}}}{2}+\frac{1-\frac{1}{\sqrt{2}}}{2}$$

$$=\frac{(\sqrt{2}+1)+(\sqrt{2}-1)}{2\sqrt{2}}=1.$$

Example 17.32 : Prove that $\dfrac{\cos A}{1-\sin A}=\tan\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right)$.

Solution: RHS $=\tan\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right)=\dfrac{\tan\frac{\pi}{4}+\tan\frac{A}{2}}{1-\tan\frac{\pi}{4}\tan\frac{A}{2}}$

$$=\frac{1+\frac{\sin A/2}{\cos A/2}}{1-\frac{\sin A/2}{\cos A/2}}=\frac{\cos A/2+\sin A/2}{\cos A/2-\sin A/2}$$

$$=\frac{\left[\cos\frac{A}{2}+\sin\frac{A}{2}\right]\left[\cos\frac{A}{2}-\sin\frac{A}{2}\right]}{\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)^2}$$

[Multiplying Numerator and Denominator by $\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)$]

$$=\frac{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}-2\cos\frac{A}{2}\sin\frac{A}{2}}$$

$$=\frac{\cos A}{1-\sin A}=\text{LHS}$$

Example 17.33 : Prove that

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\frac{\alpha-\beta}{2}.$$

Solution: $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$= \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta$

$= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= 2\{1 - \cos(\alpha - \beta)\}$

$= 2 \times 2\sin^2\frac{\alpha-\beta}{2} = 4\sin^2\frac{\alpha-\beta}{2}.$

EXERCISE 17.14

1. If $A = \frac{\pi}{3}$, verify that

   (a) $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

   (b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$

2. Find the value of sin 2A when (assuming $0 < A < \frac{\pi}{2}$)

   (a) $\cos A = \frac{3}{5}$ (b) $\sin A = \frac{12}{13}$ (c) $\tan A = \frac{16}{63}$.

3. Find the value of cos 2A when

   (a) $\cos A = \frac{15}{17}$ (b) $\sin A = \frac{4}{5}$ (c) $\tan A = \frac{5}{12}$.

4. Find the value of tan 2A when

   (a) $\tan A = \frac{3}{4}$ (b) $\tan A = \frac{a}{b}$

5. Evaluate $\sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8}$.

MODULE - IV
Functions and Trigonometric Functions

Notes

6. Prove the following :

(a) $\dfrac{1+\sin 2A}{1-\sin 2A} = \tan^2\left(\dfrac{\pi}{4}+A\right)$

7. (a) Prove that $\dfrac{\sin 2A}{1-\cos 2A} = \cos A$

(b) Prove that tanA + cot A = 2cosec 2A.

8. (a) Prove that $\dfrac{\cos A}{1+\sin A} = \tan\left(\dfrac{\pi}{4}-\dfrac{A}{2}\right)$.

(b) Prove that $(\cos\alpha+\cos\beta)^2+(\sin\alpha-\sin\beta)^2 = 4\cos^2\dfrac{\alpha-\beta}{2}$.

17.8.1 Trigonometric Functions of 3A in Terms of those of A

(a) **sin3A in terms of sinA**

Substituting 2A for B in the formula

sin(A + B) = sinA cos B + cosA sin B, we get

$\Rightarrow \sin(A + 2A) = \sin A \cos 2A + \cos A \sin 2A$

$= \sin A (1 - 2\sin^2 A) + (\cos A \times 2\sin A \cos A)$

$= \sin A - 2\sin^3 A + 2\sin A (1 - \sin^2 A)$

$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A.$

$\therefore \sin 3A = 3\sin A - 4\sin^3 A.$

(b) **cos 3A in terms of cos A**

Substituting 2A for B in the formula

cos(A + B) = cosA cos B − sinA sin B , we get

$\cos(A + 2A) = \cos A \cos 2A - \sin A \sin 2A$

$= \cos A (2\cos^2 A - 1) + (\sin A) \times 2\sin A \cos A$

$= 2\cos^3 A - \cos A - 2\cos A (1 - \cos^2 A)$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A.$

$\therefore \cos 3A = 4\cos^3 A - 3\cos A.$

(c) **tan 3A in terms of tan A**.

Putting B = 2A in the formula

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}, \quad \text{we get}$$

$$\tan(A+2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$= \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A \times \dfrac{2\tan A}{1-\tan^2 A}}$$

$$= \frac{\dfrac{\tan A - \tan^3 A + 2\tan A}{1-\tan^2 A}}{\dfrac{1-\tan^2 A - 2\tan^2 A}{1-\tan^2 A}}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

(d) **Formulae for** $\sin^3 A$ **and** $\cos^3 A$

$\sin 3A = 3 \sin A - 4\sin^3 A$

$4\sin^3 A = 3\sin A - \sin 3A$

or $\quad \sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$

Similarly, $\quad \cos 3A = 4 \cos^3 A - 3\cos A$

$\therefore \quad 3 \cos A + \cos 3A = 4 \cos^3 A.$

or $\quad \cos^3 A = \dfrac{3\cos A + \cos 3A}{4}$

Thus, we have derived the following formulae :

$\sin 3A = 3 \sin A - 4 \sin^3 A$

$\cos 3A = 4\cos^3 A - 3 \cos A$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

$$\sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

$$\cos^3 A = \frac{3\cos A + \cos 3A}{4}.$$

Example 17.34: If $A = \frac{\pi}{4}$, verify that

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Solution : (a) $\sin 3A = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}.$

$$3\sin A - 4\sin^3 A = 3\sin\frac{\pi}{4} - 4\sin^3\frac{\pi}{4}$$

$$= 3\times\frac{1}{\sqrt{2}} - 4\times\left(\frac{1}{\sqrt{2}}\right)^3$$

$$= \frac{3}{\sqrt{2}} - \frac{4}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Thus, it is verified that $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

$$4\cos^3 A - 3\cos A = 4\times\left(\frac{1}{\sqrt{2}}\right)^3 - 3\times\frac{1}{\sqrt{2}}$$

$$= \frac{4}{2\sqrt{2}} - \frac{3}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

Thus, it is verified that $\cos 3A = 4\cos^3 A - 3\cos A$.

(iii) $\tan 3A = \tan\frac{3\pi}{4} = -1.$

$$\frac{3\tan A - \tan^3 A}{1-3\tan^2 A} = \frac{3\times 1 - 1^3}{1-3\times 1^2} = \frac{2}{-2} = -1.$$

Thus, it is verified that $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$.

Example 17.35 : If $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$, prove that $\cos 3\theta = \frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$.

Solution: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$= 4\left\{\frac{1}{2}\left(a+\frac{1}{a}\right)\right\}^3 - 3\left(\frac{1}{2}\right)\left(a+\frac{1}{a}\right)$$

$$= 4\times\frac{1}{8}\left(a^3+3a^2.\frac{1}{a}+3a.\frac{1}{a^2}+\frac{1}{a^3}\right)-\frac{3a}{2}-\frac{3}{2a}$$

$$= \frac{a^3}{2}+\frac{1}{2a^3} = \frac{1}{2}\left(a^3+\frac{1}{a^3}\right).$$

Example 17.36 : Prove that

$$\sin\alpha\sin\left(\frac{\pi}{3}+\alpha\right)\sin\left(\frac{\pi}{3}-\alpha\right) = \frac{1}{4}\sin 3\alpha .$$

Solution: $\sin\alpha\sin\left(\frac{\pi}{3}+\alpha\right)\sin\left(\frac{\pi}{3}-\alpha\right)$

$$= \frac{1}{2}\sin\alpha\left[\cos 2\alpha - \cos\frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\sin\alpha\left[1-2\sin^2\alpha - \left(1-2\sin^2\frac{\pi}{3}\right)\right]$$

$$= 2\frac{1}{2}\sin\alpha\left[\sin^2\frac{\pi}{3}-\sin^2\alpha\right]$$

$$= \sin\alpha\left[\frac{3}{4}-\sin^2\alpha\right] = \frac{3\sin\alpha - 4\sin^3\alpha}{4} = \frac{1}{4}\sin 3\alpha$$

Example 17.37: Prove that

$$\cos^3 A \sin 3A + \sin^3 A \cos 3A = \frac{3}{4}\sin 4A.$$

Solution: $\cos^3 A \sin 3A + \sin^3 A \cos 3A$

$= \cos^3 A (3\sin A - 4\sin^3 A) + \sin^3 A(4\cos^3 A - 3\cos A)$

$= 3\sin A \cos^3 A - 4\sin^3 A \cos^3 A + 4\sin^3 A\cos^3 A - 3\sin^3 A\cos A$

$= 3\sin A \cos^3 A - 3\sin^3 A\cos A$

$= 3\sin A\cos A (\cos^2 A - \sin^2 A) = (3\sin A\cos A) \cos 2A$

$$= \frac{3\sin 2A}{2} \times \cos A$$

$$= \frac{3}{2}\frac{\sin 4A}{2} = \frac{3}{4}\sin 4A.$$

Example 17.38: Prove that $\cos\frac{3\pi}{9} + \sin^3\frac{\pi}{18} = \frac{3}{4}\left(\cos\frac{\pi}{9} + \sin\frac{\pi}{18}\right)$.

Solution: $\text{LHS} = \frac{1}{4}\left[3\cos\frac{\pi}{9} + \cos\frac{\pi}{3}\right] + \frac{1}{4}\left[3\sin\frac{\pi}{18} - \sin\frac{\pi}{6}\right]$

$$= \frac{3}{4}\left[\cos\frac{\pi}{9} + \sin\frac{\pi}{18}\right] + \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}\right)$$

$$= \frac{3}{4}\cos\frac{\pi}{9} + \sin\frac{\pi}{18} = \text{RHS}$$

EXERCISE 17.15

1. If $A = \frac{\pi}{3}$, verify that

   (a) $\sin 3A = 3\sin A - 4\sin^3 A$

   (b) $\cos 3A = 4\cos^3 A - 3\cos A$

   (c) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

2. Find the value of sin 3A when

(a) $\sin A = \frac{2}{3}$ (b) $\sin A = \frac{p}{q}$

3. Find the value of cos 3A when

(a) $\cos A = -\frac{1}{3}$ (b) $\cos A = \frac{c}{d}$

4. Prove that $\cos\alpha \cos\left(\frac{\pi}{3}-\alpha\right)\cos\left(\frac{\pi}{3}+\alpha\right) = \frac{1}{4}\cos 3\alpha$.

5. (a) Prove that $\sin^3\frac{2\pi}{9} - \sin^3\frac{\pi}{9} = \frac{3}{4}\left(\sin\frac{2\pi}{9} - \sin\frac{\pi}{9}\right)$

(b) Prove that $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A}$ is constant.

6. (a) Prove that $\cot 3A = \frac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$.

(b) Prove that cos 10A + cos 8A + 3cos 4A + 3 cos2A = 8 cosA $\cos^3 3A$.

17.9 TRIGONOMETRIC FUNCTIONS OF SUBMULTIPLES OF ANGLES

$\frac{A}{2}, \frac{A}{3}, \frac{A}{4}$ are called submultiples of A.

It has been proved that

$$\sin^2 A = \frac{1-\cos 2A}{2},\ \cos^2 A = \frac{1+\cos 2A}{2},\ \tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$$

Replacing A by $\frac{A}{2}$, we easily get the following formulae for the sub-multiple $\frac{A}{2}$:

$$\therefore \sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}},\ \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

and $$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

We will choose either the positive or the negative sign depending on whether corresponding value of the function is positive or negative for the value of $\frac{A}{2}$. This will be clear from the following examples.

Example 17.39 : Find the values of $\cos\frac{\pi}{12}$ and $\cos\frac{\pi}{24}$.

Solution: We use the formulae $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$ and take the positive sign, because $\cos\frac{\pi}{12}$ and $\cos\frac{\pi}{24}$ are both positive.

$$\Rightarrow \cos\frac{\pi}{12} = \pm\sqrt{\frac{1+\cos\pi/6}{2}}$$

$$= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2+\sqrt{3}}{2\times 2}}$$

$$= \sqrt{\frac{4+2\sqrt{3}}{8}}$$

$$= \sqrt{\frac{(\sqrt{3}+1)^2}{8}} \qquad \left[\because 4+2\sqrt{3} = 1+3+2\sqrt{3} = (1+\sqrt{3})^2\right].$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos\frac{\pi}{24} = \sqrt{\frac{1+\cos\pi/12}{2}}$$

$$= \sqrt{\frac{\left(1+\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{2}}$$

$$= \sqrt{\frac{2\sqrt{2}+\sqrt{3}+1}{4\sqrt{2}}}$$

$$= \sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}}$$

Example 17.40 : Find the values of $\sin\left(-\frac{\pi}{8}\right)$ and $\cos\left(-\frac{\pi}{8}\right)$.

Solution: We use the formula $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

and take the lower sign, i.e., negative sign, because $\sin\left(-\frac{\pi}{8}\right)$ is negative.

$$\sin\left(-\frac{\pi}{6}\right) = -\sqrt{\frac{1-\cos(\pi/4)}{2}}$$

$$= -\sqrt{\frac{1-1/\sqrt{2}}{2}}$$

$$= -\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} = -\frac{\sqrt{2-\sqrt{2}}}{2}$$

Similarly, $\cos\left(-\frac{\pi}{8}\right) = \pm\sqrt{\frac{1+\cos\left(-\frac{\pi}{4}\right)}{2}}$

$$= \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}$$

$$= \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$$

$$= \sqrt{\frac{2+\sqrt{2}}{4}}$$

$$= \sqrt{\frac{2+\sqrt{2}}{2}}.$$

Example 17.41: If $\cos A = \frac{7}{25}$ and $\frac{3\pi}{2} < A < 2\pi$ find the values of

(i) $\sin\frac{A}{2}$ (ii) $\cos\frac{A}{2}$ (iii) $\tan\frac{A}{2}$

Solution: $\because$ A lies in the 4th-quardrant, $\frac{3\pi}{2} < A < 2\pi$

$$\Rightarrow \quad 3\frac{\pi}{4} < \frac{A}{2} < \pi$$

$$\therefore \sin\frac{A}{2} > 0,\ \cos\frac{A}{2} < 0,\ \tan\frac{A}{2} < 0$$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1-7/25}{2}} = \sqrt{\frac{18}{50}} = \sqrt{\frac{9}{25}} = \frac{3}{5}.$$

$$\cos\frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}} = -\sqrt{\frac{1+\frac{7}{25}}{2}} = -\sqrt{\frac{32}{50}} = -\sqrt{\frac{16}{25}} = \frac{-4}{5}.$$

and $$\tan\frac{A}{2} = -\sqrt{\frac{1-\cos A}{1+\cos A}} = -\sqrt{\frac{1-\frac{7}{25}}{1+\frac{7}{25}}}$$

$$= -\sqrt{\frac{18}{32}} = -\sqrt{\frac{9}{16}} = -\frac{3}{4}.$$

Example 17.42 : Prove that following :

(a) $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$ and $\cos\frac{\pi}{10} = \frac{\sqrt{10+2\sqrt{5}}}{4}$

(b) $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$ and $\sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}$

Solution: (a) Let $A = \frac{\pi}{10} \Rightarrow 5A = \frac{\pi}{2}.$

$$\therefore \quad 2A = \frac{\pi}{2} - 3A$$

$$\therefore \quad \sin 2A = \sin\left(\frac{\pi}{2} - 3A\right) = \cos 3A$$

$$\therefore \quad 2\sin A\cos A = 4\cos^3 A - 3\cos A$$

or $\quad \cos A\ [2\ \sin A - 4\ \cos^2 A + 3] = 0$

As $\quad \cos A \neq 0$ =°i†Çò $1 - \sin^2 A = \cos^2 A$

$\therefore$ (1) becomes $\ 2\sin A - 4(1 - \sin^2 A) + 3 = 0$

$$4\ \sin^2 A + 2\ \sin A - 1 = 0$$

$$\Rightarrow \sin A = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

$$\therefore \quad \sin A = \frac{\sqrt{5}-1}{4}$$

$$\Rightarrow \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$$

Now, $\quad \cos\frac{\pi}{10} = \sqrt{1 - \sin^2\frac{\pi}{10}} = \sqrt{1 - \left(\frac{\sqrt{5}-1}{4}\right)^2}$

$$= \frac{\sqrt{10+2\sqrt{5}}}{4}$$

(b) Let $A = \frac{\pi}{10}$, $\quad 2A = \frac{\pi}{5}$

$\cos 2A = 1 - 2\ \sin^2 A.$

$$\therefore \cos\frac{\pi}{5} = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = 1 - 2\left(\frac{6-2\sqrt{5}}{16}\right) = \frac{2+2\sqrt{5}}{8}$$

$$= \frac{\sqrt{5}+1}{4}.$$

Now $\sin\frac{\pi}{5} = \sqrt{1-\cos^2\frac{\pi}{5}} = \sqrt{\frac{10-2\sqrt{5}}{4}}$.

Example 17.43 : Prove the following

$$\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$$

Solution: We have to prove that

$$\tan\alpha - \cot\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$$

or $\left(\frac{\sin\alpha}{\cos\alpha} - \frac{\cos\alpha}{\sin\alpha}\right) + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$

or $-\frac{2\cos 2\alpha}{2\sin\alpha\cos\alpha} + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$

or $-2\frac{\cos 2\alpha}{\sin 2\alpha} + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$

or $-2\cot 2\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$ ...(1)

Combining $-2\cot 2\alpha + 2\tan 2\alpha$

(1) becomes $-4\cot 4\alpha + 4\tan 4\alpha + 8\cot 8\alpha = 0$ ...(2)

Combining $-4\cot 4\alpha + 4\tan 4\alpha$

$-8\cot 8\alpha + 8\cot 8\alpha = 0 =$ RHS of (2)

EXERCISE 17.16

1. If $A = \frac{\pi}{3}$, verify that

(a) $\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$ (b) $\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$

(c) $\tan\frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$

2. Find the values of $\sin\frac{\pi}{2}$ and $\sin\frac{\pi}{24}$.

3. Determine the values of

(a) $\sin\frac{\pi}{8}$ (b) $\cos\frac{\pi}{8}$ (c) $\tan\frac{\pi}{8}$

17.10 TRIGONOMETRIC EQUATIONS

You are familiar with the equations like simple linear equations, quadratic equations in algebra.

You have also learnt how to solve the same

Thus, (i) $x - 3 = 0$ gives one value of x as a solution.

(ii) $x^2 - 9 = 0$ gives two values of x.

You must have noticed, the number of values depends upon the degree of the equation.

Now we need to consider as to what will happen in case x's and y's are replaced by trigonometric functions.

Thus solution of the equation $\sin\theta - 1 = 0$, we have

$\sin\theta = 1$ and $\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, ...$

Clearly, the solution of simple equations with only finite number of values does not necessarily hold good in case of trigonometric equations.

So, we will try to find the ways of finding solutions of such equations.

17.10.1 To find the general solution of the equation $\sin\theta = 0$

It is given that $\sin\theta = 0$

But we know that $\sin 0, \sin\pi, \sin 2\pi, ... \sin n\pi$ are equal to 0

$\therefore \theta = n\pi, \; n \in N.$

But we know $\sin(-\theta) = -\sin\theta = 0$

$\therefore \sin(-\pi), \sin(-2\pi), \sin(-3\pi), ..., \sin(-n\pi) = 0.$

$\therefore\ \theta = n\pi,\ \ n \in I.$

Thus, the general solution of equations of the types sin θ = 0 is given by θ = *n*π where *n* is an integer.

17.10.2 To find the general solution of the equation cos θ = 0

It is given that cos θ = 0

But in practice we know that $\cos\frac{\pi}{2} = 0$ Therefore, the first value of θ is

$$\theta = \frac{\pi}{2} \qquad ...(1)$$

We know that $\cos(\pi+\theta) = -\cos\theta$ or $\cos\left(\pi+\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0$.

or $$\cos\frac{3\pi}{2} = 0$$

In the same way, it can be found that

$\cos\frac{5\pi}{2}, \cos\frac{7\pi}{2}, \cos\frac{9\pi}{2}, ... \cos(2n+1)\frac{\pi}{2}$ are all zero. •

$$\therefore \theta = (2n+1)\frac{\pi}{2},\quad n \in N$$

But we know that $\cos(-\theta) = \cos\theta$

$$\therefore \cos\left(-\frac{\pi}{2}\right) = \cos\left(-\frac{3\pi}{2}\right) = \cos\left(-\frac{5\pi}{2}\right) = ... = \cos\left\{-(2n+1)\frac{\pi}{2}\right\}$$

$$\therefore \theta = (2n+1)\frac{\pi}{2},\quad n \in N$$

Therefore, $\theta = (2n+1)\frac{\pi}{2}$ is the solution of equations cos θ = 0 for all numbers whose cosine is 0.

17.10.3 To find a general solution of the equation tan θ = 0

It is given that tan θ = 0

or $\frac{\sin\theta}{\cos\theta} = 0$ or sin θ = 0.

i.e., $\theta = n\pi,\ n \in I$

We have consider above the general solution of trigonometric equations, where the right hand is zero. In the following, we take up cases where right hand side is non-zero.

17.10.4 To find a general solution of the equation sin θ = sin α

It is given that $\sin\theta = \sin\alpha$

$$\Rightarrow \quad \sin\theta - \sin\alpha = 0$$

$$\text{or} \quad 2\cos\left(\frac{\theta+\alpha}{2}\right)\sin\left(\frac{\theta-\alpha}{2}\right) = 0.$$

$$\therefore \text{ Either } \cos\left(\frac{\theta+\alpha}{2}\right) = 0 \quad \text{or} \quad \sin\left(\frac{\theta-\alpha}{2}\right) = 0$$

$$\Rightarrow \frac{\theta+\alpha}{2} = (2p+1)\frac{\pi}{2} \quad \text{or} \quad \frac{\theta-\alpha}{2} = q\pi,\ p, q \in I$$

$$\Rightarrow \theta = (2p+1)\pi - \alpha \quad \text{or} \quad \theta = 2p\pi + \alpha \qquad ...(1)$$

From (1), we get

$\theta = n\pi + (-1)^n\alpha,\ n \in I$ as the geeneral solution of the equation

$$\sin\theta = \sin\alpha$$

17.10.5 To find a general solution of the equation cos θ = cos α

It is given that $\cos\theta = \cos\alpha$

$$\Rightarrow \quad \cos\theta - \cos\alpha = 0$$

$$\Rightarrow -2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$$

$$\therefore \text{ Either } \sin\frac{\theta+\alpha}{2} = 0 \quad \text{or} \quad \sin\frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow \frac{\theta+\alpha}{2} = p\pi \quad \text{or} \quad \frac{\theta-\alpha}{2} = q\pi,\ p, q \in I$$

$$\Rightarrow \theta = 2p\pi - \alpha \quad \text{or} \quad \theta = 2p\pi + \alpha \qquad ...(1)$$

From (1), we have

$\theta = 2n\pi \pm \alpha, \quad n \in I$ as the general solution of the equation

$\cos\theta = \cos\alpha$

17.10.6 It is given that, $\tan\theta = \tan\alpha$

It is given that, $\tan\theta = \tan\alpha$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} - \dfrac{\sin\theta}{\cos\alpha} = 0$

$\Rightarrow \sin\theta\cos\alpha - \sin\alpha\cos\theta = 0$

$\Rightarrow \sin(\theta - \alpha) = 0$

$\Rightarrow \theta - \alpha = n\pi, \; n \in I$

$\Rightarrow \theta = n\pi + \alpha, \; n \in I$

Similarly, for $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$, the general solution is

$$\theta = n\pi + (-1)^n \alpha$$

and, for $\sec\theta = \sec\alpha$, the general solution is

$$\Rightarrow \theta = 2n\pi \pm \alpha$$

and, for $\cot\theta = \cot\alpha$

$\theta = n\pi + \alpha$ is its general solution.

If $\sin^2\theta = \sin^2\alpha$, then

$$\frac{1-\cos 2\theta}{2} = \frac{1-\cos 2\alpha}{2}$$

$\Rightarrow \cos 2\theta = \cos 2\alpha$

$\Rightarrow 2\theta = 2n\alpha \pm 2\alpha, \; n \in I.$

$\Rightarrow \theta = n\pi \pm \alpha.$

Similarly, if $\cos^2\theta = \cos^2\alpha$, then

$$\alpha = n\pi \pm \alpha, \qquad n \in I.$$

Again, if $\tan^2\theta = \tan^2\alpha$ then

$$\frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

$\Rightarrow$ $\cos 2\theta = \cos 2\alpha$

$\Rightarrow$ $2\theta = 2n\alpha \pm 2\alpha$

$\Rightarrow$ $\theta = n\pi \pm \alpha,\ n \in I$ is the general solution.

Example 17.44 : Find the general solution of the following equations :

(a) (i) $\sin\theta = \frac{1}{2}$ (ii) $\sin\theta = \frac{-\sqrt{3}}{2}$

(b) (i) $\cos\theta = \frac{\sqrt{3}}{2}$ (ii) $\cos\theta = -\frac{1}{2}$

(c) (i) $\cot\theta = -\sqrt{3}$ (d) $4\sin^2\theta = 1.$

Solution:

(a) (i) $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

$\therefore \theta = n\pi + (-1)^n \frac{\pi}{6}, \quad n \in I$

(ii) $\sin\theta = \frac{-\sqrt{3}}{2} = -\sin\frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin\frac{4\pi}{3}.$

$\therefore \theta = n\pi + (-1)^n \frac{4\pi}{3}, \quad n \in I$

(b) (i) $\cos\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$

$\therefore \theta = 2n\pi \pm \frac{\pi}{6}, \quad n \in I$

(ii) $\cos\theta = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$\therefore \theta = 2n\pi \pm \frac{2\pi}{3}, \quad n \in I$

(c) $\cot\theta = -\sqrt{3}$

$$\tan\theta = -\frac{1}{\sqrt{3}} = -\tan\frac{\pi}{6} = \tan\left(\pi - \frac{\pi}{6}\right) = \tan\frac{5\pi}{6}.$$

$$\therefore\ \theta = n\pi + \frac{5\pi}{6}, \qquad n \in I$$

(d) $4\sin^2\theta = 1 \Rightarrow \sin^2\theta = \frac{1}{4} = \left(\frac{1}{2}\right)^2 = \sin^2\frac{\pi}{6}$

$$\Rightarrow \sin\theta = \sin\left(\pm\frac{\pi}{6}\right)$$

$$\therefore\ \theta = n\pi \pm \frac{\pi}{6}, \qquad n \in I$$

Example 17.45 : Solve the following :

(a) $2\cos^2\theta + 3\sin\theta = 0$ (b) $\cos 4x = \cos 2x$

(c) (i) $\cos 3x = \sin 2x$ (d) $\sin 2x + \sin 4x + \sin 6x = 0$

Solution: (a) $2\cos^2\theta + 3\sin\theta = 0$

$$\Rightarrow \quad 2(1 - \sin^2\theta) + 3\sin\theta = 0$$

$$\Rightarrow \quad 2\sin^2\theta - 3\sin\theta - 2 = 0$$

$$\Rightarrow \quad (2\sin\theta + 1)(\sin\theta - 2) = 0$$

$$\Rightarrow \quad \sin\theta = -\frac{1}{2} \quad \text{or} \quad \sin\theta = 2$$

Since $\sin\theta = 2$ is not possible.

$$\therefore\ \sin\theta = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6}.$$

$$\therefore\ \theta = n\pi + (-1)^n \cdot \frac{7\pi}{6}, \quad n \in I.$$

(b) $\cos 4x = \cos 2x$

i.e., $\cos 4x - \cos 2x = 0$

$\Rightarrow \; -2\sin 3x \;\; \sin x = 0$

$\Rightarrow \; \sin 3x = 0$ or $\sin x = 0$

$\Rightarrow \; 3x = n\pi$ or $x = n\pi$

$\Rightarrow \; x = \dfrac{n\pi}{3}$ or $x = n\pi$, $n \in \text{I}$

(c) $\cos 3x = \cos 2x$.

$\Rightarrow \; \cos 3x = \cos\left(\dfrac{\pi}{2} - 2x\right)$

$\Rightarrow \; 3x = 2n\pi \pm \left(\dfrac{\pi}{2} - 2x\right), \; n \in \text{I}$

Taking positive sign only, we have

$\Rightarrow 3x = 2n\pi + \dfrac{\pi}{2} - 2x$

$\Rightarrow 5x = 2n\pi + \dfrac{\pi}{2}$

$\Rightarrow x = \dfrac{2n\pi}{5} + \dfrac{\pi}{10}.$

Now taking negative sign, we have

$3x = 2n\pi - \dfrac{\pi}{2} + 2x \;\; \Rightarrow x = 2n\pi - \dfrac{\pi}{2}, \;\; n \in \text{I}$

(d) $\sin 2x + \sin 4x + \sin 6x = 0$

or $(\sin 6x + \sin 2x) \{ \sin 4x = 0$

or $2\sin 4x \;\cos 2x \{ \sin 4x = 0$

or $\sin 4x\, [2\cos 2x \{ 1] = 0$

$\therefore \sin 4x$ or $\cos 2x = -\frac{1}{2} = \cos\frac{2\pi}{3}$.

$\Rightarrow 4x = n\pi$ or $2x = 2n\pi \pm \frac{2\pi}{3}, \quad n \in I$

$X = \frac{n\pi}{4}$ or $X = n\pi \pm \frac{\pi}{3}, \quad n \in I$.

EXERCISE 17.17

1. Find the most general value of θ satisfying :

   (i) $\sin\theta = \frac{\sqrt{3}}{2}$ (ii) $\operatorname{cosec}\theta = \sqrt{2}$

   (iii) $\sin\theta = -\frac{\sqrt{3}}{2}$ (iv) $\sin\theta = -\frac{1}{\sqrt{2}}$

2. Find the most general value of θ satisfying :

   (i) $\cos\theta = -\frac{1}{2}$ (ii) $\sec\theta = -\frac{2}{\sqrt{3}}$

   (iii) $\cos\theta = \frac{\sqrt{3}}{2}$ (iv) $\sec\theta = -\sqrt{2}$

3. 3. Find the most general value of θ satisfying :

   (i) $\tan\theta = -1$ (ii) $\tan\theta = \sqrt{3}$ (iii) $\cot\theta = -1$

4. Find the most general value of θ satisfying :

   (i) $\sin 2\theta = \frac{1}{2}$ (ii) $\cos 2\theta = \frac{1}{2}$ (iii) $\tan 3\theta = \frac{1}{\sqrt{3}}$

   (iv) $\cos 3\theta = -\frac{\sqrt{3}}{2}$ (v) $\sin^2\theta = \frac{3}{4}$ (vi) $\sin^2 2\theta = \frac{1}{4}$

   (vii) $4\cos^2\theta = 1$ (viii) $\cos^2 2\theta = \frac{3}{4}$.

5. Find the general solution of the following :

   (i) $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$ (ii) $4\cos^2\theta - 4\sin\theta = 1$.

   (iii) $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$.

KEY WORDS

- An angle is generated by the rotation of a ray.
- The angle can be negative or positive according as rotation of the ray is clockwise or anticlockwise.
- A degree is one of the measures of an angle and one complete rotation generates an angle of 360°.
- An angle can be measured in radians, 360° being equivalent to 2π radians.
- If an arc of length l subtends an angle of θ radians at the centre of the circle with radius r, we have $l = r\,\theta$
- If the coordinates of a point P of a unit circle are (x, y) then the six trigonometric functions are defined as $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$, $\sec\theta = \frac{1}{x}$ and $\operatorname{cosec}\theta = \frac{1}{y}$.

  The coordinates (x, y) of a point P can also be written as $(\cos\theta, \sin\theta)$.

  Here θ is the angle which the line joining centre to the point P makes with the positive direction of x-axis.
- The values of the trigonometric functions $\sin\theta$ and $\cos\theta$ when θ takes values $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ are given by

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

- Graphs of $\sin\theta$, $\cos\theta$ are continous every where
  - Maximum value of both $\sin\theta$ and $\cos\theta$ is 1.
  - Minimum value of both $\sin\theta$ and $\cos\theta$ is -1.
  - Period of these functions is 2π.
- $\tan\theta$ and $\cot\theta$ can have any value between $-\infty$ and $+\infty$.
  - The function $\tan\theta$ has discontinuities (breaks) at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ in $(0, 2\pi)$.
  - Its period is π.
  - The graph of $\cot\theta$ has discontinuities (breaks) at $0, \pi, 2\pi$.

    Its period is π.
- $\sec\theta$ cannot have any value numerically less than 1.
- It has breaks at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ It repeats itself after 2π.
- $\operatorname{cosec}\theta$ cannot have any value between -1 and $+1$.

  It has discontinuities (breaks) at $\pi, 2\pi$.

  It repeats itself after 2π.
- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$.

  $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$.

  $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B},\ \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

  $$\cot(A+B) = \frac{\cot A\cot B - 1}{\cot B + \cot A},\ \cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A}$$
- $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

  $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

  $2\cos A\cos B = \cos(A+B) - \cos(A-B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

- $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$

  $\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$

  $\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

  $\cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$

- $\sin(A+B) \sin(A-B) = \sin^2 A - \sin^2 B$

  $\cos(A+B) \cos(A-B) = \cos^2 A - \sin^2 B.$

- $\sin 2A = 2 \sin A \cos A = \frac{2\tan A}{1+\tan^2 A}$

- $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

  $= \frac{1-\tan^2 A}{1+\tan^2 A}$

- $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

- $\sin^2 A = \frac{1-\cos 2A}{2}, \quad \cos^2 A = \frac{1+\cos 2A}{2}, \quad \tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$

- $\sin 3A = 3\sin A - 4\sin^3 A, \quad \cos 3A = 4\cos^3 A - 3\cos A$

  $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

- $\sin^3 A = \frac{3\sin A - \sin 3A}{4}, \quad \cos^3 A = \frac{3\cos A + \cos 3A}{4}$

- $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}, \quad \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$,

  $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$

- $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}, \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$
- $\sin\theta = 0 \Rightarrow \theta = n\pi, \quad n \in I.$

  $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}, \quad n \in I.$

  $\tan\theta = 0 \Rightarrow \theta = n\pi, \quad n \in I.$
- $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha, \quad n \in I.$
- $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha, \quad n \in I.$
- $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi \pm \alpha, \quad n \in I.$
- $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha, \quad n \in I.$
- $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha, \quad n \in I.$
- $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha, \quad n \in I.$

SUPPORTIVE WEBSITES

http://www.wikipedia.org

http://mathworld.wolfram.com

PRACTICE EXERCISE

1. A train is moving at the rate of 75 km/hour along a circular path of radius 2500 m. Through how many radians does it turn in one minute ?
2. Find the number of degrees subtended at the centre of the circle by an arc whose length is 0.357 times the radius.
3. The minute hand of a clock is 30 cm long. Find the distance covered by the tip of the hand in 15 minutes.
4. Prove that

   (a) $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

(b) $\dfrac{1}{\sec\theta+\tan\theta} = \sec\theta-\tan\theta$

(c) $\dfrac{\tan\theta}{1+\tan^2\theta}-\dfrac{\cot\theta}{1+\cot^2\theta}=2\sin\theta\cos\theta$

(d) $\dfrac{1+\sin\theta}{1-\sin\theta}=(\tan\theta+\sec\theta)^2$

(e) $\sin^8\theta-\cos^8\theta=(\sin^2\theta-\cos^2\theta)(1-2\sin^2\theta\cos^2\theta)$

(f) $\sqrt{\sec^2\theta+\text{cosec}^2\theta}=\tan\theta+\cot\theta$

5. If $\theta=\dfrac{\pi}{4}$ verify that $\sin 3\theta=3\sin\theta-4\sin^3\theta$.

6. Evaluate :

(a) $\sin\dfrac{25\pi}{6}$ (b) $\sin\dfrac{21\pi}{4}$ (c) $\tan\left(\dfrac{3\pi}{4}\right)$

(d) $\sin\dfrac{17}{4}\pi$ (e) $\cos\dfrac{19}{3}\pi$

7. Draw the graph of cos x from $x=-\dfrac{\pi}{2}$ and $x=\dfrac{3\pi}{2}$.

8. Define a periodic function of x and show graphically that the period of tan x is π, i.e. the position of the graph from $x=\pi$ to 2π is repetition of the portion from $x=0$ to π.

9. Prove that $\tan(A+B)\times\tan(A-B)=\dfrac{\cos^2 B-\cos^2 A}{\cos^2 B-\sin^2 A}$.

10. Prove that $\cos\theta-\sqrt{3}\sin\theta=2\cos\left(\theta+\dfrac{\pi}{3}\right)$.

11. If $A+B=\dfrac{\pi}{4}$

Prove that (1 + tanA) (1+tanB) = 2 and (cot A − 1) (cot B − 1).

12. Prove each of the following:

(i) $\dfrac{\sin(A-B)}{\cos A\cos B}+\dfrac{\sin(B-C)}{\cos B\cos C}+\dfrac{\sin(C-A)}{\cos C\cos A}=0$

(ii) $\cos\left(\dfrac{\pi}{10}-A\right)\cos\left(\dfrac{\pi}{10}+A\right)+\cos\left(\dfrac{2\pi}{5}-A\right)\cos\left(\dfrac{2\pi}{5}+A\right)=\cos 2A.$

(iii) $\cos\dfrac{2\pi}{9}\cos\dfrac{4\pi}{9}\cos\dfrac{9\pi}{9}=-\dfrac{1}{8}.$

(iv) $\cos\dfrac{13\pi}{45}+\cos\dfrac{17\pi}{45}+\cos\dfrac{43\pi}{45}=0.$

(v) $\tan\left(A+\dfrac{\pi}{6}\right)+\cot\left(A-\dfrac{\pi}{6}\right)=\dfrac{1}{\sin 2A-\sin\dfrac{\pi}{3}}$

(vi) $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}=\tan\theta$

(viii) $\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}=\tan 2\theta+\sec 2\theta$

(ix) $\cos^2 A+\cos^2 A\left(A+\dfrac{\pi}{3}\right)+\cos 2\left(A-\dfrac{\pi}{3}\right)=\dfrac{3}{2}.$

(x) $\dfrac{\sec 8A-1}{\sec 4A-1}=\dfrac{\tan 8A}{\tan 2A}$

(xi) $\cos\dfrac{\pi}{30}\cos\dfrac{7\pi}{30}\cos\dfrac{11\pi}{30}\cos\dfrac{13\pi}{30}=\dfrac{11}{16}$

(xii) $\sin\dfrac{\pi}{10}+\sin\dfrac{13\pi}{10}=-\dfrac{1}{2}.$

13. Find the general value of 'θ' satisfying

(a) $\sin\theta=\dfrac{1}{\sqrt{2}}$ (b) $\sin\theta=\dfrac{\sqrt{3}}{2}$

(c) $\sin\theta=-\dfrac{1}{\sqrt{2}}$ (d) $\operatorname{cosec}\theta=\sqrt{2}$

14. Find the general value of 'θ' satisfying

(a) $\cos\theta = \frac{1}{2}$ (b) $\sec\theta = \frac{2}{\sqrt{3}}$

(c) $\cos\theta = -\frac{\sqrt{3}}{2}$ (d) $\sec\theta = -2$

15. Find the most general value of 'θ' satisfying

(a) $\tan\theta = 1$ (b) $\tan\theta = -1$ (c) $\cot\theta = -\frac{1}{\sqrt{3}}$

16. Find the general value of 'θ' satisfying

(a) $\sin^2\theta = \frac{1}{2}$ (b) $4\cos^2\theta = 1$ (c) $2\cot^2\theta = \mathrm{cosec}^2\theta$

17. Solve the following for θ :

(a) $\cos p\theta = \cos q\theta$ (b) $\sin 9\theta = \sin\theta$ (c) $\tan 5\theta = \cot\theta$

18. Solve the following for θ :

(a) $\sin m\theta = \sin n\theta = 0$

(b) $\tan m\theta + \cot n\theta = 0$

(c) $\cos\theta + \cot 2\theta + \cot 3\theta = 0$

(d) $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

ANSWERS

EXERCISE 17.1

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{12}$ (iii) $\frac{5\pi}{12}$ (iv) $\frac{7\pi}{12}$

(v) $\frac{3\pi}{2}$

2. (i) 45^0 (ii) 15^0 (iii) 9^0

(iv) 3^0 (v) 120^0

3. $\frac{\pi}{4}, \frac{13\pi}{36}, \frac{14\pi}{36}$ 4. $\frac{5\pi}{6}$ 6. $\frac{\pi}{3}$

EXERCISE 17.2

1. (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{5\pi}{6}$

2. (a) 36^0 (b) 30^0 (c) 20^0

3. $\frac{1}{6}$ radian; 9.55^0 4. $\frac{1}{5}$ radian; 5. 95.54m

6. (a) 0.53m (b) 38.22 cm (c) 0.002 radian

   (d) 12.56m (e) 31.4 cm (f) 3.75 radian

   (g) 6.28m (h) 2 radian (i) 19.11 m

EXERCISE 17.3

1. (i) – ive (ii) – ive (iii) – ive (iv) + ive

   (v) + ive (vi) – ive (vii) + ive (viii) – ive

2. (i) zero (ii) zero (iii) $-\frac{1}{2}$

   (iv) –1
   (v) 1 (vi) Not defined (vii) Not defined (viii) 1

EXERCISE 17.4

2. $\sin\theta = \frac{1}{\sqrt{5}}, \cos\theta = \frac{2}{\sqrt{5}}, \cot\theta = 2, \text{cosec}\,\theta = \sqrt{5}, \sec\theta = \frac{\sqrt{5}}{2}$

3. $\sin\theta = \frac{a}{b}, \cos\theta = \frac{\sqrt{b^2 - a^2}}{a}, \sec\theta = \frac{b}{\sqrt{b^2 - a^2}}, \tan\theta = \frac{a}{\sqrt{b^2 - a^2}},$

   $\cot\theta = \frac{\sqrt{b^2 - a^2}}{a}$

6. $\frac{2m}{1 + m^2}$

EXERCISE 17.5

1. (i) $4\frac{1}{4}$ (ii) $6\frac{1}{2}$ (iii) -1

   (iv) $\frac{22}{3}$ (v) zero

EXERCISE 17.6

1. 1, −1 (3) Graph of $y = 2\sin\theta$, $[0, \pi]$

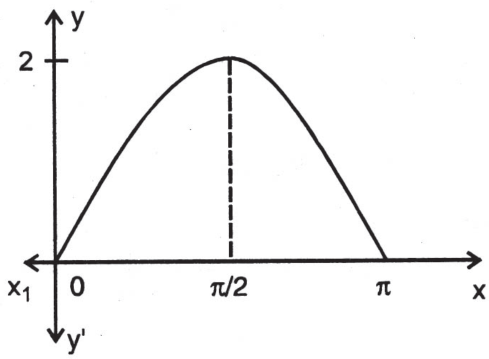


Fig. 17.35

4. (a) $\frac{7\pi}{6}, \frac{11\pi}{6}$ (b) $\frac{4\pi}{3}, \frac{5\pi}{3}$

5. $y = 2\sin x$ from $-\pi$ to π

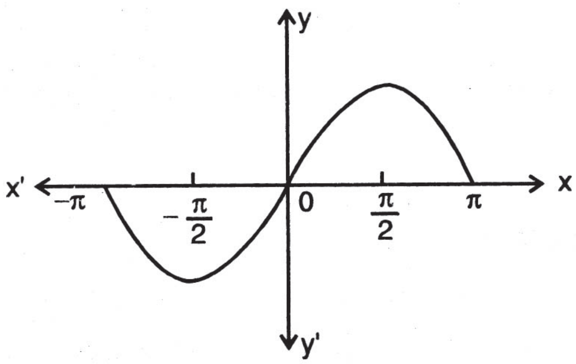


Fig. 17.36

EXERCISE 17.7

1. (a) $y = \cos\theta - \frac{\pi}{4}$ to $\frac{\pi}{4}$

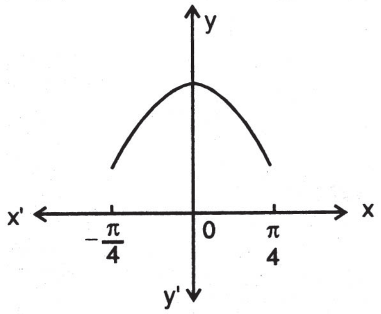


Fig. 17.37

(b) $y = 3\cos\theta$; 0 to 2π

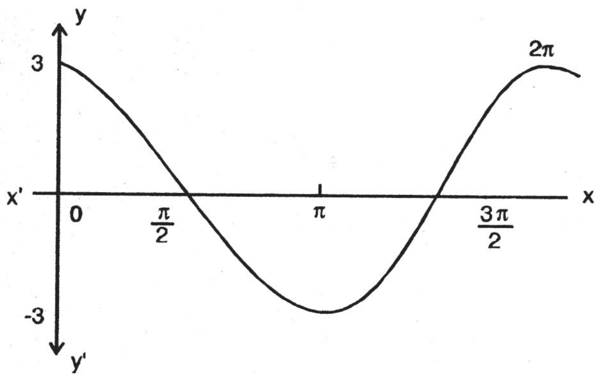


Fig. 17.38

(c) $y = \cos 3\theta$ $-\pi$ to π

$\cos\theta = 0.87$

$\theta = \frac{\pi}{6}, -\frac{\pi}{6}$

$\cos\theta = -0.87$

$\theta = \frac{5\pi}{6}, -\frac{5\pi}{6}$

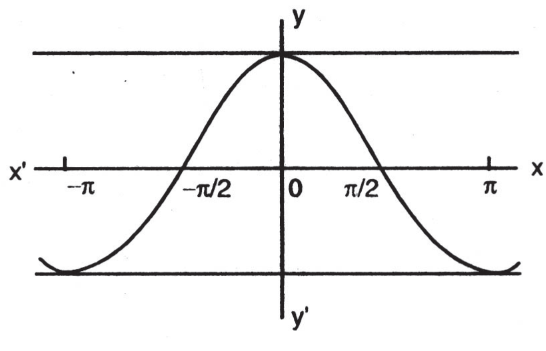


Fig. 17.39

(d) Graph y = cos θ in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, lies below the x-axis.

(e) $y = \cos\theta$

θ lies in 2π to 4π.

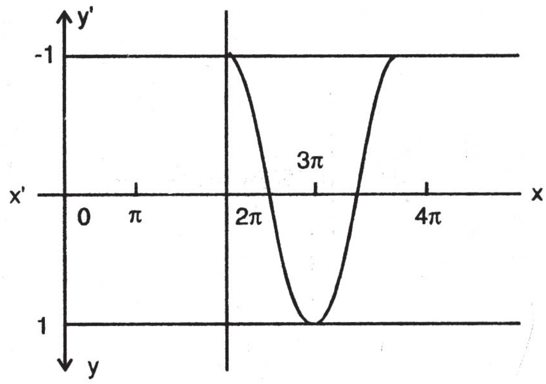


Fig. 17.40

EXERCISE 17.8

1. (a) Infinite

(b) At $\frac{\pi}{2}, \frac{3\pi}{2}$ there are breaks in graphs.

(c) $y = \tan 2\theta$ $-\pi$ to π ,

At $\theta = \frac{\pi}{3}$, $\tan\theta = 1.7$

2. (a) Infinite (b) $\cot\theta = -1$ at $\theta = \frac{3\pi}{4}$

EXERCISE 17.9

1. (a) $y = \sec\theta$

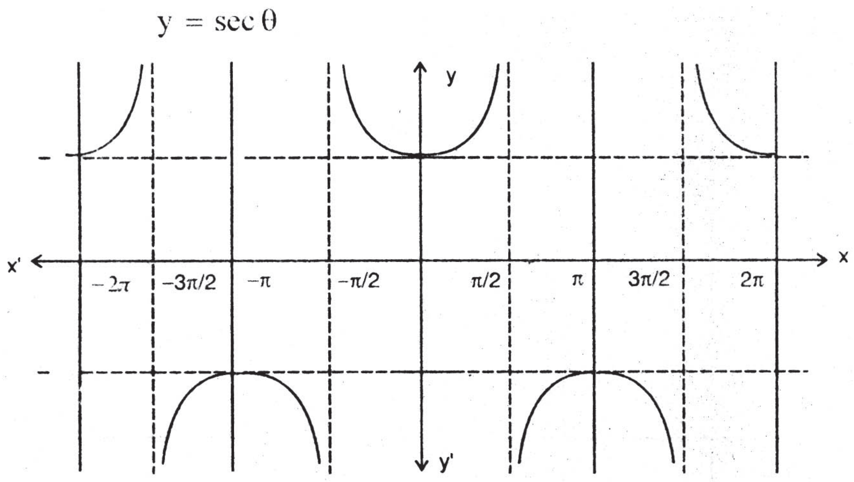


Fig. 17.41

Points of discontinuity of sec 2θ are at $\frac{\pi}{4}, \frac{3\pi}{4}$ in the interval $[0, 2\pi]$.

(b) In tracing the graph from 0 to -2π, use cosec $(-\theta) = -$cosec 2θ.

EXERCISE 17.10

1. (a) Period is $\frac{2\pi}{3}$

   (b) Period is $\frac{2\pi}{2} = \pi$

   (c) Period is y is $\frac{\pi}{3}$

   (d) $y = \sin^2 2x = \frac{1-\cos 4x}{2} = \frac{1}{2} - \frac{1}{2}\cos 4x$; Period of y is $\frac{2\pi}{4}$ i.e. $\frac{\pi}{2}$.

   (e) $y = 3\cot\left(\frac{x+1}{3}\right)$, Period of y is $\frac{\pi}{\frac{1}{3}} = 3\pi$

EXERCISE 17.11

1. (a) (i) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (ii) $\frac{\sqrt{3}}{2}$ (c) $\frac{21}{221}$

2. (a) $\frac{\sqrt{3}-1}{2\sqrt{2}}$

EXERCISE 17.12

1. (i) $\frac{\cos^2 A - \sin^2 A}{2}$ (ii) $-\frac{1}{4}$

2. (c) $-\frac{\left(\sqrt{3}+1\right)}{2\sqrt{2}}$

EXERCISE 17.13

1. (a) $\sin 5\theta - \sin \theta$ (b) $\cos 2\theta - \cos 6\theta$

   (c) $\cos\frac{\pi}{3} + \cos\frac{\pi}{6}$ (d) $\sin\frac{\pi}{2} + \sin\frac{\pi}{6}$

2. (a) $2 \sin 5\theta \cos \theta$ (ii) $2 \cos 5\theta \sin \theta$

   (c) $2 \sin 3\theta \sin \theta$ (d) $2 \cos 6\theta \cos \theta$

EXERCISE 17.14

2. (a) $\frac{24}{25}$ (b) $\frac{120}{169}$ (c) $\frac{2016}{4225}$
3. (a) $\frac{161}{289}$ (b) $-\frac{7}{25}$ (c) $\frac{119}{169}$
4. (a) $\frac{24}{7}$ (b) $\frac{2ab}{b^2 - a^2}$ 5. 1

EXERCISE 17.15

2. (a) $\frac{22}{27}$ (b) $\frac{(3pq^2 - 4p^3)}{q^3}$

3. (a) $\frac{23}{27}$ (b) $\frac{4c^3 - 3cd^2}{d^3}$

EXERCISE 17.16

2. (a) $\frac{\sqrt{3}-1}{2\sqrt{2}}, \frac{\sqrt{\left(4-\sqrt{2}-\sqrt{6}\right)}}{2\sqrt{2}}$

3. (a) $\frac{\sqrt{2-\sqrt{2}}}{2}$ (b) $\frac{\sqrt{2+\sqrt{2}}}{2}$ (c) $\sqrt{2}-1$

EXERCISE 17.17

1. (i) $\theta = n\pi + (-1)^n \frac{\pi}{3}, \quad n \in I$ (ii) $\theta = n\pi + (-1)^n \frac{\pi}{4}, \quad n \in I$

   (iii) $\theta = n\pi + (-1)^n \frac{4\pi}{3}, \quad n \in I$ (iv) $\theta = n\pi + (-1)^n \frac{5\pi}{4}, \quad n \in I$

2. (i) $\theta = 2n\pi \pm \frac{2\pi}{3}, \quad n \in I$ (ii) $\theta = 2n\pi \pm \frac{5\pi}{6}, \quad n \in I$

   (iii) $\theta = 2n\pi \pm \frac{\pi}{6}, \quad n \in I$ (iv) $\theta = 2n\pi \pm \frac{3\pi}{4}, \quad n \in I$

3. (i) $\theta = n\pi + \frac{3\pi}{4}, \quad n \in I$ (ii) $\theta = n\pi + \frac{\pi}{3}, \quad n \in I$

   (iii) $\theta = n\pi - \frac{\pi}{4}, \quad n \in I$

4. (i) $\theta = \frac{n\pi}{2}, (-1)^n \frac{\pi}{12}, \quad n \in I$ (ii) $\theta = n\pi \pm \frac{\pi}{6}, \quad n \in I$

   (iii) $\theta = \frac{n\pi}{3} + \frac{\pi}{18}, \quad n \in I$ (iv) $\theta = \frac{2n\pi}{3} \pm \frac{5\pi}{18}, \quad n \in I$

   (v) $\theta = n\pi \pm \frac{\pi}{3}, \quad n \in I$ (vi) $\theta = \frac{n\pi}{2} \pm \frac{\pi}{12}, \quad n \in I$

   (vii) $\theta = n\pi \pm \frac{\pi}{3}, \quad n \in I$ (viii) $\theta = \frac{n\pi}{2} \pm \frac{\pi}{12}, \quad n \in I$

5. (i) $\theta = 2n\pi \pm \frac{5\pi}{6}, \quad n \in I$ (ii) $\theta = n\pi + (-1)^n \frac{\pi}{6}, \quad n \in I$

   (iii) $\theta = 2n\pi \pm \frac{\pi}{3}, \quad n \in I$

PRACTICE EXERCISE

1. $\frac{1}{2}$ radian   2. 20.45^0 3. 15π cm.

6. (a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) -1 (d) $\frac{1}{\sqrt{2}}$ (e) $\frac{1}{2}$

7.

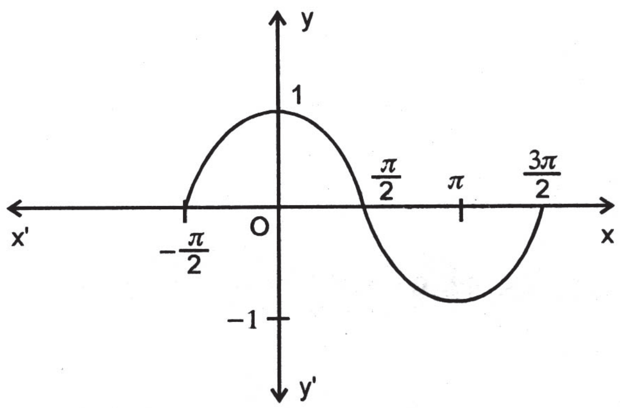
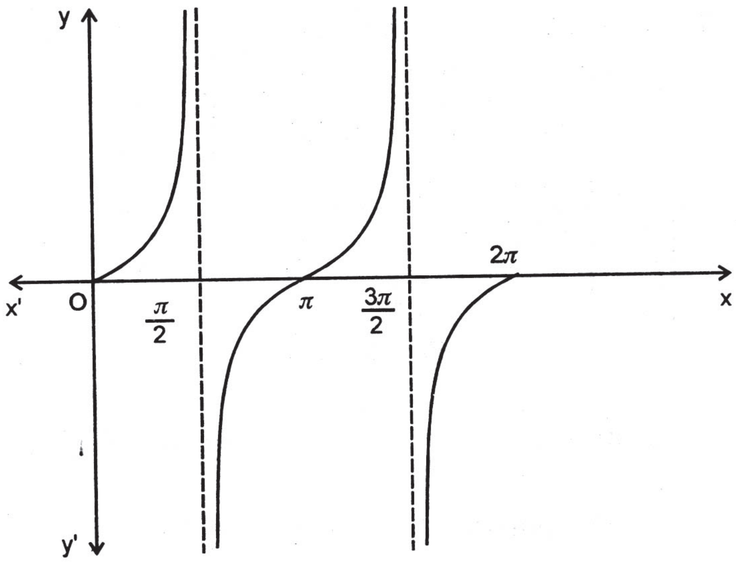


Fig. 17.42

8.



y = sec 2θ

Fig. 17.43

13. (a) $\theta = n\pi + (-1)^n \frac{\pi}{4}, \quad n \in I$ (b) $\theta = n\pi + (-1)^n \frac{\pi}{3}, \quad n \in I$

(c) $\theta = n\pi + (-1)^n \frac{5\pi}{4}, \quad n \in I$ (d) $\theta = n\pi + (-1)^n \frac{\pi}{4}, \quad n \in I$

14. (a) $\theta = 2n\pi \pm \frac{\pi}{3}, \quad n \in I$ (b) $\theta = 2n\pi \pm \frac{\pi}{6}, \quad n \in I$

(c) $\theta = 2n\pi \pm \frac{5\pi}{6}, \quad n \in I$ (d) $\theta = 2n\pi \pm \frac{2\pi}{3}, \quad n \in I$

15. (a) $\theta = n\pi + \frac{\pi}{4}, \quad n \in I$

(b) $\theta = n\pi \pm \frac{\pi}{3}, \quad n \in I$

(c) $n\pi + \frac{2\pi}{3}, \quad n \in I.$

16. (a) $\theta = n\pi \pm \frac{\pi}{4}, \quad n \in I$

(b) $\theta = n\pi \pm \frac{\pi}{3}, \quad n \in I$

(c) $\theta = n\pi \pm \frac{\pi}{4}, \quad n \in I$

17. (a) $\theta = \frac{2n\pi}{p \mp q}, \quad n \in I$

(b) $\theta = \frac{n\pi}{4}$ or $(2n+1)\frac{\pi}{10}, \quad n \in I$

(c) $\theta = (2n+1)\frac{\pi}{12}, \quad n \in I$

18. (a) $\theta = \frac{(2k+1)\pi}{m-n}$ or $\frac{2k\pi}{m+n}, \quad k \in I$

(b) $\theta = \frac{(2k+1)}{2(m-n)}, k \in I$

(c) $\theta = (2n+1)\frac{\pi}{4}$ or $2n\pi \pm \frac{2\pi}{3}, \quad n \in I$

(d) $\theta = \frac{2n\pi}{5}$ or $\theta = n\pi \pm \frac{\pi}{2}, \quad n \in I$ or $\theta = (2n-1)\pi, \quad n \in I.$

Chapter

18

INVERSE TRIGONOMETRIC FUNCTIONS

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- define inverse trigonometric functions;
- state the condition for the inverse of trigonometric functions to exist;
- define the principal value of inverse trigonometric functions;
- find domain and range of inverse trigonometric functions;
- state the properties of inverse trigonometric functions; and
- simplify expressions involving inverse trigonometric functions.

PREREQUISITES

- Knowledge of function and their types, domain and range of a function.
- Formulae for trigonometric functions of sum, difference, multiple and sub-multiples of angles.

INTRODUCTION

From the previous topic functions, recall that if f is a function say from A to B, in general $f^{-1}(b)$ $(b \in B)$ may correspond with one element, more than one element or may not with any element in A.

Also recall that, if $f : A \to B$ be one-one and onto then for each $b \in B$ then $f^{-1}(b)$ will correspond with a single element in A.

We therefore have a correspondence that assigns to each $b \in B$ and one element $f^{-1}(b)$ in A.

Thus f^{-1} is a function from B to A.

Hence f^{-1} is called the inverse function of 'f'.

If f is one-one and onto then f^{-1} is also a function

Example : $f : A \to B$ is a function.

$f^{-1}(b) = \{1, 2\}$, Hence f^{-1} is not a function.

Hence, the inverse of f does not exist.

Example : Let $f : R \to R$ be defined by $f(x) = 2x + 3$

We know that, f is one-one and onto.

Hence, the inverse of f exist.

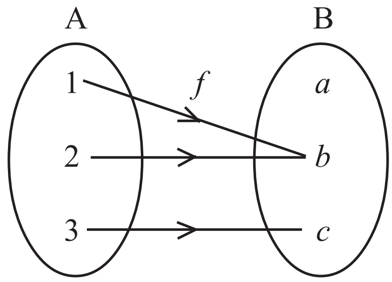


Fig. 18.1

In this lesson, we will learn more about inverse trigonometric functions, its domain and range and simplifying expression that involve inverse trigonometric functions.

18.1 IS INVERSE OF EVERY FUNCTION POSSIBLE?

Take two ordered pairs of a function (x_1, y) and (x_2, y)

If we invert them, we will get (y, x_1) and (y, x_2)

This is not a function because the first member of the two ordered pairs is the same.

Now let us take another function :

$$\left(\sin\frac{\pi}{2}, 1\right), \left(\sin\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\sin\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$$

Writing the inverse, we have

$$\left(1, \sin\frac{\pi}{2}\right), \left(\frac{1}{\sqrt{2}}, \sin\frac{\pi}{4}\right), \left(\frac{\sqrt{3}}{2}, \sin\frac{\pi}{3}\right)$$

which is a function.

Let us consider some examples from daily life.

f: Student $\rightarrow$ Score in Mathematics

Do you think f^{-1} will exist ?

It may or may not be because the moment two students have the same score, f^{-1} will cease to be a function. Because the first element in two or more ordered pairs will be the same. So we conclude that

every function is not invertible.

Example 18.1 : If f:R $\rightarrow$R defined by $f(x) = x^3 + 4$ What will be f^{-1}?

Solution : In this case f is one-to-one and onto both.

$\Rightarrow$ f is invertible.

Let $y = x^3 + 4$

$$\therefore\ y-4=x^3 \Rightarrow x=\sqrt[3]{y-4}$$

So f^{-1}, inverse function of f i.e., $f^{-1}(y) = \sqrt[3]{y-4}$.

The functions that are one-to-one and onto will be invertible.

Let us extend this to trigonometry :

Take $y = \sin x$. Here domain is the set of all real numbers. Range is the set of all real numbers lying between -1 and 1, including -1 and 1 i.e. $-1 \le y \le 1$.

We know that there is a unique value of y for each given number x.

In inverse process we wish to know a number corresponding to a particular value of the sine

Suppose $\quad y = \sin x = \frac{1}{2}$

$$\Rightarrow \sin x = \sin\frac{\pi}{6} = \sin\frac{5\pi}{6} = \sin\frac{13\pi}{6} = ...$$

x may have the values as $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, ...$

Thus there are infinite number of values of x.

$y = \sin x$ can be represented as

$$\left(\frac{\pi}{6}, \frac{1}{2}\right), \left(\frac{5\pi}{6}, \frac{1}{2}\right), ...$$

The inverse relation will be

$$\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right), ...$$

It is evident that it is not a function as first element of all the ordered pairs is $\frac{1}{2}$ which contradicts the definition of a function.

Consider $y = \sin x$, where $x \in \mathrm{R}$ (domain) and $y \in [-1, 1]$ or $-1 \le y \le 1$ which is called range. This is many-to-one and onto function, therefore it is not invertible.

Can $y = \sin x$ be made invertible and how? Yes, if we restrict its domain in such a way that it becomes one-to-one and onto taking x as

(i) $\frac{-\pi}{2} \le x \le \frac{\pi}{2},\ y \in [-1, 1]$ or

(ii) $\frac{3\pi}{2} \le x \le \frac{5\pi}{2},\ y \in [-1, 1]$ or

(iii) $-\frac{5\pi}{2} \le x \le -\frac{3\pi}{2},\ y \in [-1, 1]$ etc.,

Now consider the inverse function $y = \sin^{-1} x$

We know the domain and range of the function. We interchange domain and range for the inverse of the function. Therefore,

(i) $\frac{-\pi}{2} \le y \le \frac{\pi}{2},\ x \in [-1, 1]$ or

(ii) $\frac{3\pi}{2} \le y \le \frac{5\pi}{2},\ x \in [-1, 1]$ or

(iii) $-\frac{5\pi}{2} \le y \le -\frac{3\pi}{2},\ x \in [-1, 1]$ etc.,

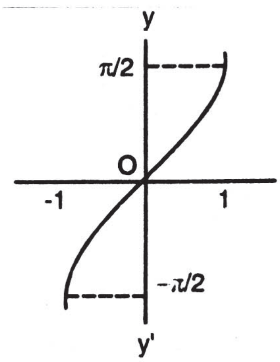
Here we take the least numerical value among all the values of the real number whose sine is x which is called the principle value of $\sin^{-1}x$.

For this the only case is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, Therefore, for principal value of $y = \sin^{-1}x$, the domain is $[-1, 1]$ i.e., $x \in [-1, 1]$ and range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

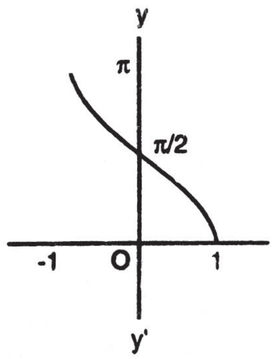
Similarly, we can discuss the other inverse trigonometric functions.

	Function	Domain	Range (Principal value)
1.	$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3.	$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1} x$	R	$[0, \pi]$
5.	$y = \sec^{-1} x$	$x \geq 1$ or $x \leq -1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
6.	$y = \operatorname{cosec}^{-1} x$	$x \geq 1$ or $x \leq -1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

18.2 GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS



$y = \sin^{-1} x$



$y = \cos^{-1} x$

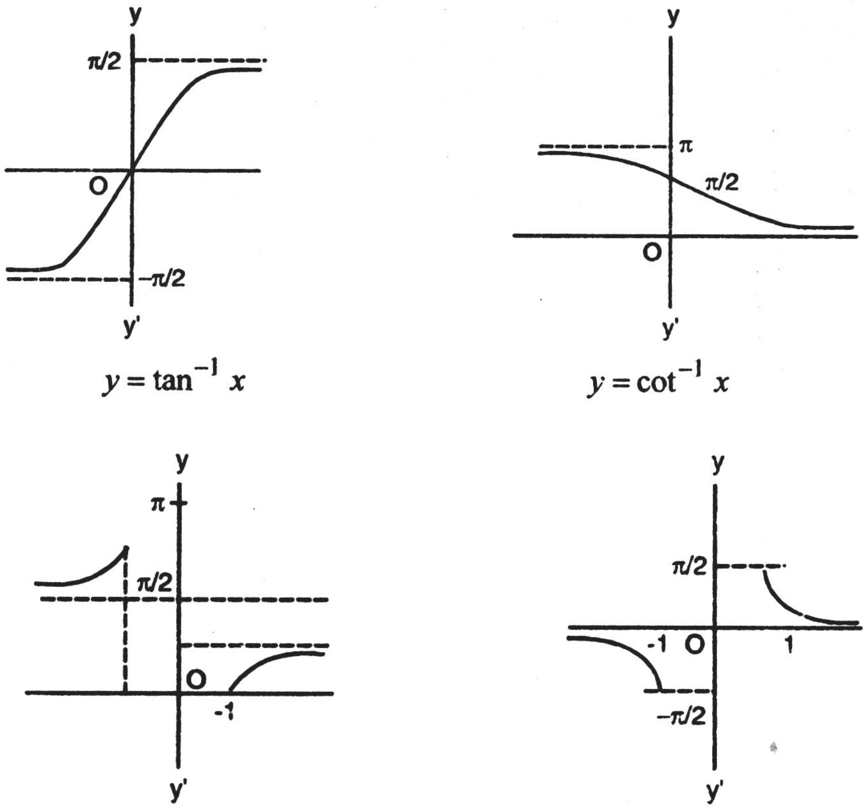


Fig. 18.2

Example 18.2 : Find the principal value of each of the following :

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$ (iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution: (i) Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \sin\theta = \sin\left(\frac{\pi}{4}\right) \Rightarrow \theta = \frac{\pi}{4}$$

(ii) Let $\cos^{-1}\left(-\frac{1}{2}\right) = \theta$

$$\Rightarrow \cos\theta = -\frac{1}{2} = \cos\left(\pi - \frac{\pi}{3}\right) \Rightarrow \cos\left(\frac{2\pi}{3}\right) \text{ or } \theta = \frac{2\pi}{3}$$

(iii) Let $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \theta$

or $\frac{-1}{\sqrt{3}} = \tan\theta$ or $\tan\theta = \tan\left(\frac{-\pi}{6}\right)$

$\Rightarrow \theta = \frac{-\pi}{6}$.

Example 18.3 : Find the principal value of each of the following :

(a) (i) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\tan^{-1}(-1)$

(b) Find the value of the following using the principal value :

$$\sec\left[\cos^{-1}\frac{\sqrt{3}}{2}\right].$$

Solution: (a) (i) Let $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$, then

$\Rightarrow \frac{1}{\sqrt{2}} = \cos\theta$ or $\cos\theta = \cos\frac{\pi}{4}$

$\Rightarrow \theta = \frac{\pi}{4}$

(ii) Let $\tan^{-1}(-1) = \theta$, then

$-1 = \tan\theta$ or $\tan\theta = \tan\left(-\frac{\pi}{4}\right)$

$\Rightarrow \theta = -\frac{\pi}{4}$

(b) Let $\cos^{-1}\frac{\sqrt{3}}{2} = \theta$, then

$\frac{\sqrt{3}}{2} = \cos\theta$ or $\cos\theta = \cos\left(\frac{\pi}{6}\right)$

$\Rightarrow \theta = \dfrac{\pi}{6}.$

$\therefore\ \sec\left[\cos^{-1}\dfrac{\sqrt{3}}{2}\right] = \sec\theta = \sec\left(\dfrac{\pi}{6}\right) = \dfrac{2}{\sqrt{3}}.$

Example 18.4 : Simplify the following :

(i) $\cos[\sin^{-1}x]$ (ii) $\cot[\text{cosec}^{-1}x]$

Solution: (i) Let $[\sin^{-1}x] = \theta$

$\Rightarrow\ x = \sin\theta.$

$\therefore\ \cos\ [\sin^{-1}x] = \cos\ \theta\ = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$ $(\because \sin\theta = x)$

(ii) Let $\text{cosec}^{-1}x = \theta$

$\Rightarrow\ x = \text{cosec}\ \theta$

$\cot(\text{cosec}^{-1}x) = \cot\theta = \sqrt{\text{cosec}^2\theta - 1}$

$= \sqrt{x^2-1}$ $(\because \text{cosec}\ \theta = x)$

EXERCISE 18.1

1. Find the principal value of each of the following :

(a) $\cos^{-1}\dfrac{\sqrt{3}}{2}$ (b) $\text{cosec}^{-1}(-\sqrt{2})$ (c) $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

(d) $\tan^{-1}(-\sqrt{3})$ (e) $\cot^{-1}(1)$

2. Evaluate each of the following :

(a) $\cos\left(\cos^{-1}\dfrac{1}{3}\right)$ (b) $\text{cosec}^{-1}\left(\text{cosec}\dfrac{\pi}{4}\right)$ (c) $\cos\left(\text{cosec}^{-1}\dfrac{2}{\sqrt{3}}\right)$

(d) $\tan(\sec^{-1}\sqrt{2})$ (e) $\text{cosec}\left[\cot^{-1}(-\sqrt{3})\right]$

3. Simplify each of the following expressions :

(a) $\sec(\tan^{-1}x)$ (b) $\tan(\text{cosec}^{-1}x/2)$ (c) $\cot(\text{cosec}^{-1}x^2)$

(d) $\cos(\cot^{-1}x^2)$ (e) $\tan\left(\sin^{-1}\sqrt{1-x^2}\right)$

18.3 PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Property 1: $\sin^{-1}(\sin\theta) = \theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Solution: Let $\sin\theta = x$

$$\Rightarrow \quad \theta = \sin^{-1}x$$

$$= \sin^{-1}(\sin\theta) = \theta$$

Also $\sin(\sin^{-1}x) = x$

Similarly, we can prove that

(i) $\cos^{-1}(\cos\theta) = \theta, \quad 0 \le \theta \le \pi.$

(ii) $\tan^{-1}(\tan\theta) = \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Property 2: (i) $\text{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$ (ii) $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

(iii) $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$

Solution: (i) Let $\text{cosec}^{-1}x = \theta$ (ii) Let $\cot^{-1}x = \theta$

$\Rightarrow \quad x = \text{cosec}\,\theta$ $\qquad \Rightarrow \quad x = \cot\theta$

$\Rightarrow \quad \left(\frac{1}{x}\right) = \sin\theta$ $\qquad \Rightarrow \quad \frac{1}{x} = \tan\theta$

$\therefore \quad \theta = \sin^{-1}\left(\frac{1}{x}\right) \qquad \Rightarrow \theta = \tan^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \text{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right) \qquad \therefore \quad \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right).$

(iii) $\sec^{-1}x = \theta$

$\Rightarrow \quad x = \sec\theta$

$\therefore \quad \frac{1}{x} = \cos\theta \qquad \text{or} \quad \theta = \cos^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right).$

Property 3: (i) $\sin^{-1}(-x) = -\sin^{-1}x$ (ii) $\tan^{-1}(-x) = -\tan^{-1} x$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x.$

Solution: (i) Let $\sin^{-1}(-x) = \theta$

$\Rightarrow -x = \sin\theta \quad \text{or} \quad x = -\sin\theta = \sin(-\theta)$

$\therefore \quad -\theta = \sin^{-1} x \quad \text{or} \quad \theta = -\sin^{-1} x.$

or $\sin^{-1}(-x) = -\sin^{-1} x.$

(ii) Let $\tan^{-1}(-x) = \theta$

$\Rightarrow -x = \tan\theta \quad \text{or} \quad x = -\tan\theta = \tan(-\theta)$

$\Rightarrow \quad -\theta = \tan^{-1} x \quad \text{or} \quad \theta = -\tan^{-1} x.$

$\therefore \quad \tan^{-1}(-x) = -\tan^{-1} x.$

(iii) Let $\cos^{-1}(-x) = \theta$

$\Rightarrow -x = \cos\theta \quad \text{or} \quad x = -\cos\theta = \cos(\pi-\theta)$

$\Rightarrow \quad \cos^{-1} x = \pi - \theta.$

$\therefore \quad \cos^{-1}(-x) = \pi - \cos^{-1} x.$

Property 4: (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$.

Solution: (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

Let $\sin^{-1}x = \theta \Rightarrow x = \sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$

or $\cos^{-1}x = \left(\frac{\pi}{2}-\theta\right)$

$\Rightarrow \theta + \cos^{-1}x = \frac{\pi}{2}$ or $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

(ii) Let $\cot^{-1}x = \theta \Rightarrow x = \cot\theta = \tan\left(\frac{\pi}{2}-\theta\right)$

$\therefore \tan^{-1}(x) = \frac{\pi}{2} - \theta$ or $\theta + \tan^{-1}x = \frac{\pi}{2}$

or $\cot^{-1}x + \tan^{-1}x = \frac{\pi}{2}$.

(iii) Let $\text{cosec}^{-1}x = \theta$

$\Rightarrow x = \text{cosec}\,\theta = \sec\left(\frac{\pi}{2}-\theta\right)$

$\therefore \sec^{-1}(x) = \frac{\pi}{2} - \theta$ or $\theta + \sec^{-1}x = \frac{\pi}{2}$

$\Rightarrow \text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$.

Property 5

(i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$.

(ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$.

Solution : (i) Let $\tan^{-1}x = \theta,\ \tan^{-1}y = \phi \Rightarrow x = \tan\theta,\ y = \tan\phi$.

We have to prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

By substituting the above values on L.H.S. and R.H.S., we have

LHS $= \theta + \phi$ and RHS $= \tan^{-1}\left[\frac{\tan\theta + \tan\phi}{1-\tan\theta.\tan\phi}\right]$

$= \tan^{-1}\left[\tan(\theta+\phi)\right] = \theta+\phi =$ LHS.

$\therefore$ The result holds.

Simiarly (ii) can be proved.

Property 6:

(i) $2\tan^{-1} x = \sin^{-1}\left[\frac{2x}{1+x^2}\right] = \cos^{-1}\left[\frac{1-x^2}{1+x^2}\right] = \tan^{-1}\left[\frac{2x}{1+x^2}\right]$

(i) (ii) (iii) (iv)

Let $x = \tan\theta$

Substituting in (i), (ii), (iii), and (iv) we get

$2\tan^{-1} x = 2\tan^{-1}(\tan\theta) = 2\theta$ .... (i)

$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$

$= \sin^{-1}\left(2\sin\theta\cos\theta\right)$

$= \sin^{-1}\left(\sin 2\theta\right) = 2\theta$ .... (ii)

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\right)$$

$$= \cos^{-1}(\cos^2\theta-\sin^2\theta)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= 2\theta \qquad \text{...(iii)}$$

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$= 2\theta \qquad \text{...(iv)}$$

From (i), (ii), (iii) and (iv), we get

$$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right).$$

Property 7:

(i) $$\sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$$

$$= \sec^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right]$$

$$= \cot^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right] = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

(ii) $$\cos^{-1}x = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$$

$$= \operatorname{cosec}^{-1}\left[\frac{1}{\sqrt{1-x^2}}\right]$$

$$= \cot^{-1}\left[\frac{x}{\sqrt{1-x^2}}\right]$$

$$= \sec^{-1}\left[\frac{1}{x}\right]$$

Proof: Let $\sin^{-1}x = \theta \Rightarrow \sin\ \theta = x$

(i) $\cos\theta = \sqrt{1-x^2},\ \tan\theta = \dfrac{x}{\sqrt{1-x^2}},\ \sec\theta = \dfrac{1}{\sqrt{1-x^2}},$

$\cot\theta = \dfrac{\sqrt{1-x^2}}{x}$ and $\operatorname{cosec}\theta = \dfrac{1}{x}$

$$\therefore\ \sin^{-1}x = \theta = \cos^{-1}(\sqrt{1-x^2}) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{x}\right).$$

(ii) Let $\cos^{-1}x = \theta \Rightarrow x = \cos\ \theta$

$\therefore\ \sin\theta = \sqrt{1-x^2},\ \tan\theta = \dfrac{\sqrt{1-x^2}}{x},\ \sec\theta = \dfrac{1}{x},\ \cot\theta = \dfrac{x}{\sqrt{1-x^2}}$

and $\operatorname{cosec}\ \theta = \dfrac{1}{\sqrt{1-x^2}}$

$$\cos^{-1} x = \sin^{-1}\left(\sqrt{1-x^2}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \sec^{-1}\left(\frac{1}{x}\right)$$

Property 8

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

(ii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$

(iii) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

(iv) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right]$

Solution: (i) Let $x = \sin\theta$, $y = \sin\phi$, then

LHS $= \theta + \phi$,

RHS $= \sin^{-1}[\sin\theta\cos\phi + \cos\theta\sin\phi]$

$= \sin^{-1}[\sin(\theta + \phi)] = \theta + \phi.$

$\therefore$ LHS = RHS.

(ii) Let $x = \cos\theta$, $y = \cos\phi$

LHS $= \theta + \phi$

RHS $= \cos^{-1}[\cos\theta\cos\phi - \sin\theta\sin\phi]$

$= \cos^{-1}[\cos(\theta + \phi)] = \theta + \phi.$

$\therefore$ LHS = RHS.

(iii) Let $x = \sin\theta$, $y = \sin\phi$

LHS = $\theta - \phi$

RHS $= \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

$$= \sin^{-1}\left[\sin\theta\sqrt{1-\sin^2\phi} - \sin\phi\sqrt{1-\sin^2\theta}\right]$$

$$= \sin^{-1}\left[\sin\theta\cos\phi - \cos\theta\sin\phi\right]$$

$$= \sin^{-1}\left[\sin(\theta-\phi)\right] = \theta-\phi.$$

$\therefore$ LHS = RHS.

(iv) Let $x = \cos\theta$, $y = \cos\phi$

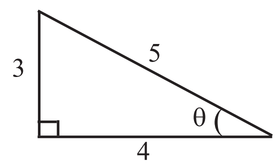
$\therefore$ LHS = $\theta - \phi$

RHS $= \cos^{-1}\left[\cos\theta\cos\phi + \sin\theta\sin\phi\right]$

$$= \cos^{-1}\left[\cos(\theta-\phi)\right] = \theta-\phi$$

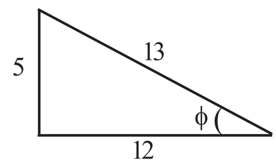
$\therefore$ LHS = RHS.

Example 18.5 : Evaluate: $\cos\left[\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13}\right]$



Solution: $\sin^{-1}\left(\frac{3}{5}\right) = \theta$ and $\sin^{-1}\left(\frac{5}{13}\right) = \phi$, then

$\sin\theta = \frac{3}{5}$ and $\sin\phi = \frac{5}{13}$.



$\Rightarrow \cos\theta = \frac{4}{5}$ and $\cos\phi = \frac{12}{13}$.

$\therefore$ The given expression becomes $\cos[\theta + \phi]$

$= \cos\theta\cos\phi - \sin\theta\sin\phi$

$(\because \cos(A+B) = \cos A\cos B - \sin A\sin B)$

$$= \frac{4}{5}\cdot\frac{12}{13} - \frac{3}{5}\cdot\frac{5}{13} = \frac{33}{65}.$$

Example 18.6 : Prove that

$$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{2}{9}\right).$$

Solution: Applying the formula :

$$\tan^{-1}(x)+\tan^{-1}(y) = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ we have}$$

$$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right)$$

$$= \tan^{-1}\left(\frac{20}{90}\right)=\tan^{-1}\left(\frac{2}{9}\right).$$

Example 18.7 : Prove that

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$$

Solution: Applying the property

$$\cos^{-1}x+\cos^{-1}y = \cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right), \text{ we have}$$

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{4}{5}\times\frac{12}{13}-\sqrt{1-\frac{16}{25}}\sqrt{1-\frac{144}{169}}\right)$$

$$= \cos^{-1}\left(\frac{33}{65}\right).$$

Example 18.8 : Prove that

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

Solution: Let $\sqrt{x} = \tan\theta$ then

$$\text{LHS} = \tan^{-1}\sqrt{x} = \tan^{-1}(\tan\theta) = \theta$$

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta)$$

$$= \frac{1}{2}\times 2\theta = \theta.$$

$\therefore$ LHS = RHS.

Example 18.9: Solve the equation

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x, \quad x > 0$$

Solution: Let $x = \tan\theta$, then

$$\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\tan^{-1}(\tan\theta)$$

$$\Rightarrow \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right) = \frac{1}{2}\theta.$$

$$\left(\because \tan(A-B) = \frac{\text{Tan A} - \text{Tan B}}{1+\text{Tan A Tan B}}\right)$$

$$\Rightarrow \frac{\pi}{4}-\theta = \frac{1}{2}\theta \quad \Rightarrow \frac{\pi}{4} = \frac{1}{2}\theta+\theta = \frac{3}{2}\theta.$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}\times\frac{2}{3} = \frac{\pi}{6}$$

$$\therefore\ x = \tan\theta \Rightarrow x = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

Example 18.10: Show that

$$\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

Solution: Let $x^2 = \cos 2\theta$, then

$$2\theta = \cos^{-1}(x^2)$$

$$\Rightarrow \theta = \frac{1}{2}\cos^{-1}x^2$$

Substituting $x^2 = \cos 2\theta$ in L.H.S. of the given equation, we have

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$$

By dividing with cos θ in both Numerator and Denominator.

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= \frac{\pi}{4}+\theta$$

$$= \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2).$$

= R.H.S.

EXERCISE 18.2

1. Evaluate each of the following :

(a) $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$ (b) $\cot(\tan^{-1}\alpha+\cot^{-1}\alpha)$

(c) $\tan\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right)$ (d) $\tan\left(2\tan^{-1}\frac{1}{5}\right)$

(e) $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$.

2. If $\cos^{-1}x+\cos^{-1}y=\beta$, prove that
$x^2-2xy\cos\beta+y^2=\sin^2\beta$

3. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, prove that
$x^2+y^2+z^2+2xyz=1$

4. Prove each of the following:

(a) $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)=\frac{\pi}{2}$,

(b) $\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}$.

(c) $\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\frac{3}{5}=\tan^{-1}\frac{27}{11}$

(d) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

5. Show that

$$\sec^2(\tan^{-1}2)+\mathrm{cosec}^2(\cot^{-1}2)=10$$

KEY WORDS

- Inverse of a trigonometric function exists if we restrict the domain of it.

(i) $\sin^{-1}x = y$ if $\sin y = x$; $-1 \le x \le 1$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

(ii) $\cos^{-1}x = y$ if $\cos y = x$; $-1 \le x \le 1$, $0 \le y \le \pi$,

(iii) $\tan^{-1}x = y$ if $\tan y = x$; $x \in R$, $\left(-\frac{\pi}{2} < y < \frac{\pi}{2}\right)$.

(iv) $\cot^{-1}x = y$ if $\cot y = x$; $x \in R$, $[0 < y < \pi]$

(v) $\sec^{-1}x = y$ if $\sec y = x$ where $x \ge 1$, $0 \le y < \frac{\pi}{2}$ or $x \le -1$, $\frac{\pi}{2} < y \le \pi$.

(vi) $\mathrm{cosec}^{-1}x = y$ if $\mathrm{cosec}\, y = x$ where $x \ge 1$, $0 < y \le \frac{\pi}{2}$

or $x \le -1$, $-\frac{\pi}{2} \le y < 0$.

- Graphs of inverse trigonometric functions can be represented in the given intervals by interchanging the axes as in case of $y = \sin x$, etc.

- **Properties :**

(i) $\sin^{-1}(\sin\theta) = \theta$, $\tan^{-1}(\tan\theta) = \theta$, an $(\tan^{-1}\theta) = \theta$ and $\sin(\sin^{-1}\theta) = \theta$.

(ii) $\mathrm{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$, $\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$, $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$.

(iii) $\sin^{-1}(-x) = -\sin^{-1}x$, $\tan^{-1}(-x) = -\tan^{-1}x$, $\cos^{-1}(-x) = \pi - \cos^{-1}x$.

(iv) $\sin^{-1}(x) + \cos^{-1}x = \frac{\pi}{2}$, $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, $\mathrm{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$.

(v) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

(vi) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(vii) $\sin^{-1}x = \cos^{-1}(\sqrt{1-x^2}) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right).$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right).$$

(viii) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right].$

(ix) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp x\sqrt{1-x^2}\sqrt{1-y^2}\right]$

SUPPORTIVE WEBSITES

http : //www.wikipedia.org

http:// math world . wolfram.com

PRACTICE EXERCISE

1. Prove each of the following :

(a) $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left(\frac{77}{85}\right)$

(b) $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{9}\right) = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$

(c) $\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{27}{11}\right)$

2. Prove each of the following :

(a) $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{23}{11}\right).$

(b) $\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}2.$

(c) $\tan^{-1}\left(\frac{1}{8}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}\left(\frac{1}{3}\right)$

3. (a) Prove that $2\sin^{-1}x=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

(b) Prove that $2\cos^{-1}x=\cos^{-1}(2x^2-1)$.

(c) Prove that $\cos^{-1}x=2\sin^{-1}\left(\sqrt{\frac{1-x}{2}}\right)=2\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$

4. Prove the following :

(a) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}$.

(b) $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)=\frac{\pi}{4}-x$.

(c) $\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)$ « 0

5. Prove the following :

(a) $\tan^{-1}2x+\tan^{-1}3x=\pi/4$

(b) $2\tan^{-1}(\cos x)=\tan^{-1}(2\text{cosec } x)$

(c) $\cos^{-1}x+\sin^{-1}\left(\frac{1}{2}x\right)=\frac{\pi}{6}$

(d) $\cot^{-1}x-\cot^{-1}(x+2)=\frac{\pi}{12}, x>0$

ANSWERS

EXERCISE 18.1

1. (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$ (c) $-\frac{\pi}{3}$

(d) $-\frac{\pi}{3}$ (e) $\frac{\pi}{4}$

2. (a) $\frac{1}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{1}{2}$ (d) 1

(e) -2

3. (a) $\sqrt{1+x^2}$ (b) $\frac{2}{\sqrt{x^2-4}}$

(c) $\sqrt{x^4-1}$ (d) $\frac{x^2}{\sqrt{x^4+1}}$

(e) $\sqrt{\frac{1-x}{x}}$

EXERCISE 18.2

1. (a) 1 (b) 0

(c) $\frac{x+y}{1-xy}$ (d) $\frac{5}{12}$

(e) $-\frac{7}{17}$

PRACTICE EXERCISE

5. (a) $\frac{1}{6}$ (b) $\frac{\pi}{4}$

(c) ± 1 (d) $\sqrt{3}$

Chapter 19

PROPERTIES OF TRIANGLES

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- derive sine formula, cosine formula and projection formula
- apply these formulae to solve problems.

PREREQUISITES

- Trigonometric and inverse trigonometric functions.
- Formulae for sum and difference of trigonometric functions of real numbers.
- Trigonometric functions of multiples and sub-multiples of real numbers.

INTRODUCTION

We have so far considered Trigonometry as a subject useful to study the trigonometric functions and their properties in a modern view point. But one of the main aims of learning Trigonometry is to determine the relation between the sides and angles of a given triangle.

The purpose of learning this chapter is to develop the necessary rules and methods for determining the rest of the sides and angles of a triangle, given one of two sides and/or angles.

Notation

In a ΔABC, the angles corresponding to the vertices A, B, and C are denoted by A, B, and C and the sides opposite to these vertices are denoted by a, b and c respectively. These angles and sides are called six elements of the triangle.

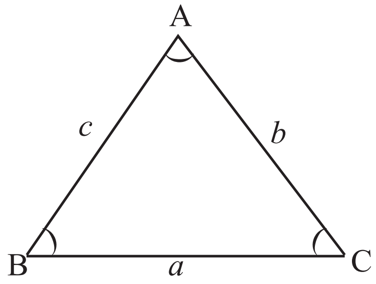


Fig. 19.1

19.1 SINE RULE

In ΔABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$,

where R is the circumradius.

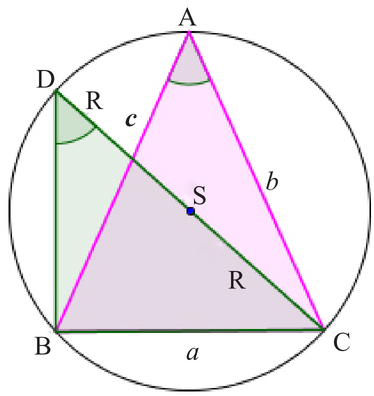


Fig. 19.2

Case (i) : $\angle A$ is acute (see Fig. 19.2).

S is the centre of the circumcircle and CD is its diameter.

Then CS = SD = R and CD = 2R. Join BD.

Then $\angle DBC = \dfrac{\pi}{2}$ and ΔDBC is a right angled triangle.

Then $\angle BAC = \angle BDC$, ($\because$ angles in the same segment)

$\therefore \sin A = \sin \angle BAC = \sin \angle BDC$

$$= \frac{BC}{CD} = \frac{a}{2R}.$$

$$\therefore \frac{a}{\sin A} = 2R.$$

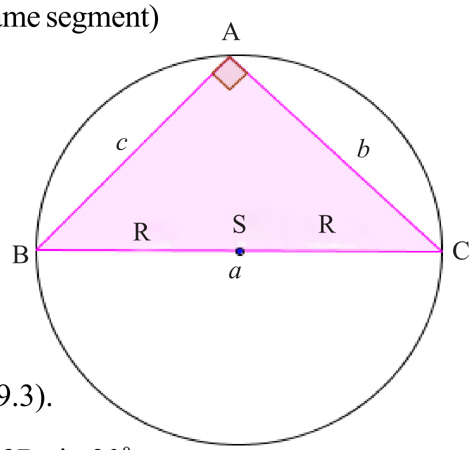


Fig. 19.3

Case (ii) : $\angle A$ is a right angle (see Fig. 19.3).

Then $BC = a = 2R = 2R.1 = 2R \sin 90^0$

$\therefore a = 2R \sin A$. Hence $\dfrac{a}{\sin A} = 2R$.

Case (iii) : $\underline{|A}$ is obtuse (see Fig. 19.4).

$\underline{|DBC}$ is right angle. ($\because$ angle in the semi circle)

In the cyclic quadrilateral BACD,

$$\underline{|BDC} = 180^0 - \underline{|BAC} = 180^0 - A$$

In ΔBDC, $\sin A = \sin(180^0 - A)$

$$= \sin \underline{|BDC} = \frac{BC}{CD} = \frac{a}{2R}.$$

Hence $\dfrac{a}{\sin A} = 2R$.

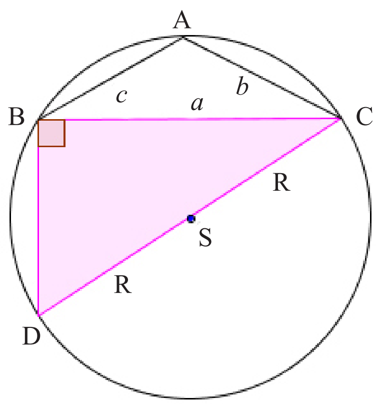


Fig. 19.4

In a similar way, we can prove

$$\frac{b}{\sin B} = 2R, \quad \frac{c}{\sin C} = 2R$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

$$a = 2R \sin A, \quad b = 2R \sin B, \quad c = 2R \sin C$$

(or) $\sin A = \dfrac{a}{2R}, \ \sin B = \dfrac{b}{2R}, \ \sin C = \dfrac{c}{2R}$

Let us take some examples :

Example 19.1: Prove that $a \cos\dfrac{B-C}{2} = (b+c)\sin\dfrac{A}{2}$, using sine-formula.

Solution: R.H.S. $= (b+c)\sin\dfrac{A}{2}$

We know that, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (say)

$\Rightarrow$ $a = k \sin A, \ b = k \sin B, \ c = k \sin C$

$\therefore$ RHS : $k(\sin B + \sin C) \cdot \sin\dfrac{A}{2}$

$$= k.2\sin\frac{B+C}{2}.\cos\frac{B-C}{2}\cdot\sin\frac{A}{2}$$

Now $\dfrac{B+C}{2} = 90 - \dfrac{A}{2}$ $\qquad$ ($\because$ $A + B + C = \pi$

$$\therefore \sin\frac{B+C}{2} = \cos\frac{A}{2} \qquad \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}=90^0)$$

$$\therefore \text{RHS} = 2k\cos\frac{A}{2}\cos\frac{B-C}{2}\cdot\sin\frac{A}{2}$$

$$= k.\sin A.\cos\frac{B-C}{2}$$

$$= a\cos\frac{B-C}{2} = \text{L.H.S}$$

Example 19.2 : Using sine formula, prove that

$$a(\cos C - \cos B) = 2(b-c)\cos^2\frac{A}{2}.$$

Solution: We have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$$a = k\sin A, \quad b = k\sin B, \quad c = k\sin C$$

$$\therefore \quad \text{RHS: } 2k(\sin B - \sin C).\cos^2\frac{A}{2}$$

$$= 2k\left[2\cos\frac{B+C}{2}.\sin\frac{B-C}{2}\right].\cos^2\frac{A}{2}$$

$$= 4k\sin\frac{A}{2}.\sin\frac{B-C}{2}.\cos^2\frac{A}{2}$$

$$= 2a\sin\frac{B-C}{2}.\cos\frac{A}{2}$$

$$= 2a\sin\frac{B+C}{2}.\sin\frac{B-C}{2} \qquad \left(\cos\frac{A}{2} = \sin\left(\frac{B+C}{2}\right)\right)$$

$$= a(\cos C - \cos B)$$

$$= \text{L.H.S.}$$

Example 19.3: Using sine formula, prove that

$$a\sin A - b\sin B = c\sin(A-B)$$

Solution: We have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

L.H.S. $= k \sin A \, . \, \sin A - k \sin B . \; \sin B$

$$= k\,[\sin^2 A - \sin^2 B]$$

$$= k \sin(A + B) \, . \, \sin(A - B)$$

$A + B = \pi - C \Rightarrow \sin(A + B) = \sin C$

$\therefore$ L.H.S $= k \sin C \sin(A - B)$

$$= c \sin(A - B) = \text{R.H.S.} \qquad (\because \; k \sin C = c)$$

Example 19.4 : In any triangle, show that

$$a(b \cos C - \cos B) = b^2 - c^2.$$

Solution: We have, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say)

L.H.S : $k \sin A\,[k \sin B \cos C - k \sin C \, . \, \cos B]$

$$= k^2 \sin A \, . \, [\sin(B - C)]$$

$$= k^2 \sin(B+C) \, . \, \sin(B - C) \qquad [\because \; \sin A = \sin(B + C)]$$

$$= k^2 (\sin^2 B - \sin^2 C)$$

$$= k^2 \sin^2 B - k^2 \sin^2 C$$

$$= b^2 - c^2 = \text{R.H.S}$$

EXERCISE 19.1

1. Using sine-formula, show that each of the following hold :

(i) $\dfrac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}} = \dfrac{a-b}{a+b}$

(ii) $b \cos B + c \cos C = a \cos(B - C)$

(iii) $a \sin \dfrac{B-C}{2} = (b-c) \cos \dfrac{A}{2}$

(iv) $\dfrac{b+c}{b-c} = \tan\dfrac{B+C}{2} \cdot \cot\dfrac{B-C}{2}$

(v) $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C$

2. In any triangle if $\dfrac{a}{\cos A} = \dfrac{b}{\cos B}$ prove that the tiangle is isosceles.

19.2 COSINE RULE

We shall now derive the cosine rule connecting the sides a, b, c of ΔABC with the cosines of its angles A, B, C.

$$\text{In } \Delta ABC, \; b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a^2 = (2R\sin A)^2$$
$$= 4R^2[\sin(B+C)]^2$$
$$= 4R^2(\sin B\cos C + \cos B\sin C)^2 \qquad (\because \sin A = \sin(B+C))$$
$$= 4R^2\{\sin^2 B(1-\sin^2 C) + \sin^2 C(1-\sin^2 B) + 2\sin B\sin C\cos B\cos C\}$$
$$= 4R^2\{\sin^2 B + \sin^2 C + 2\sin B\sin C(\cos B\cos C - \sin B\sin C)\}$$
$$= 4R^2\{\sin^2 B + \sin^2 C + 2\sin B\sin C\cos(B+C)\}$$
$$= b^2 + c^2 - 2bc\cos A.$$

The proofs of the other two results are similar.

Note

(i) The rule is also known as the '**law of cosines**' and the rules in it are called '**cosine rules**'.

(ii) From the cosine rules, we can write $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$,

$\cos B = \dfrac{c^2+a^2-b^2}{2ca}$ and $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$.

These rules are used to find the three angles of a triangle when its sides are given.

19.3 PROJECTION FORMULA

In ΔABC, $a = b \cos C + c \cos B$.

Proof: From the cosine rules, we have

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore b\cos C + c \cos B = b\left(\frac{a^2 + b^2 - c^2}{2ab}\right) + c\left(\frac{c^2 + a^2 - b^2}{2ca}\right)$$

$$= \frac{a^2 + b^2 - c^2 + c^2 + a^2 - b^2}{2a} = \frac{2a^2}{2a} = a.$$

Similarly, we can prove that $b = c\cos A + a\cos C$ and $c = a\cos B + b\cos A$

Note: These three rules are called the '**projection rules**'.

Let us take some examples to show its application

Example 19.5 : In any triangle ABC, show that

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

Solution:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\text{L.H.S : } \frac{b^2 + c^2 - a^2}{2abc} + \frac{c^2 + a^2 - b^2}{2abc} + \frac{a^2 + b^2 - c^2}{2abc}$$

$$= \frac{1}{2abc}[b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2]$$

$$= \frac{a^2 + b^2 + c^2}{2abc} = \text{R.H.S}$$

Example 19.6 : If $\underline{|A} = 60^0$, show that in ΔABC

$$(a + b + c)(b + c - a) = 3bc$$

Solution: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ ...(i)

$$A = 60^0 \Rightarrow \cos A = \cos 60^0 = \frac{1}{2}$$

$\therefore$ (i) becomes $\frac{1}{2} = \frac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow$ $b^2 + c^2 - a^2 = bc$

or $b^2 + c^2 + 2bc - a^2 = 3bc$

or $(b + c)^2 - a^2 = 3bc$

or $(b + c + a)(b + c - a) = 3bc$ (by adding $2bc$ on both sides)

Example 19.7 : If the sides of a triangle are 3 cm, 5 cm and 7 cm find the greatest angle of a triangle.

Solution: Here $a = 3$ cm, $b = 5$ cm, $c = 7$ cm

We know that in a triangle, the angle opposite to the largest side is greatest

$\therefore$ $\angle C$ is the greatest angle.

$$\therefore \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{9 + 25 - 49}{30} = -\frac{15}{30} = -\frac{1}{2}$$

$$\therefore \quad \cos C = -\frac{1}{2} \Rightarrow C = \frac{2\pi}{3}$$

$\therefore$ The greatest angle of the triangle is $\frac{2\pi}{3}$ or 120^0.

Example 19.8 : In ΔABC if $\angle A = 60^0$. Prove that $\frac{b}{c+a} + \frac{c}{a+b} = 1$.

Solution : $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

or $\cos 60^0 = \frac{1}{2} = \frac{b^2 + c^2 - a^2}{2bc}$

$\therefore$ $b^2 + c^2 - a^2 = bc$

or $b^2 + c^2 = a^2 + bc$ ...(i)

R.H.S : $\frac{b}{c+a}+\frac{c}{a+b}=\frac{ab+b^2+c^2+ac}{(c+a)(a+b)}$

$=\frac{ab+ac+a^2+bc}{(c+a)(a+b)}$ [Using(i)]

$=\frac{a(a+b)+c(a+b)}{(c+a)(a+b)}$

$=\frac{(a+b)+(a+c)}{(c+a)(a+b)}=1$

EXERCISE 19.2

1. In any triangle ABC, show that

(i) $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0$

(ii) $(a^2-b^2+c^2)\tan B=(b^2-c^2+a^2)\tan C=(c^2-a^2+b^2)\tan A$

(iii) $\frac{k}{2}[\sin 2A+\sin 2B+\sin 2C]=\frac{a^2+b^2+c^2}{2abc}$

where $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$

(iv) $(b^2-c^2)\cot A=(c^2-a^2)\cot B=(a^2-b^2)\cot C=0$

2. The sides of a triangle are $a=9$ cm, $b=8$ cm, $c=4$ cm Show that

$6\cos C=4+3\cos B.$

Example 19.9 : In any triangle ABC, show that

$(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$

Solution: L.H.S : $b\cos A+c\cos A+c\cos B+a\cos B+a\cos C+b\cos C$

$=(b\cos A+a\cos B)+(c\cos A+a\cos C)+(c\cos B+b\cos C)$

$=c+b+a$

$=a+b+c=$ R.H.S

Example 19.10 : In any ΔABC, prove that

$$\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Solution: L.H.S : $\frac{1-2\sin^2 A}{a^2} - \frac{1-2\sin^2 B}{b^2}$

$$= \frac{1}{a^2} - \frac{2\sin^2 A}{a^2} - \frac{1}{b^2} + \frac{2\sin^2 B}{b^2}$$

$$= \frac{1}{a^2} - \frac{1}{b^2} - 2k^2 + 2k^2 = \frac{1}{a^2} - \frac{1}{b^2} \quad \left(\because \frac{\sin A}{a} = \frac{\sin B}{b} = k\right)$$

= R.H.S

Example 19.11: In Δ ABC, if a cos A = b cos B, where $a \neq b$ prove that Δ ABC is a right angled triangle.

Solution: a cos A = b cos B

$$a\left[\frac{b^2+c^2-a^2}{2bc}\right] = b\left[\frac{c^2+a^2-b^2}{2ca}\right]$$

or $\quad a^2[b^2 + c^2 - a^2] = b^2\,[a^2 + c^2 - b^2]$

or $\quad a^2b^2 + a^2c^2 - a^4 = a^2b^2 + b^2c^2 - b^4$

or $\quad c^2(a^2 - b^2) = (a^2 - b^2)\ (a^2 + b^2)$

$\Rightarrow \quad c^2 = a^2 + b^2$

$\therefore$ Δ ABC •is a right triangle.

Example 19.12 : If $a = 2$, $b = 3$, $c = 4$, find cos A, cos B and cos C.

Solution: $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{9+16-4}{2\times3\times4} = \frac{21}{24} = \frac{7}{8}$

$$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{16+4-9}{2\times4\times2} = \frac{11}{16}$$

and $\quad \cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{4+9-16}{2\times2\times3} = -\frac{3}{12} = -\frac{1}{4}$

EXERCISE 19.3

1. If $a = 3$, $b = 4$ and $c = 5$ find cos A, cos B and cos C.
2. The sides of a triangle are $7\,\text{cm}$, $4\sqrt{3}\,\text{cm}$ and $\sqrt{13}\,\text{cm}$. Find the smallest angle of the triangle.
3. If $a : b : c = 7 : 8 : 9$, prove that

   cos A : cos B : cos C = 14 : 11 : 6.
4. If the sides of a triangle are $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. Show that the greatest angle of the triangle is 120°.
5. In a triangle, b cos A = a cos B, prove that the triangle is isosceles.
6. Deduce sine formula from the projection formula.

19.4 HALF ANGLE FORMULAE AND AREA OF A TRIANGLE

We have learnt in elementary geometry that, if the base '*b*' and the altitude '*h*' are given, then the area of triangle, denoted by $\Delta = \frac{1}{2}bh$

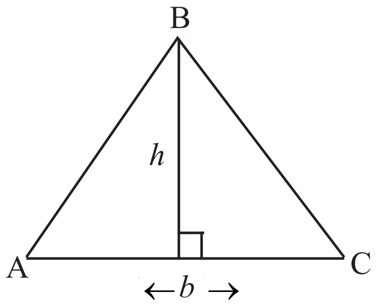


Fig. 19.5

However, if the 3 sides of a triangle *a*, *b* and *c* are given, then the area of triangle $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Where $S = \frac{a+b+c}{2}$ (half of the perimeter)

In ΔABC,

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

Similarly $\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$ and $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

Similarly $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$ and $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Similarly $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$ and $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Let us take some examples, to show the applications of above results.

Example 19.13 : If the sides of a triangle are 13, 14, 15 then find area of that triangle.

Let $a = 13,\ b = 14,\ c = 15.$

Solution : Area of the triangle $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

where $S = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$

$\therefore \quad \Delta = \sqrt{21(21-13)(21-14)(21-15)}$

$= \sqrt{21\times 8\times 7\times 6}$

$= 84$ sq.units.

Example 19.14 : In ΔABC, if $(a + b + c)\ (b + c - a) = 3bc$, find A.

Solution: We know that, $a + b + c = 2s$ and $b + c = 2s - a$.

$(\underline{a + b + c})\ (\underline{b + c} - a) = 3bc$

$\Rightarrow \quad 2s(2s - a - a) = 3bc$

$\Rightarrow \quad 2s(2s - 2a) = 3bc$

$\Rightarrow \quad 4s(s - a) = 3bc$

$$\Rightarrow \quad \frac{s(s-a)}{bc} = \frac{3}{4}$$

$$\therefore \cos^2\frac{A}{2} = \frac{3}{4} \qquad \left(\therefore \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}\right)$$

$$\Rightarrow \cos\frac{A}{2} = \frac{\sqrt{3}}{2} = \cos 30^0.$$

$$\therefore \frac{A}{2} = 30^0 \Rightarrow A = 60^0$$

EXERCISE 19.3

1. If $a = 4$, $b = 5$, $c = 7$, find $\cos\frac{B}{2}$.

2. If $\tan\frac{A}{2} = \frac{5}{6}$ and $\tan\frac{C}{2} = \frac{2}{5}$, determine the relation between a, b, c.

3. If $\cot\frac{A}{2} = \frac{b+c}{a}$, find angle B.

KEY WORDS

- It is possible to find out the unknown elements of a triangle, if the relevent elements are given by using

  Sine-formula :

  (i) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

  Cosine formulae :

  (ii) $\cos A = \frac{b^2+c^2-a^2}{2bc}$,

  $\cos B = \frac{c^2+a^2-b^2}{2ca}$,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Projection formulae :

$a = b \cos C + c \cos B$

$a = c \cos A + a \cos C$

$a = a \cos B + b \cos A$

Half Angle formulae :

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Area of Triangle :

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

where $S = \dfrac{a+b+c}{2}$

SUPPORTIVE WEB SITES

- http : //www.wikipedia.org
- http:// math world . wolfram.com

PRACTICE EXERCISE

In a triangle ABC, prove the following (1-10) :

1. $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0$

2. $a \cos A + b \cos B + c \cos C = 2a \sin B \sin C$

3. $\frac{b^2-c^2}{a^2}.\sin 2A+\frac{c^2-a^2}{b^2}.\sin 2B+\frac{a^2-b^2}{c^2}.\sin 2C=0$

4. $\frac{c^2+a^2}{b^2+c^2}=\frac{1+\cos B\cos(C-A)}{1+\cos A\cos(B-C)}$

5. $\frac{c-b\cos A}{b-c\cos A}=\frac{\cos B}{\cos C}$

6. $\frac{a-b\cos C}{c-b\cos A}=\frac{\sin C}{\sin A}$

7. $(a+b+c)\left[\tan\frac{A}{2}+\tan\frac{B}{2}\right]=2c\cot\frac{C}{2}$

8. $\sin\left(\frac{A-B}{2}\right)=\frac{a-b}{c}\cos\frac{C}{2}$

9. (i) $b\cos B+c\cos C=a\cos(B-C)$

(ii) $a\cos A+b\cos B=c\cos(A-B)$

10. $b^2=(c-a)^2\cos^2\frac{B}{2}+(c+a)^2\sin^2\frac{B}{2}$

11. In a triangle, if $b=5$, $c=6$ $\tan\frac{A}{2}=\frac{1}{\sqrt{2}}$, then show that $a=\sqrt{41}$.

12. In any ΔABC, show that

$$\frac{\cos A}{\cos B}=\frac{b-a\cos C}{a-b\cos C}$$

13. If $\cot\frac{A}{2}:\cot\frac{B}{2}:\cot\frac{C}{2}=3:5:7$, then show that $a:b:c=6:5:4$.

14. Prove that $\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4\Delta}$

15. Express $a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}$ interms of s, a, b, c.

ANSWERS

EXERCISE 19.3

1. $\cos A = \frac{4}{5}$

   $\cos B = \frac{3}{5}$

   $\cos C = 0$

2. The smallest angle of the triangle is 30^0

EXERCISE 19.4

1. $\sqrt{\frac{6}{7}}$

2. *a*, *b*, *c* are in Arithmetic Progression (AP)

3. 90^0

Chapter 20

LIMITS AND CONTINUITY

LEARNING OUTCOMES

After studying this lesson, student will be able to

- Define limit of a function, Right hand limit and Left hand limit.
- Use standard limits and L' - hospital's rule evaluate limits.
- Define the continuity of a function at a point and in an interval.
- Interprete geometrically the continuity of a function at a point

PREREQUISITES

- Relations, functions, trigonometric functions, exponential functions and logarithemic functions.

INTRODUCTION

Calculus can be considered as the subject that studies the problems of change. This mathematical discipline stems from the 17th century investigations of Isaac Newton (1642 - 1727) and Gottfried Leibnitz (1646 - 1716) and today its stands "Language of Science and Technology".

Basic Notion of a 'Limit' was conceived in 1680 by Newton and Leibnitz simultaneously, while they were facing with the creation of calculus. There were other mathematicians of the same era who proposed other definitions of the intuitive concept of Limit of Course there are evidences that the idea of 'Limit' was first known to "Archemedes" (287 - 212 B.C.)

It is Angustin - Louis (1789 - 1857) who formulated the definition and presented the arguments with greater care than his predecessors in his monumental work. 'Coursed Analyse'. But the concept of Limit Still remained elusive.

The definition of limit today, was given by Karl Weierstrass (1815 - 1897)

Intervals and neighbourhoods

Which are very much useful in studying Limits and Continuity.

Intervals

Let $a, b \in \mathbf{R}$ such that $a \leq b$. Then the set

(i) $\{x \in \mathbf{R} : a \leq x \leq b\}$, denoted by $[a, b]$ is called a closed interval.

(ii) $\{x \in \mathbf{R} : a < x < b\}$, denoted by (a, b) is called an Open Interval.

In similar way

(iii) $(a, b] = \{x \in \mathbf{R} : a < x \leq b\}$ open closed interval

(iv) $[a, b) = \{x \in \mathbf{R} : a \leq x < b\}$ closed open interva.

The intervals are said to be intervals of finite length $b - a$.

Neighbourhoods :

Let a be a real number and let δ be a Positive real number. Then set of all real numbers tying between $a - \delta$ and $a + \delta$ is called the neighbourhood of a radius 'δ' and is denoted by $N_\delta(a)$.

Thus

$$N_\delta(a) = (a - \delta\ ,\ a + \delta) = \{x, \mathbf{R} \setminus a - \delta < x < a + \delta\}$$

The set $N_\delta(a) - \{a\}$ is called deleted nbd of a of radius δ. The set $(a - \delta, a)$ is called the left nbd of a and the $(a, a + \delta)$ is known as the right nbd of a.

If δ is very small and x lies in the interval $(a - \delta, a)$. Then x is said to approach to a from the left and we write $x \to a^-$.

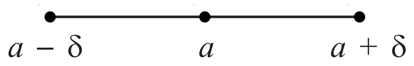
If x, $(a, a + \delta)$, then approach to a from the right which is denoted by $x \to a^+$.

Consider the statement $|x - a| < \delta$. We have

$$|x - a| < \delta \Leftrightarrow -\delta < x - a < \delta$$

$$\Leftrightarrow a - \delta| < x < a + \delta \Leftrightarrow x.\ N_\delta(a)$$

Thus $|x - a| < \delta$ means that x lies in the nbd of 'a' of radius δ as shown in fig.



$(a - \delta, a) \cup (a, a + \delta)$ or $(a - \delta, a + \delta) \setminus \{a\}$

Note : 1) Any interval (c, d) is a neighbourhood of some $a \in (c, d)$, infact, take

$$a = \frac{c+d}{2} \quad \text{and} \quad \delta = \frac{d-c}{2} > 0$$

Then $(a - \delta, a + \delta) = \left[\frac{c+d}{2} - \frac{d-c}{2}, \frac{c+d}{2} + \frac{d-c}{2}\right)$

$$= (c, d)$$

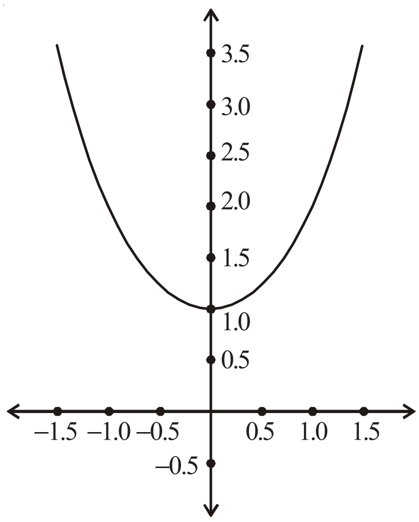
Therefore (c, d) is the δ - neighbourhood of a.

MODULE - V
Calculus

Notes

20.1 LIMITS

1. Consider the function $f(x) = x^2 + 1$, $x \in \mathbf{R}$. Here we observe that as x takes values very close to 'O'. The value of $f(x)$ approaches 1.



In this case, we say that $f(x)$ tends 1 as x tends to 'O' and we write it as

$$\underset{x \to 0}{\text{Lt}}\ f(x) = 1$$

That is limit of $f(x)$ is 1 as x tends to '0'.

2. Let us define $f : (\mathbf{R} \setminus \{1\}) \to \mathbf{R}$ by $f(x) = \dfrac{x^2 - 1}{x - 1}$, $x \neq 1$ in the following table, we compute the values of $f(x)$ for certain values on either. Side of $x = 1$

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1

We can werite $f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x+1)(x-1)}{x-1} = x + 1$, beacuse $x - 1 \neq 0$ and so division by $(x - 1)$ is possible.

We see that as x closer to 1, the corresponding value of $f(x)$ also get closer to z, However in this case $f(x)$ is not defined at $x = 1$. The idea can be expressed by saying that the limiting value of $f(x)$ is 2 when x approaches to 1.

3. Let $f : (\mathbf{R} \setminus \{2\}) \to \mathbf{R}$ be defined by

$$f(x) = \frac{x^2 + 3x - 10}{x - 2}$$

Here is a table of values of x near 2 and corresponding $f(x)$.

x	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1
$f(x)$	6.9	6.99	6.999	6.9999	7.0001	7.001	7.01	7.1

Though f is not defined at 2, but $f(x)$ is approaching to 7 as 'x' is nearing to 2 the same can be seen in above table.

4. Find $\underset{x\to 3}{\text{Lt}} f(x)$, where $f(x) = \dfrac{x^2-9}{x-3}$ solve by substitution method.

Step 1: x close to a say $x = a + h$ h is very small +ve number clearly as $x \to a$, $h \to 0$	For $f(x) = \dfrac{x^2-9}{x-3}$ we write $x = 3 + h$ so that as $x \to 3$, $h \to 0$
Step:2 Simpllify $f(x) = f(a + h)$	Now $f(x) = f(3 + h)$ $= \dfrac{(3+h)^2-9}{3+h-3}$ $= \dfrac{h^2+6h}{h}$ $= h + 6$
Step 3 : Put $h = 0$ and get the required result	$\underset{x\to 3}{\text{Lt}} f(x) = \underset{h\to 0}{\text{Lt}}(6+h)$ As $x \to 0$, $h \to 0$ Thus $\underset{x\to 3}{\text{Lt}} f(x) = 6+0=6$ by putting $h = 0$.

Remark : $f(3)$ is not defined, however in this case the limit of the function $f(x)$ as $x \to 3$ is 6

Discuss other methods of finding limits of different types of function.

Example 1 : $\underset{x\to 1}{\text{Lt}}\ f(x)$, where $f(x) = \begin{cases} \dfrac{x^3-1}{x^2-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

Here, for $x \neq 1$, $f(x) = \dfrac{x^3-1}{x^2-1}$

$$= \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$$

Solve by Method of factors.

Step 1: Factorise $g(x)$ and $h(x)$	**Sol.** $f(x) = \dfrac{x^3-1}{x^2-1}$ $= \dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$ ($\because x \neq 1$, $x - 1 \neq 0$ and as such can be cancelled)
Step 2 : $f(x) = \dfrac{x^2+x+1}{x+1}$	Simplify $f(x)$
Step 3: Putting the value of x, we get the required limit.	$\therefore \underset{x\to 1}{\text{Lt}} \dfrac{x^3-1}{x^2-1} = \dfrac{1+1+1}{1+1} = \dfrac{3}{2}$ Also $f(1) = 1$ (given) In this case $\underset{x\to 1}{\text{Lt}}\ f(x) \neq f(1)$

20.1.1 Definition of the Limit

Let E $\subseteq$ **R** and f : E$\to$**R**. Let $a \in$ **R** be such that $((a - r, a + r) \setminus \{a\}) \cap$ E is non empty for every $r > 0$, if there exists a real number l satisfying the condition below then l is aid to be a limit of f at a.

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - l| < \varepsilon$. Whenever $x \in E$ and $0 < |x - a| < \delta$, in this case, we say that limit of the function $f(x)$ as x tends to 'a' exists and it is 'l' and we write it as.

$\underset{x \to a}{\text{Lt}} f(x) = l$ or $f(x) \to l$ as $x \to a$

If such an l does not exist, we say that $\underset{x \to a}{\text{Lt}} f(x)$ does not exist.

20.1.2 Right and Left Hand Limits

We studied the limit of function f at a given point $x = a$ as the approaching value of $f(x)$ when x tends to 'a' there are two ways x could approach 'a' either from the left of 'a' or from the right of 'a' this naturally to two limits namely the 'right hand limit' and the 'left hand limit' we denote the right hand limit of 'f' at 'a' by $\underset{x \to a^+}{\text{Lt}} f(x)$. Similarly the left hand limit of f at 'a' by $\underset{x \to a^-}{\text{Lt}} f(x)$.

Therefore $x \to a^-$ is equivalent to $x = a - h$ where $h > 0 \ni h \to 0$

Similarly $x \to a^+$ is equivalent to $x = a + h$ where $h \to 0$.

Definition (Right and left Limits)

Let $E \subseteq R$ and let $f : E \to R$.

(i) Suppose $a \in R$ is $E \cap (a, a + r)$ is non empty for every $r > 0$. We say that $l \in R$ is a right hand limit of f at 'a' and we write $\underset{x \to a^+}{\text{Lt}} f(x) = l$ if given $\varepsilon > 0 \ \exists \ a, \ \delta > 0 \ni |f(x) - l| < \varepsilon$ whenever $0 < x - a < \delta$ and $x \in E$.

(ii) Suppose $a \in R$ is such that $E \cap (a - r, a)$ is non empty for every $r > 0$. We say that $m \in R$ is a left hand limit of f at 'a' and we write $\underset{x \to a^-}{\text{Lt}} f(x) = m$ if $\ni |f(x) - m| < \varepsilon$ whenever $0 < a - x < \delta$ and $x \in E$.

Note :

I $\lim_{x\to a^+} f(x) = \ell$ and $\lim_{x\to a^-} f(x) = \ell$ $\Rightarrow \lim_{x\to a} f(x) = \ell$

II $\lim_{x\to a^+} f(x) = \ell_1$ and $\lim_{x\to a^-} f(x) = \ell_2$ $\Rightarrow \lim_{x\to a} f(x)$ does not exist

III $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ does not exist $\Rightarrow \lim_{x\to a} f(x)$ does not exist

20.2 BASIC THEOREMS ON LIMIT

1. $\lim_{x\to a} cx = c \lim_{x\to a} x$, c being a constant.

   To verify this, consider the function $f(x) = 5x$.

   We observe that in $\lim_{x\to 2} 5x$, 5 being a constant is not affected by the limit.

   $\therefore \lim_{x\to 2} 5x = 5 \lim_{x\to 2} x$

   $= 5 \times 2 = 10$

2. $\lim_{x\to a}[g(x) + h(x) + p(x) + ...] = \lim_{x\to a} h(x) + \lim_{x\to a} p(x) + ...$

   where $g(x)$, $h(x)$, $p(x)$, ... are any function.

3. $\lim_{x\to a} [f(x).g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

   To verify this, consider $f(x) = 5x^2 + 2x + 3$

   and $g(x) = x + 2$.

   Then $\lim_{x\to 0} f(x) = \lim_{x\to 0} (5x^2 + 2x + 3)$

$$= 5\lim_{x\to 0} x^2 + 2\lim_{x\to 0} x + 2 = 2$$

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} (x+2) = \lim_{x\to a} x + 2 = 2$$

$$\therefore \lim_{x\to 0} (5x^2+2x+3) = \lim_{x\to 0} (x+2) = 6$$

Again $\lim_{x\to 0} [f(x).g(x)] = \lim_{x\to 0} [(5x^2+2x+3)(x+2)]$

$$= \lim_{x\to 0} (5x^3+12x^2+7x+6)$$

$$= 5\lim_{x\to 0} x^3 + 12\lim_{x\to 0} x^2 + 7\lim_{x\to 0} x + 6$$

$$= 6$$

From (i) and (ii)

$$\lim_{x\to 0} [(5x^2+2x+3)(x+2)] = \lim_{x\to 0} (5x^2+2x+3) \lim_{x\to 0} (x+2)$$

4. $\lim_{x\to a}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$

To verify this, consider the function $\lim_{x\to a} g(x) \neq 0$

we have $\lim_{x\to -1} (x^2+5x+6) = (-1)^2 + 5(-1) + 6$

$$= 1 - 5 = +6$$

$$= 2$$

and $\lim_{x\to -1} (x+2) = -1+2$

$$= 1$$

$$\frac{\lim_{x\to -1} (x^2+5x+6)}{\lim_{x\to -1} (x+2)} = \frac{2}{1} = 2$$

Also $\lim\limits_{x\to 1-}\dfrac{(x^2+5x+6)}{x+2} = \lim\limits_{x\to -1}\dfrac{(x+3)(x+2)}{x+2}$ $\left[\begin{array}{l}\because x^2+5x+6\\ =x^2+3x+2x+6\\ =x(x+3)+2(x+3)\\ =(x+3)(x+2)\end{array}\right]$

$\lim\limits_{x\to -1}(x+3)$

$= -1 + 3 = 2$

From (i) and (ii) $\lim\limits_{x\to -1}\dfrac{x^2+5x+6}{x+2}\ \dfrac{\lim\limits_{x\to -1}(x^2+5x+6)}{\lim\limits_{x\to -1}(x+2)}$

We have seen above that there are many ways that two given functions may be combined to form a new function. The limit of the combined function as $x\to a$ can be calculated from the limits of the given functions. To sum up, we state below some basic results on limits, which can be used to find the limit of the functions combined with basic operations.

If $\lim\limits_{x\to a} f(x) = \ell$ and $\lim\limits_{x\to a} g(x) = m$, then

(i) $\lim\limits_{x\to a} k\,f(x) = k\lim\limits_{x\to a} f(x) = k\ell$ where k is a constant.

(ii) $\lim\limits_{x\to a}[f(x)\pm g(x)] = \lim\limits_{x\to a} f(x)\pm \lim\limits_{x\to a} g(x)\ \ell+m$

(iii) $\lim\limits_{x\to a}[f(x)\cdot g(x)] = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x)\ \ell\cdot m$

(iv) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{\ell}{m}$ provided $\lim\limits_{x\to a} g(x)\neq 0$

The above results can be easily extended in case of more than two functions.

Example 20.1 : $f(x) = \begin{array}{l}1-x \text{ if } x\le 1\\ 1+x \text{ if } x>1\end{array}$; $a=1$

Solution : LHL $= \underset{x\to a^-}{\text{Lt}} f(x) = \underset{x\to 1^-}{\text{Lt}} f(x)$

$= \underset{x\to 1}{\text{Lt}}(1-x) = 0$

RHL $= \underset{x\to a^+}{\text{Lt}} f(x) = \underset{x\to 1^+}{\text{Lt}} f(x)$

$= \underset{x\to 1^+}{\text{Lt}} (1+x) = 1+2 = 2$

$\therefore$ LHL $\neq$ RHL $\underset{x\to 1}{\text{Lt}} f(x)$ does not exist.

Example 20.2 : $f(x) = \begin{cases} x+2 & \text{if } -1 < x \le 3 \\ x^2 & \text{if } 3 < x < 5 \end{cases}$; $a = 3$. Find LHL, RHL at a point mentioned against them.

LHL $= \underset{x\to 3^-}{\text{Lt}} f(x) = \underset{x\to 3}{\text{Lt}}\, x+2 = 3+2 = 5$

RHL $= \underset{x\to 3^+}{\text{Lt}} f(x) = \underset{x\to 3}{\text{Lt}}\, x^2 = 3^2 = 9$

$\therefore$ LHL $\neq$ RHL $\Rightarrow \underset{x\to 3}{\text{Lt}} f(x)$ does not exist.

Example 20.3 : Evaluate LHL, RHL

(i) $\underset{x\to 3^+}{\text{Lt}} \dfrac{|x-3|}{x-3}$

here $\underset{x\to 3^+}{\text{Lt}} \dfrac{|x-3|}{x-3} = \underset{h\to 0}{\text{Lt}} \dfrac{|(3+h)-3|}{[(3+h)-3]}$

$= \underset{h\to 0}{\text{Lt}} \dfrac{|h|}{h}$

$= \underset{h\to 0}{\text{Lt}} \dfrac{h}{h}$ (as $h > 0$ so $|h| = h$)

$= 1$

(ii) $\underset{x\to 3^-}{\text{Lt}} \dfrac{|x-3|}{x-3} = \underset{h\to 0}{\text{Lt}} \dfrac{|(3-h)-3|}{[(3-h)-3]} = \underset{h\to 0}{\text{Lt}} \dfrac{|-h|}{h}$

$= \underset{h\to 0}{\text{Lt}} \dfrac{h}{-h} = -1$

From (i), (ii) $\underset{x\to 3^+}{\text{Lt}} \dfrac{|x-3|}{x-3} \neq \underset{x\to 3^-}{\text{Lt}} \dfrac{|x-3|}{x-3}$

Example 20.4 : If a function $f(x)$ is defined as

$$f(x) = \begin{cases} x, & x \le x < \frac{1}{2} \\ 0, & x = \frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 \end{cases}$$

Examine the existence of $\underset{x\to\frac{1}{2}}{\text{Lt}} f(x)$

Solution : Here $f(x) = \begin{cases} x, & 0 \le x < \frac{1}{2} & \text{...(i)} \\ 0, & x = \frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1 & \text{...(ii)} \end{cases}$

$$\underset{x\to\left(\frac{1}{2}\right)^-}{\text{Lt}} f(x) = \underset{h\to 0}{\text{Lt}}\, f\left(\frac{1}{2}-h\right)$$

$$= \underset{x\to 0}{\text{Lt}}\left(\frac{1}{2}-h\right) \qquad \left[\because \frac{1}{2}-h < \frac{1}{2} \text{ and from (i)} \quad f\left(\frac{1}{2}-h\right) = \frac{1}{2}-h\right]$$

$$= \frac{1}{2}-0 = \frac{1}{2} \qquad \text{...(iii)}$$

$$\underset{x\to\left(\frac{1}{2}\right)^+}{\text{Lt}} f(x) = \underset{h\to 0}{\text{Lt}}\, f\left(\frac{1}{2}+h\right)$$

$$= \underset{h\to 0}{\text{Lt}}\left[\left(\frac{1}{2}+h\right)-1\right] \qquad \left[\therefore \frac{1}{2}^{\text{th}} > \frac{1}{2} \text{ and from(ii)}, \quad f\left(\frac{1}{2}+h\right) = \left(\frac{1}{2}+h\right)-1\right]$$

$$= \frac{1}{2}+(-1) = -\frac{1}{2} \qquad \text{...(iv) from (iii) \& (iv)}$$

LHL $\neq$ RHL $\quad \because$ its does not exist.

20.2.1 Evaluation of Limits

In this section we will discuss various methods of Evaluating limits.

(i) Algebraic Limits

(ii) Trigonometric Limits

(iii) Exponential and logarithmic limits

20.2.2 Algebraic Limits

Let $f(x)$ be an algebraic function and 'a' be a real number. Then $\underset{x\to a}{\text{Lt}} f(x)$ is known as an algebraic limit.

Example: $\underset{x\to 1}{\text{Lt}} \dfrac{x^3-1}{x-1}$, $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$, $\underset{x\to 2}{\text{Lt}} \dfrac{x^5-32}{x-2}$ etc.

Algebraic Limit have the following Methods.

i) Direct substitution Method

ii) Factorisation Method

iii) Rationalisation Method

iv) Using stand Result

v) Method of evaluating limits when $x\to\infty$.

(i) Direct Substitution Method

Example 20.5 : $\underset{x\to 1}{\text{Lt}} (3x^2+4x+5) = 3(1)^2+4(1)+5 = 12$

Example 20.6 : $\underset{x\to 0}{\text{Lt}} \dfrac{\cos x}{1+\sin x} = \dfrac{\cos 0}{1+\sin 0} = \dfrac{1}{1+0} = 1$

Example 20.7 : Evaluate $\underset{x\to 0}{\text{Lt}} (x^2+3x+7)$

Solution : $\underset{x\to 0}{\text{Lt}} (x^2+3x+7)$

$= (0 + 0 + 7)$

$= 7.$

Example 20.8 : Evaluate $\underset{x \to 1}{\text{Lt}} \left[(x+1)^2 + 2\right]$

Solution : $\underset{x \to 1}{\text{Lt}} \left[(x+1)^2 + 2\right]$

$= [(1 + 1)^2 + 2] = [2^2 + 2]$

$= 4 + 2 = 6$

Example 20.9 : Evaluate $\underset{x \to 0}{\text{Lt}} \left[(2x+1)^3 - 5\right]$

Solution : $\underset{x \to 0}{\text{Lt}} \left[(2x+1)^3 - 5\right]$

$= [(2 \times 0 + 1)^3 - 5]$

$= [1 - 5]$

$= - 4$

Example 20.10 : Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{1}{x^2 - 3x + 2}$

Solution : $\underset{x \to 0}{\text{Lt}} \dfrac{1}{x^2 - 3x + 2}$

$$= \frac{1}{0^2 - 3.0 + 2} = \frac{1}{0 - 0 + 2} = \frac{1}{2}$$

Example 20.11 : Evaluate $\underset{x \to 2}{\text{Lt}} \left[\dfrac{2}{x+1} - \dfrac{3}{x}\right]$

Solution : $\underset{x \to 2}{\text{Lt}} \left[\dfrac{2}{x+1} - \dfrac{3}{x}\right]$

$$= \left[\frac{2}{2+1} - \frac{3}{2}\right]$$

$$= \left[\frac{2}{3} - \frac{3}{2}\right]$$

$$= \left[\frac{4-9}{6}\right]$$

$$= \frac{-5}{6}.$$

Example 20.12 : Evaluate $\underset{x \to 1^-}{\text{Lt}} \frac{3x+1}{x-10}$

Solution : $\underset{x \to 1^-}{\text{Lt}} \frac{3x+1}{x-10}$

$$= \frac{3(-1)+5}{-1-10} = \frac{-3+5}{-11} = \frac{2}{-11}$$

$$= -\frac{2}{11}.$$

Example 20.13 : Evaluate $\underset{x \to 0}{\text{Lt}} \frac{px+q}{ax+b}$

Solution : $\underset{x \to 0}{\text{Lt}} \frac{px+q}{ax+b}$

$$= \frac{p(0)+q}{a(0)+b}$$

$$= \frac{q}{b}.$$

Example 20.14 : Find $\underset{x \to 0}{\text{Lt}} \left[\frac{(e^x + x - 1)}{x}\right]$

Solution : $\underset{x \to 0}{\text{Lt}} \left(\frac{e^x + x - 1}{x}\right) = \underset{x \to 0}{\text{Lt}} \left(\frac{e^x - 1}{x} + \frac{x}{x}\right)$

$$= \underset{x \to 0}{\text{Lt}} \left(\frac{e^x - 1}{x}\right) + \underset{x \to 0}{\text{Lt}} (1)$$

$$= 1 + 1 \qquad \left[\because \underset{x\to 0}{\text{Lt}} \frac{e^x - 1}{x} = 1\right]$$

$$= 2.$$

Example 20.15 : Find $\underset{x\to 0}{\text{Lt}} \frac{(1+x)e^x - 1}{x}$

Solution : $\underset{x\to 0}{\text{Lt}} \frac{(1+x)e^x - 1}{x} = \underset{x\to 0}{\text{Lt}} \frac{e^x + xe^x - 1}{x}$

$$= \underset{x\to 0}{\text{Lt}} \left[\frac{e^x - 1}{x} + \frac{xe^x}{x}\right]$$

$$= \underset{x\to 0}{\text{Lt}} \left(\frac{e^x - 1}{x}\right) + \underset{x\to 0}{\text{Lt}} \left(\frac{xe^x}{x}\right)$$

$$= 1 + \underset{x\to 0}{\text{Lt}} (e^x) = 1 + 1 = 2.$$

Example 20.16 : Show that $\underset{x\to 0^+}{\text{Lt}} \left\{\frac{2|x|}{x} + x + 1\right\} = 3$

Solution : $\underset{x\to 0^+}{\text{Lt}} \left\{\frac{2|x|}{x} + x + 1\right\}$

$\underset{x\to 0^+}{\text{Lt}} \frac{2x}{x} + x + 1$ Since $|x| = x$ for $x > 0$

$\underset{x\to 0^+}{\text{Lt}} (2 + x + 1) = \underset{x\to 0}{\text{Lt}} (2 + 0 + 1) = 3$

$\therefore \underset{x\to 0^+}{\text{Lt}} \left\{\frac{2|x|}{x} + x + 1\right\} = 3.$

Example 20.17 : Compute $\underset{x\to 2^+}{\text{Lt}} ([x] + x)$ and $\underset{x\to 2^-}{\text{Lt}} ([x] + x)$

Solution : $\underset{x\to 2^+}{\text{Lt}} [x + x]$ Replace x by $2 + h$; $h \to 0$

$$\underset{h\to 0}{\text{Lt}} [2 + h] + 2 - h = \underset{h\to 0}{\text{Lt}} 2 + 2 + h - h = 4$$

$$\underset{x\to 2^-}{\text{Lt}}\,[x]+x \quad \text{Replace } x \text{ by } 2-h \text{ where } h\to 0$$

$$= \underset{h\to 0}{\text{Lt}}[2-h]+2-h = 2+2 = 4.$$

Example 20.18 : $\underset{x\to 0}{\text{Lt}}\left[\dfrac{(1+x)^{3/2}-1}{x}\right]$

Solution : $\underset{x\to 0}{\text{Lt}}\left[\dfrac{(1+x)^{3/2}-1}{x}\right] = \underset{x\to 0}{\text{Lt}}\left[\dfrac{(1+x)^{3/2}-1^{3/2}}{(1+x)-1}\right]$

$$\underset{1+x\to 1}{\text{Lt}}\frac{(1+x)^{3/2}-1^{3/2}}{(1+x)-1} = \frac{3}{2}(1) = \frac{3}{2}$$

Example 20.19 : $\underset{x\to 0}{\text{Lt}}\dfrac{\sin ax}{x\cos x}$

Solution : $\underset{x\to 0}{\text{Lt}}\dfrac{\sin ax}{x\cos x} = \underset{x\to 0}{\text{Lt}}\dfrac{\sin ax}{ax}\times\dfrac{a}{\cos x}$

$$= 1\times\frac{a}{1} = a.$$

Example 20.20 : $\underset{x\to 0}{\text{Lt}}\dfrac{1-\cos^2 mx}{\sin^2 nx}$

Solution : $\underset{x\to 0}{\text{Lt}}\dfrac{1-\cos 2mx}{\sin^2 nx} = \underset{x\to 0}{\text{Lt}}\dfrac{2\sin^2 mx}{\sin^2 nx}$

$$= \underset{x\to 0}{\text{Lt}}\ 2.\left[\frac{\sin mx}{nx}\right]^2 . \left[\frac{nx}{\sin nx}\right]\times\frac{m^2x^2}{n^2x^2}$$

$$= 2.\frac{m^2}{n^2} = \frac{2m^2}{n^2}$$

Example 20.21 : $\underset{x\to\infty}{\text{Lt}}\dfrac{8|x|+3x}{3|x|-2x}$

Solution : $x\to\infty \Rightarrow x>0 \qquad \because\ |x| = x$

$$= \underset{x\to\infty}{\text{Lt}}\frac{8|x|+3x}{3|x|-2x} = \underset{x\to\infty}{\text{Lt}}\frac{8|x|+3x}{3|x|-2x}$$

$$= \underset{x\to\infty}{\text{Lt}}\frac{11x}{x} = 11$$

Example 20.22 : Compute the following $\underset{x\to 0}{\text{Lt}}\left[\dfrac{(1+x)^{\frac{1}{8}}-(1-x)^{\frac{1}{8}}}{x}\right]$.

Solution : $\underset{x\to 0}{\text{Lt}}\left[\dfrac{(1+x)^{\frac{1}{8}}-1}{x}-\dfrac{(1-x)^{\frac{1}{8}}-1}{x}\right]$

$$=\underset{x\to 0}{\text{Lt}}\frac{(1+x)^{\frac{1}{8}}-1}{(1+x)-1}+\underset{x\to 0}{\text{Lt}}\frac{(1-x)^{\frac{1}{8}}-1}{(1-x)-1}$$

$$=\frac{1}{8}(1)^{\frac{1}{8}-1}+\frac{1}{8}(1)^{\frac{1}{8}-1}$$

$$=\frac{1}{8}+\frac{1}{8}=\frac{2}{8}=\frac{1}{4}.$$

Example 20.23 : $\underset{x\to 0}{\text{Lt}}\dfrac{x\sin a-a\sin x}{x-a}$

Solution : $\underset{x\to 0}{\text{Lt}}\dfrac{x\sin a-a\sin x}{x-a}$

$$=\underset{x\to a}{\text{Lt}}\frac{x\sin a-a\sin a+a\sin a-a\sin x}{x-a}$$

$$=\underset{x\to a}{\text{Lt}}\frac{\sin a(x-a)-a(\sin x-a\sin a)}{x-a}$$

$$=\underset{x\to a}{\text{Lt}}\frac{\sin a(x-a)}{(x-a)}-\underset{x\to a}{\text{Lt}}\frac{a2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{x-a}$$

$$=\sin a-a.2\cos a\frac{1}{2}$$

$$=\sin a-a\cos a.$$

Notes

Example 20.24 : $\underset{x\to a}{\text{Lt}}\left[\frac{\tan x-\tan a}{(x-a)}\right]$ $\tan(x-a)=\frac{\tan x-\tan a}{1+\tan x\tan a}$

Solution : $\underset{x\to a}{\text{Lt}}\left[\frac{\tan x-\tan a}{(x-a)}\right]=\underset{x\to a}{\text{Lt}}\frac{\frac{\sin x}{\cos x}-\frac{\sin a}{\cos a}}{(x-a)}$

$$=\underset{x\to a}{\text{Lt}}\frac{\sin x\cos a-\cos x\sin a}{(x-a)\cos x\cos a}$$

$$=\underset{x\to a}{\text{Lt}}\frac{\sin(x-a)}{(x-a)}\times\underset{x\to a}{\text{Lt}}\frac{1}{\cos x\cos a}$$

$$=1.\frac{1}{\cos a.\cos a}=\frac{1}{\cos^2 a}=\sec^2 a$$

Example 20.25 : $\underset{x\to 0}{\text{Lt}}\frac{\cos ax-\cos bx}{x^2}$

Solution : $\underset{x\to 0}{\text{Lt}}\left[\frac{\cos ax-\cos bx}{x^2}\right]=\underset{x\to 0}{\text{Lt}}\frac{2\sin\left(\frac{ax+bx}{2}\right)\sin\left(\frac{bx-ax}{2}\right)}{x^2}$

$$=2.\underset{x\to 0}{\text{Lt}}\frac{\sin(a+b)\frac{x}{2}}{x}\,.\,\underset{x\to 0}{\text{Lt}}\sin\left[\frac{(b-a)\frac{x}{2}}{x}\right]$$

$$=2.\underset{x\to 0}{\text{Lt}}\frac{\sin(b+a)\frac{x}{2}}{(b+a)\frac{x}{2}}\times\underset{x\to 0}{\text{Lt}}\frac{\sin(b-a)\frac{x}{2}}{(b-a)\frac{x}{2}}\times\left(\frac{b+a}{2}\right)\times\left(\frac{b-a}{2}\right)$$

$$=2\cdot 1\cdot 1\cdot\left(\frac{b+a}{2}\right)\cdot\left(\frac{b-a}{2}\right)\quad=2\cdot\frac{(b^2-a^2)}{4}$$

Example 20.26 : Evaluate $\underset{x\to 0}{\text{Lt}}\frac{1-\cos 2x}{3\tan^2 x}$

Solution : $\underset{x\to 0}{\text{Lt}}\frac{1-\cos 2x}{3\tan^2 x}=\underset{x\to 0}{\text{Lt}}\frac{2\sin^2 x}{3.\frac{\sin^2 x}{\cos^2 x}}$

$$= \frac{2}{3} \underset{x\to 0}{\text{Lt}} \cos^2 x$$

$$= \frac{2}{3} \underset{x\to 0}{\text{Lt}} (\cos x)^2$$

$$= \frac{2}{3} \times 1$$

$$= \frac{2}{3}$$

$$\therefore \underset{x\to 0}{\text{Lt}} \frac{1-\cos 2x}{3\tan^2 x} = \frac{2}{3}.$$

Example 20.27 : Evaluate $\underset{x\to 1}{\text{Lt}} \dfrac{\cos\frac{\pi}{2}x}{1-x}$

Solution : $x \to 1 \Rightarrow 1-x \to 0$

Let $1-x = h \Rightarrow x = 1-h$

$$\underset{x\to 1}{\text{Lt}} \frac{\cos\frac{\pi}{2}x}{1-x} = \underset{h\to 0}{\text{Lt}} \frac{\cos\frac{\pi}{2}(1-h)}{h}$$

$$= \underset{h\to 0}{\text{Lt}} \frac{\cos\left(\frac{\pi}{2}-\frac{\pi h}{2}\right)}{h} \qquad \cos(90-\theta) = \sin\theta$$

$$= \underset{h\to 0}{\text{Lt}} \frac{\sin\frac{\pi h}{2}}{\frac{\pi h}{2}} \times \frac{\pi}{2}$$

$$= \frac{\pi}{2} \underset{\frac{\pi h}{2}\to 0}{\text{Lt}} \frac{\sin\frac{\pi h}{2}}{\frac{\pi h}{2}} \qquad \left(h \to 0 \Rightarrow \frac{\pi h}{2} \to 0\right)$$

$$= \frac{\pi}{2} \times 1 = \frac{\pi}{2}.$$

(ii) Factorisation Method

Consider $\underset{x \to a}{\text{Lt}} \dfrac{f(x)}{g(x)}$, putting $x = a$ the rationalisation function $\dfrac{f(x)}{g(x)}$.

Takes the form $\dfrac{0}{0}, \dfrac{\infty}{\infty}$ etc. Then $(x - a)$ is factor of both $f(x)$ and $g(x)$.

In such a case we factorise the numerator and denominator then cancel out the common factor $(x - a)$. After cancelling out the common factor $x - a$, we again put $x - a$ in the given expression. This process is repeated till we get a meaningful number.

Example 20.29 : $\underset{x \to 2}{\text{Lt}} \dfrac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$ put $x = 2$ we get form $\dfrac{0}{0}$

$$= \underset{x \to 2}{\text{Lt}} \frac{(x-1)(x-2)(x-3)}{(x-2)(x-4)} = \frac{1}{2}$$

Example 20.30 : $\underset{x \to 3}{\text{Lt}} \dfrac{x^3 - 7x^2 + 15x - 9}{x^4 - 5x^3 + 27x - 27}$ $x = 3$ we get $\left(\text{form } \dfrac{0}{0}\right)$

$$= \underset{x \to 3}{\text{Lt}} \frac{(x-3)(x^2 - 4x + 3)}{(x-3)(x^3 + 2x^2 - 6x + 9)} \Rightarrow \left(\text{form } \frac{0}{0}\right)$$

$$= \underset{x \to 3}{\text{Lt}} \frac{x^2 - 4x + 3}{x^3 - 2x^2 - 6x + 9} \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x \to 3}{\text{Lt}} \frac{(x-3)(x-1)}{(x-3)(x^2 + x - 3)} \left[\text{form } \frac{0}{0}\right]$$

$$= \underset{x \to 3}{\text{Lt}} \frac{(x-1)}{x^2 + x - 3} = \frac{3-1}{9+3-3} = \frac{2}{9}.$$

Example 20.31 : Evaluate $\underset{x \to 2}{\text{Lt}} \dfrac{x^3 - 8}{x - 2}$

Solution : $\underset{x \to 2}{\text{Lt}} \dfrac{x^3 - 8}{x - 2} = \underset{x \to 2}{\text{Lt}} \dfrac{(x-2)(x^2 + 2x + 4)}{x - 2}$

$$= \underset{x\to 2}{\text{Lt}}\,(x^2+2x+4)$$

$$= 2^2 + 2 \times 2 + 4 = 4 + 4 + 4$$

$$= 12.$$

Example 20.32 : Evaluate $\underset{x\to a}{\text{Lt}}\,\dfrac{x^2-a^2}{x-a}$.

Solution : $\underset{x\to a}{\text{Lt}}\,\dfrac{(x+a)(x-a)}{(x-a)} = (a+a) = 2a$

Example 20.33 : Evaluate: $\underset{x\to \frac{1}{3}}{\text{Lt}}\,\dfrac{9x^2-1}{3x-1}$

Solution : $\underset{x\to \frac{1}{3}}{\text{Lt}}\,\dfrac{9x^2-1}{3x-1}$

$$= \underset{x\to \frac{1}{3}}{\text{Lt}}\,\frac{(3x-1)(3x+1)}{(3x-1)}$$

$$= \left(3\times\frac{1}{3}+1\right)$$

$$= (1 + 1)$$

$$= 2$$

Example 20.34 : $\underset{x\to a}{\text{Lt}}\,\dfrac{\tan(x-a)}{x^2-a^2}$

Solution : $\underset{x\to a}{\text{Lt}}\,\dfrac{\tan(x-a)}{(x-a)}\times\dfrac{1}{x+a}$

$$= \underset{x\to a}{\text{Lt}}\,\frac{\tan(x-a)}{x-a}\times \underset{x\to a}{\text{Lt}}\,\frac{1}{x+a}$$

$$1\times\frac{1}{a+a} = \frac{1}{2a}$$

Example 20.35 : $\underset{x\to 2}{\text{Lt}}\left[\frac{1}{x-2}-\frac{4}{(x-2)x+2}\right]$

Solution : $\underset{x\to 2}{\text{Lt}}\left[\frac{1}{x-2}-\frac{4}{(x-2)(x+2)}\right]$

$$= \underset{x\to 2}{\text{Lt}}\left[\frac{1}{(x-2)}\left\{1-\frac{4}{x+2}\right\}\right]$$

$$= \underset{x\to 2}{\text{Lt}}\frac{1}{(x-2)}\left[\frac{x+2-4}{x+2}\right] = \frac{1}{(x-2)}\left[\frac{x-2}{x+2}\right]$$

$$= \underset{x\to 2}{\text{Lt}}\left[\frac{1}{2+2}\right] = \frac{1}{4}$$

$$= \frac{1}{4}$$

Example 20.36 : Evaluate $\underset{x\to 0}{\text{Lt}}\frac{e^{2x}-1}{x}$

Solution : $\underset{2x\to 0}{\text{Lt}}\left(\frac{e^{2x}-1}{x}\right)\times 2 \quad \underset{2x\to 0}{\text{Lt}}\left(\frac{e^{2x}-1}{x}\right)\times 2 \qquad \left(\therefore \underset{x\to 0}{\text{Lt}}\frac{e^{x}-1}{x}=1\right)$

$$= 1\times 2 = 2$$

Example 20.37 : Evaluate $\underset{x\to 0}{\text{Lt}}\frac{\sin ax}{\sin bx}$

Solution : $\underset{x\to 0}{\text{Lt}}\frac{\sin ax}{\sin bx} = \underset{x\to 0}{\text{Lt}}\frac{\frac{\sin ax}{ax}\times ax}{\frac{\sin bx}{bx}\times bx}$

$$= \frac{1\times a}{1\times b} = \frac{a}{b}.$$

(iii) Rationalisation Method

Particularly used when either Numerator or denominator OR both involve expressions consisting of square roots as explained.

Example 20.38 : Evaluate $\underset{x \to a}{\text{Lt}} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

When $x = 0$ we get Indeterminate form $\dfrac{0}{0}$

Now

$$\underset{x \to a}{\text{Lt}} \frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$$

$$= \underset{x \to a}{\text{Lt}} \frac{\left(\sqrt{a+2x}+\sqrt{3x}\right)}{\left(\sqrt{3a+x}-2\sqrt{x}\right)} \times \left(\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{3a+x}+2\sqrt{x}}\right) \times \left(\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}}\right)$$

$$= \underset{x \to a}{\text{Lt}} \frac{(a-x)\left(\sqrt{3a+x}+2\sqrt{x}\right)}{(a-x)\left(\sqrt{a+2x}-3\sqrt{x}\right)} = \underset{x \to a}{\text{Lt}} \left(\frac{\sqrt{3a+x}+\sqrt{2x}}{3\sqrt{a+2x}+\sqrt{3x}}\right)$$

$$= \frac{4\sqrt{a}}{2(3\sqrt{3a})} = \frac{2}{3\sqrt{3}}$$

Example 20.39 : Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{x}{\sqrt{1+x}-1}$

Solution : $\underset{x \to 0}{\text{Lt}} \dfrac{x}{\sqrt{1+x}-1}$

$$= \underset{x \to 0}{\text{Lt}} \left(\frac{x}{\sqrt{1+x}-1}\right)\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right)$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x\sqrt{1+x}+1}{(1+x-1)} = \underset{x \to 0}{\text{Lt}} \frac{x\left(\sqrt{1+x}+1\right)}{x}$$

$$= \underset{x \to 0}{\text{Lt}} \left(\sqrt{1+x}+1\right) = \sqrt{1+0}+1 = 1+1 = 2$$

Example 20.40 : Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

Solution : Rationalise the factor containing square root, simplify, put the value of x, get the result.

$$= \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \times \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \frac{(1+x)-(1-x)}{x[\sqrt{1+x}+\sqrt{1-x}]}$$

$$= \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \frac{2}{\left(\sqrt{1+x}\right)+\left(\sqrt{1-x}\right)}$$

$$\underset{x\to 0}{\text{Lt}} \frac{2x}{x\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{1+1} = \frac{2}{2} = 1.$$

Example 20.41 : Evaluate $\underset{x\to 3}{\text{Lt}} \dfrac{\sqrt{12-x}-x}{\sqrt{6+x}-3}$

Solution : Rationalizing Numerator & Denominator we get

$$\underset{x\to 3}{\text{Lt}} \frac{\sqrt{12-x}-x}{\sqrt{6+x}-3} = \frac{\left(\sqrt{12-x}-x\right)}{\left(\sqrt{6+x}-3\right)} \times \frac{\left(\sqrt{12-x}+x\right)}{\left(\sqrt{6+x}+3\right)} \times \frac{\left(\sqrt{6+x}+3\right)}{\left(\sqrt{12-x}+x\right)}$$

$$= \underset{x\to 3}{\text{Lt}} \frac{\left(12-x-x^2\right)}{6+x-9} \cdot \underset{x\to 3}{\text{Lt}} \frac{\sqrt{6+x}+3}{\sqrt{12-x}+x}$$

$$\underset{x\to 3}{\text{Lt}} \frac{-(x+4)(x-3)}{(x-3)} \cdot \underset{x\to 3}{\text{Lt}} \frac{\sqrt{6+x}+3}{\sqrt{12-x}+x} \quad x \neq 3$$

$$= (-3+4)\,\frac{6}{6} = -7.$$

Example 20.42 : $\underset{x\to 0}{\text{Lt}} \left(\dfrac{3^x-1}{\sqrt{1+x}-1} \right)$

Solution : $\underset{x\to 0}{\text{Lt}} \left(\dfrac{3^x-1}{\sqrt{1+x}-1} \right) = \underset{x\to 0}{\text{Lt}} \dfrac{3^x-1}{\sqrt{1+x}-1} \times \dfrac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$$= \underset{x\to 0}{\text{Lt}} \frac{3^x-1}{|1+x-1|} \times \sqrt{1+x}+1$$

$$= \underset{x\to 0}{\text{Lt}} \frac{3^x - 1}{x} \cdot \underset{x\to 0}{\text{Lt}} \left(\sqrt{1+x}+1\right)$$

$$= \log_e 3\left(\sqrt{1+0}+1\right)$$

$$= \log_e 3\ (1+1) = 2\log_e 3$$

$$= \log_e 3^2 \Rightarrow \log_e 9$$

Example 20.43 : $\underset{x\to\infty}{\text{Lt}} \left(\sqrt{x^2+x}-x\right)$

Solution : $\underset{x\to\infty}{\text{Lt}} \left(\sqrt{x^2+x}-x\right)$

$$= \underset{x\to\infty}{\text{Lt}} \frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\sqrt{x^2+x}+x}$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x}$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{x}{\sqrt{x^2+x}+x}$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{x}{x\left[\sqrt{1+\frac{1}{x}}+1\right]}$$

$$= \underset{x\to\infty}{\text{Lt}} \frac{1}{x\left[\sqrt{1+\frac{1}{x}}+1\right]}$$

As $x \to \infty,\ \frac{1}{x} \to 0$

$$\frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}.$$

20.3 FINDING LIMITS OF SOME OF THE IMPORTANT FUNCTIONS

(i) Prove that $\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$ where n is a positive integer.

Proof : $\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$

$$= \lim_{h\to 0}\frac{\left(a^n + na^{n-1}h + \frac{n(n-1)}{2!}a^{n-2}h^2 + ... + h^n\right) - a^n}{h}$$

$$= \lim_{h\to 0}\frac{h\left(n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + ... + h^{n-1}\right)}{h}$$

$$= \lim_{h\to 0}\left[n\,a^{n-1} + \frac{n(n-1)}{2!}a^{n-2}h + ... + h^{n-1}\right]$$

$$= na^{n-1} + 0 + 0 + ... + 0$$

$$= na^{n-1}$$

$$\lim_{x\to a}\frac{x^n - a^n}{x-a} = n \cdot a^{n-1}$$

Note : However, the result is true for all n.

(ii) Prove that (a) $\lim_{x\to 0}\sin x = 0$

and (b) $\lim_{x\to 0}\cos x = 1$

Proof : Consider a unit circle with centre B, in which $\angle C$ is a right angle and $\angle ABC = x$ radians.

Now $\sin x$ = A C and $\cos x$ = BC

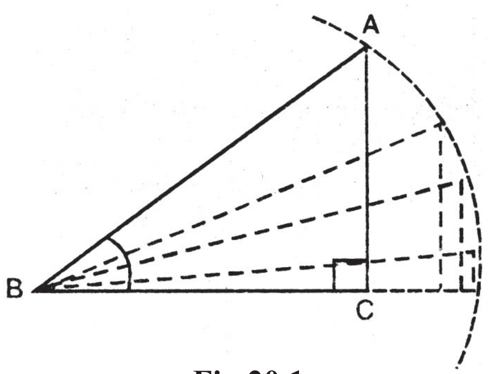


Fig 20.1

As x decreases, A goes on coming nearer and nearer to C.

i.e., when $x \to 0$, A $\to$ C

or when $x \to 0$, AC $\to 0$

and BC $\to$ AB, i.e., BC $\to 1$

$\therefore$ When $x \to 0$ $\sin x \to 0$ and $\cos x \to 1$

Thus we have

$$\lim_{x \to 0} \sin x = 0 \quad \text{and} \quad \lim_{x \to 0} \cos x = 1$$

(iii) Prove that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof : Draw a circle of radius 1 unit and with centre at the origin *O*. Let B(1, 0) be a point on the circle. Let A be any other point on the circle. Draw AC $\perp$ OX.

Let $\angle AOX = x$ radians, where $0 < x < \frac{\pi}{2}$

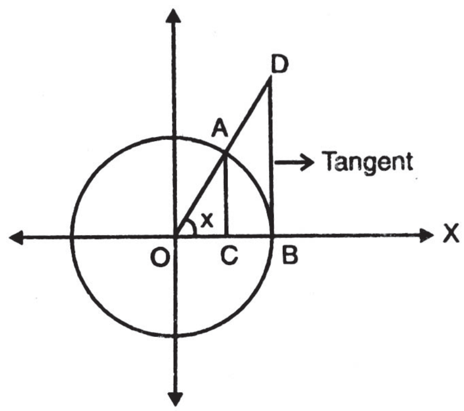


Fig. 20.2

Draw a tangent to the circle at B meeting OA produced at D. Then BD $\perp$ OX.

Area of ΔAOC < area of sector OBA < area of ΔOBD.

or $\frac{1}{2} OC \times AC < \frac{1}{2} x(1)^2 < \frac{1}{2} OB \times BD$

$\left[\because \text{ area of triangle } = \frac{1}{2}\text{base} \times \text{height and area of sector} = \frac{1}{2}\theta r^2\right]$

$\therefore \quad \frac{1}{2}\cos x \ \sin x < \frac{1}{2}x < \frac{1}{2}\cdot 1 \cdot \tan x$

$$\left[\because \ \cos x = \frac{OC}{OA}, \sin x = \frac{AC}{OA} \text{ and } \tan x = \frac{BD}{OB}, OA = 1 = OB\right]$$

i.e., $\cos x \ \frac{x}{\sin x} < \frac{\tan x}{\sin x}$ [Dividing throughout by $\frac{1}{2}\sin x$]

or $\quad \cos x \ \frac{x}{\sin x} < \frac{1}{\cos x}$

or $\quad \frac{1}{\cos x} > \frac{\sin x}{x} < \cos x$

i.e., $\quad \cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$

Taking limit as $x \to 0$, we get

$$\lim_{x\to 0} \cos x < \lim_{x\to 0} \frac{\sin x}{x} < \lim_{x\to 0} \frac{1}{\cos x}$$

or $\quad 1 < \lim_{x\to 0} \frac{\sin x}{x} < 1$

$$\left[\because \ \lim_{x\to 0} \cos x = 1; \ \lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{1} = 1\right]$$

Thus, $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Note : In the above results, it should be kept in mind that the angle x must be expressed in radians.

(iv) Prove that $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e.$

Proof : By Binomial theorem, when $|x| < 1$, we get

$$(1+x)^{\frac{1}{x}} = \left[1+\frac{1}{x}.x+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3+....\infty\right]$$

$$=\left[1+1+\frac{(1-x)}{2!}+\frac{(1-x)\ (1-2x)}{3!}+.......\infty\right]$$

$$\therefore\ \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0}\left[1+1+\frac{1-x}{2!}+\frac{(1-x)\ (1-2x)}{3!}+......\infty\right]$$

$$=\left[1+1+\frac{1}{2!}+\frac{1}{3!}+.......\infty\right]$$

$= e$ (By definition)

$$\therefore\ \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

(v) Prove that

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}\log\ (1+x)=\lim_{x\to 0}\log\ (1+x)^{1/x}$$

$= \log e$ (Using $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$)

$= 1$

(vi) Prove that $\lim_{x\to 0}\frac{e^x-1}{x}=1$

Proof : We know that $e^x=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+.........\right)$

$$\therefore\quad e^x-1=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+.........-1\right)$$

$$=\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+.........\right)$$

$$\therefore \quad \frac{e^x - 1}{x} = \frac{\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)}{x} \quad \text{[Dividing throughout } x\text{]}$$

$$= \frac{x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + ...\right)}{x}$$

$$\therefore \quad \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + ...\right)$$

$$= 1 + 0 + 0 + ... = 1$$

$$\therefore \quad \lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

Using Standard Results

1. If $n \in Q$ then $\underset{x \to a}{\text{Lt}} \frac{x^n - a^n}{x - a} = na^{n-1}$

2. $\underset{x \to a}{\text{Lt}} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$

3. $\underset{x \to 1}{\text{Lt}} \frac{(x + x^2 + x^3 + x^4 + ... + x^n) - n}{x - 1} = \frac{n(n+1)}{2}$

4. $\underset{x \to -a}{\text{Lt}} \frac{x^9 + a^9}{x + 9} = 9$ find 'a'

$$= \underset{x \to -a}{\text{Lt}} \frac{x^9 - (-a)^9}{x - (-a)} = 9$$

$$= 9(-a)^{9-1} = 9 \Rightarrow 9a^8 = 9 \Rightarrow a^8 = 1$$

$$\Rightarrow a = \pm 1$$

iv) If $a > 0$, $n \in R$. Then $\underset{x \to a}{\text{Lt}}\, x^n = a^n$.

Trigonometric Limits

i) $\underset{x\to 0}{\text{Lt}} \cos x = 1$

iii) $\underset{x\to 0}{\text{Lt}} \frac{\tan x}{x} = 1$

ii) $\underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} = 1$

iv) $\underset{x\to 0}{\text{Lt}} \frac{\tan^{-1} x}{x} = 1$

v) $\underset{x\to 0}{\text{Lt}} \frac{\sin^{-1} x}{x} = 1$

vii) $\underset{x\to a}{\text{Lt}} \frac{\sin(x-a)}{x-a} = 1$

vi) $\underset{x\to a}{\text{Lt}} \frac{\tan(x-a)}{x-a} = 1$

Exponential and logarithmic limits

i) $\underset{x\to 0}{\text{Lt}} \frac{a^x - 1}{x} = \log_e a$

ii) $\underset{x\to 0}{\text{Lt}} \frac{\log(1+x)}{x} = 1$

iii) $\underset{x\to 0}{\text{Lt}} \frac{e^x - 1}{x} = 1$

Particular Cases

i) $\underset{x\to 0}{\text{Lt}} (1+x)^{\frac{1}{x}} = e$

ii) $\underset{x\to \infty}{\text{Lt}} \left(1+\frac{1}{x}\right)^x = e$

iii) $\underset{x\to 0}{\text{Lt}} (1+\lambda x)^{\frac{1}{x}} = e^{\lambda}$

iv) $\underset{x\to \infty}{\text{Lt}} \left[1+\frac{\lambda}{x}\right]^x = e^{\lambda}$

Evaluate : $\underset{x\to \infty}{\text{Lt}} \frac{ax^2 + bx + c}{dx^2 + ex + f} = \underset{x\to \infty}{\text{Lt}} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x} + \frac{f}{x^2}}$

$$= \frac{a+0+0}{d+0+0} = \frac{a}{d}.$$

Using Limits by 'L' Hospital's Rule

If $f(x)$ and $g(x)$ be two functions of x such that

i) $\underset{x\to a}{\text{Lt}}\, f(x) = \underset{x\to a}{\text{Lt}}\, g(x) = 0$

ii) Both are continuous at $x = a$.

iii) Both are differentiable at $x = a$

iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$ then

$$\underset{x\to a}{\text{Lt}} \frac{f(x)}{g(x)} = \underset{x\to a}{\text{Lt}} \frac{f'(x)}{g'(x)}, \text{ provided that } g(a) \neq 0$$

The above rules also applicable if $\underset{x\to a}{\text{Lt}}\, f(x) = \infty$ and $\underset{x\to a}{\text{Lt}}\, g(x) = \infty$.

Example 20.44 : Evaluate $\underset{x\to a}{\text{Lt}} \frac{x^a - a^x}{x^x - a^a} = \frac{0}{0}$ form

$$= \underset{x\to a}{\text{Lt}} \frac{ax^{a-1} - a^x \log a}{x^x(1+\log x) - 0} \quad \text{[Using L' Hospital Rule]}$$

$$= \frac{a^a - a^a \log a}{a^a(1+\log a)} = \frac{1-\log a}{1+\log a}.$$

Some times following expansions are very useful to evaluate limits:

a) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$

b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

c) $a^x = 1 + x + (\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + ...$

d) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$

e) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...$

f) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

g) $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + ...$

h) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + ...$

i) $\sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1}{2}.\frac{3}{4}\frac{x^5}{4} + \frac{1}{2}.\frac{3}{4}.\frac{5}{6}\frac{x^7}{7} + ...$

j) $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + ...$

h) $\sec x = 1 + \frac{x^2}{2!} + 5\frac{x^4}{4!} + ...$

Find Right and Left Limits at the point '*a*'.

EXERCISE 20.1

1. Evaluate the following.

a) $\underset{x \to 2}{\text{Lt}} \left[2(x+3)7\right]$ b) $\underset{x \to 1}{\text{Lt}} \left[(x+3)^2 - 16\right]$

c) $\underset{x \to 1}{\text{Lt}}\ (3x+1)(x+1)$

2. Find the limits of the following.

a) $\underset{x \to 1}{\text{Lt}} \frac{x+2}{x+1}$ b) $\underset{x \to \frac{1}{3}}{\text{Lt}} \frac{9x^2 - 1}{3x - 1}$

c) $\underset{x \to 1}{\text{Lt}} \left[\frac{1}{x-1} - \frac{2}{x^2 - 1}\right]$ d) $\underset{x \to 1}{\text{Lt}} \frac{x^4 - 1}{x - 1}$

e) $\underset{x \to 0}{\text{Lt}} \frac{px+q}{ax+b}$

3. Compute the following

a) Evaluate $\underset{x\to a}{\text{Lt}}\ \dfrac{x^2-a}{x-a}$ b) $\underset{x\to 3}{\text{Lt}}\ \dfrac{1}{x+1}$

c) $\underset{x\to 2}{\text{Lt}}\left[\dfrac{2}{x+1}-\dfrac{3}{x}\right]$ d) $\underset{x\to 0}{\text{Lt}}\left[\sqrt{x}+x^{5/2}\right]\ x>0$

e) $\underset{x\to 2^-}{\text{Lt}}\ \sqrt{2-x}\ \ x<2$ What is $\underset{x\to 2}{\text{Lt}}\ \sqrt{2-x}$?

f) $\underset{x\to 1}{\text{Lt}}\left[\dfrac{2x+1}{3x^2-4x+5}\right]$ g) $\underset{x\to 0}{\text{Lt}}\left[\dfrac{(1+x)^{3/2}-1}{x}\right]$

4. a) Find $\underset{x\to 0}{\text{Lt}}\left[\dfrac{\sqrt{1+x}-1}{x}\right]$

b) $\underset{x\to 0}{\text{Lt}}\left(\dfrac{e^x-1}{\sqrt{1+x}-1}\right)$ c) $\underset{x\to 0}{\text{Lt}}\ \dfrac{e^{3x}-1}{x}\ \ x\neq 0$

d) $\underset{x\to 0}{\text{Lt}}\ \dfrac{\tan(x-a)}{x^2-a^2}$ e) $\underset{x\to 0}{\text{Lt}}\ \dfrac{1-\cos mx}{1-\cos nx}\ \ n\neq 0$

5. Compute the following Limits

a) $\underset{x\to 3}{\text{Lt}}\ \dfrac{x^2+3x+2}{x^2-6x+9}$ b) $\underset{x\to\infty}{\text{Lt}}\ \dfrac{8|x|+3x}{3|x|-2x}$

EXERCISE 20.2

1. Find the RHL and LHL of the function.

a) $f(x)\ \begin{matrix} x+2 \text{ if } -1<x\le 3 \\ x^2 \quad \text{if} \quad 3<x<5 \end{matrix}\ ;\quad a=3$

b) $f(x)=\begin{cases} x^2 & \text{if } x\le 1 \\ x & \text{if } 1<x\le 2; \\ x-3 & \text{if } x>2 \end{cases}\qquad a=2$

2. Compute the following Limits.

a) $\underset{x \to 0}{\text{Lt}} \dfrac{\sin(a+bx)-\sin(a-bx)}{x}$

b) $\underset{x \to 0}{\text{Lt}} \dfrac{\cos ax - \cos bx}{x^2}$

c) $\underset{x \to 0}{\text{Lt}} \dfrac{1-\cos 2mx}{\sin^2 nx}$

d) $\underset{x \to a}{\text{Lt}} \dfrac{x\sin a - a\sin x}{x-a}$

3. Evaluate

a) $\underset{x \to 2}{\text{Lt}} \dfrac{\sqrt{3-x}-1}{2-x}$

b) $\underset{x \to 1}{\text{Lt}} \left[\dfrac{1}{x-1} - \dfrac{2}{x^2+1}\right]$

c) $\underset{x \to 0}{\text{Lt}} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$

4. Find the value of 'a' such that $\underset{x \to 2}{\text{Lt}} f(x)$ exists where

$$f(x) = \begin{cases} ax+5, & x < 2 \\ x-1, & x \geq 2 \end{cases}$$

5. Find the value of $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{-x}}{x}$.

6. Evaluate $\underset{\theta \to 0}{\text{Lt}} \dfrac{1-\cos 4\theta}{1-\cos 6\theta}$

7. Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{\sin ax}{\tan bx}$

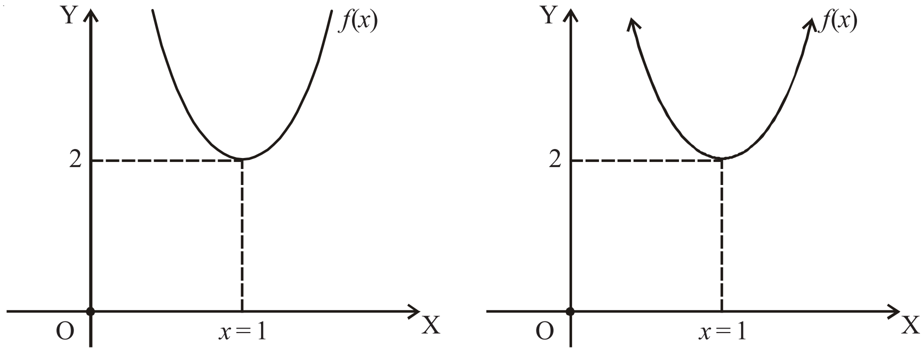
8. Evaluate $\underset{x \to 0}{\text{Lt}} \dfrac{1-\cos 2x}{3\tan^2 x}$

9. Find the value $\underset{x \to \frac{\pi}{2}}{\text{Lt}} (\sec x - \tan x)$

10. Find $\underset{x \to 0}{\text{Lt}} \dfrac{(1+x)e^x - 1}{x}$

20.4 INTRODUCTION OF CONTINUITY

As we have discussed in limits it exists at neighbouring points, Graphically it could be stated as, shown in figure.



But when we say that the function $f(x)$ is continuous at a point $x = a$, it means that at point $(a, f(a))$ the Graph of the function has no holes or gaps. That is, its graphs is unbroken at a point $(a, f(a))$.

Graphically it could be stated as, shown in figure there

$\underset{x\to a^-}{\text{Lt}} f(x) = 2$ and $f(1) = 2$

$\underset{x\to 1}{\text{Lt}} f(x) = f(1)$ hence, $f(x)$ is continuous.

20.4.1 CONTINUITY OF A FUNCTION

A function $f(x)$ is said to be continuous at $x = a$; where $a \in$ domain of $f(x)$ if

$$\underset{x\to a^-}{\text{Lt}} f(x) = \underset{x\to a^+}{\text{Lt}} f(x) = f(a)$$

i.e., LHL = RHL = value of a function at $x = a$ OR $\underset{x\to a}{\text{Lt}} f(x) = f(a)$.

If $f(x)$ is not continuous at $x = a$ we say that $f(x)$ is discontinuous at $x = a$ any of the following cases :

(i) $\underset{x\to a^-}{\text{Lt}} f(x)$ and $\underset{x\to a^+}{\text{Lt}} f(x)$ exists but are not equal.

(ii) $\underset{x\to a^-}{\text{Lt}} f(x)$ and $\underset{x\to e^-}{\text{Lt}} f(x)$ exists and are equal but not equal to $f(a)$.

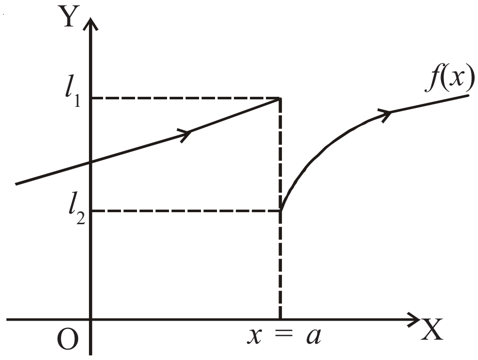
(iii) $f(a)$ is not defined

(iv) At least one of the limit does not exists.

20.4.2 GRAPHICAL VIEW

(i) $\underset{x\to a^-}{\text{Lt}} f(x)$ and $\underset{x\to a^+}{\text{Lt}} f(x)$

exists but are not equal.



Here $\underset{x\to a^-}{\text{Lt}} f(x) = l_1$,

$\underset{x\to a^+}{\text{Lt}} f(x) = l_2$

$\therefore$ $\underset{x\to a^-}{\text{Lt}} f(x)$ and $\underset{x\to a^+}{\text{Lt}} f(x)$

exists but are not equal.

Thus $f(x)$ is dis continuous at $x = a$.

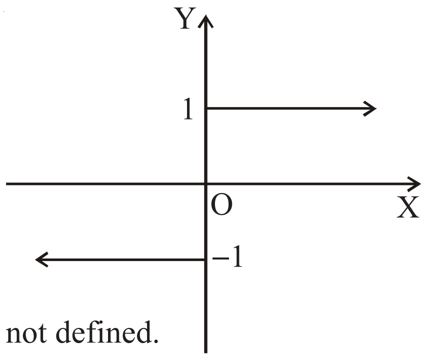
It does not matter whether $f(a)$ exists or not.

Example 20.45 : If $f(x) = \dfrac{|x|}{x}$ is discontinuous at $x = 0$.

if $\begin{matrix} x > 0 \text{ we get } +1 \\ x < 0 \text{ we get } -1 \end{matrix}$ $\therefore$ $\underset{x\to 0^+}{\text{Lt}} f(x) \neq \underset{x\to 0^-}{\text{Lt}} f(x)$

(i) Graphically

$$f(x) = \frac{|x|}{x} \begin{cases} \dfrac{x}{x} & x > 0\ 1,\ x > 0 \\ \dfrac{-x}{x} & x < 0,\ -1,\ x < 0 \end{cases}$$



and $f(0) = \dfrac{0}{0}$ (indeterminant form) not defined.

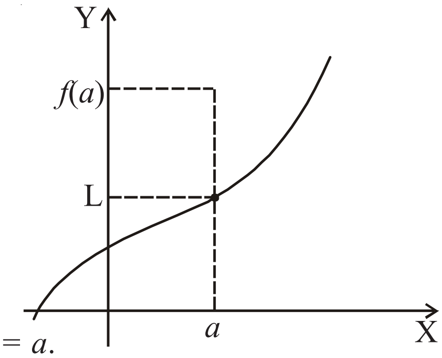
Thus $\underset{x\to 0}{\text{Lt}} f(x) \Rightarrow$ it does not exits and hence, function is discontinuous.

(ii) $\underset{x\to a^-}{\text{Lt}} f(x)$ and $\underset{x\to a^+}{\text{Lt}} f(x)$ exists and are equal but not equal to $f(a)$.

Here $\underset{x\to a^-}{\text{Lt}} f(x) = \underset{x\to a^+}{\text{Lt}} f(x) = l$

$f(a)$ is also defined but $f(a) \neq$ L

So again limit of $f(x)$ exists at $x = a$.



But it is not continuous at $x = a$.

20.4.3 PROPERTIES OF CONTINUOUS FUNCTION

Let $f(x)$ and $g(x)$ are continous functions at $x = a$ then

(i) C $f(x)$ is continuous at $x = a$ where C is any constant

(ii) $f(x) \pm g(x)$ is continuous at $x = a$

(iii) $f(x)\,.\,g(x)$ is continuous at $x = a$.

(iv) $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$

(vi) $f(x)$ be mapping such that $f : [a, b] \to [a, b]$

$\Rightarrow f(x) = x$ for some $x \in [a, b]$

20.5 IMPORTANT RESULTS ON CONTINUITY

By using the properties mentioned above, we shall now discuss some important results on continuity.

(i) Consider the function $f(x) = px + q, x \in$ R

The domain of this functions is the set of real numbers. Let a be any arbitary real number. Taking limit of both sides of (i), we have

$\lim_{x\to a} f(x) = \lim_{x\to a} (px + q) = pa + q$ [value of $px + q$ at $x = a$.]

$\therefore$ $px + q$ is continuous at $x = a$.

Similarly, if we consider $f(x) = 5x^2 + 2x + 3$, we can show that it is a continuous function.

In general $f(x) = a_0 + a_1 x + a_2x^2 + ... + a_{n-1} x^{n-1} + a_n x^n$.

where $a_0, a_1, a_2, ... a_n$ are constants and n is a non-negative integer,

we can show that $a_0, a_1x, a_2x^2, ... a_n x^n$ are all continuos at a point $x = c$ (where c is any real number) and by property (ii), their sum is also continuous at $x = c$.

$\therefore$ $f(x)$ is continuous at any point c.

Hence every polynomial function is continuous at every point.

(ii) Consider a function $f(x) = \dfrac{(x+1)\ (x+3)}{x-5}$, $f(x)$ is not defined when $x - 5 = 0$ i.e, at $x = 5$.

Since $(x + 1)$ and $(x + 3)$ are both continuous, we can say that $(x + 1) (x + 3)$ is also continuous. [Using property iii]

$\therefore$ Denominator of the function $f(x)$, i.e.,$(x - 5)$ is also continuous.

$\therefore$ Using the property (iv), we can say that the function $\dfrac{(x+1)\ (x+3)}{x-5}$ is continuous at all points except at $x = 5$.

In general if $f(x) = \dfrac{p(x)}{g(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$, then $f(x)$ is continuous if $p(x)$ and $q(x)$ both are continuous.

Example 20.46 : Prove that $f(x) = \sin x$ is a continous function.

Solution : Lt $f(x) = \sin x$

The domain of $\sin x$ is '**R**'

Let 'a' be a any orbitrary real number

$$\underset{x\to a}{\text{Lt}} f(x) = \underset{h\to 0}{\text{Lt}} f(a+h)$$

$$= \underset{h\to 0}{\text{Lt}} \sin(a+h)$$

$$= \underset{h\to 0}{\text{Lt}} \left[\sin a.\cos h + \cos a \sin h\right]$$

$$= \sin a \underset{h\to 0}{\text{Lt}} \cos h + \cos a \underset{h\to 0}{\text{Lt}} \sin h$$

$$= \sin a \times 1 + \cos a \ . \ 0$$

$$= \sin a \qquad \text{...(i)}$$

Also $f(a) = \sin a$ ...(ii)

From (i) & (ii) $\underset{x\to a}{\text{Lt}} f(x) = f(a)$ sin x is continuous at $x = a$.

$\therefore$ sin x is continuous at $x = a$ and 'a' is an aribitrary point.

Therefore $f(x) = \sin x$ is continuous.

Example 20.47 : Check the continuity of the function f given below at 1 and 2

$$f(x) = \begin{cases} x+1 & \text{if} \quad x \le 1 \\ 2x & \text{if} \quad 1 < x < 2 \\ 1+x^2 & \text{if} \quad x \ge 2 \end{cases}$$

Solution : We have $\underset{x\to 1^+}{\text{Lt}} f(x) = \underset{x\to 1^+}{\text{Lt}} 2x = 2 = f(1)$

$$\underset{x\to 1^-}{\text{Lt}} f(x) = \underset{x\to 1^-}{\text{Lt}} (x+1) = 2$$

Hence $\underset{x\to 1}{\text{Lt}} f(x) = f(1)$

Therefore f is continuous at 1

Similarly $\underset{x\to 2^+}{\text{Lt}} f(x) = \underset{x\to 2^+}{\text{Lt}} (1+x^2) = 5 = f(2)$

$$\underset{x\to 2^-}{\text{Lt}} f(x) = \underset{x\to 2^-}{\text{Lt}} 2x = 4$$

Hence $\underset{x\to 2^+}{\text{Lt}} f(x)$ and $\underset{x\to 2^-}{\text{Lt}} f(x)$ exist

but are not equal so that f is not continuous at 2.

Example 20.48 : Prove that tan x is continuous when $0 \le x < \frac{\pi}{2}$

Sol: Let $f(x) = \tan x$; The domain of tan x is

$$\mathbf{R} - (2x+1)\frac{\pi}{2},\ n \in \mathrm{I}$$

Let $a \in \mathbf{R} - (2x+1)\frac{\pi}{2}$, be arbitrary

$$\underset{x\to a}{\mathrm{Lt}}\, f(x) = \underset{h\to 0}{\mathrm{Lt}}\, f(a+h) = \underset{h\to 0}{\mathrm{Lt}}\, \tan(a+h)$$

$$= \underset{h\to 0}{\mathrm{Lt}}\, \frac{\sin(a+h)}{\cos(a+h)}$$

$$= \underset{h\to 0}{\mathrm{Lt}}\, \frac{\sin\cos h + \cos a \sin h}{\cos a \cos h - \sin a \sin h}$$

$$= \frac{\sin a \underset{h\to 0}{\mathrm{Lt}} \cos h + \cos a \underset{h\to 0}{\mathrm{Lt}} \sin h}{\cos a \underset{h\to 0}{\mathrm{Lt}} \cos h - \sin a \underset{h\to 0}{\mathrm{Lt}} \sin h}$$

$$= \frac{\sin a . 1 + \cos a \times 0}{\cos a \times 1 - \sin a \times 0}$$

$$= \frac{\sin a}{\cos a} = \tan a \quad \text{...(i)} \quad [\because\ \forall\, a \in \text{Domain of } \tan x,\ \cos a \ne 0]$$

Also $f(a) = \tan a$ ...(ii)

From (i) and (ii) $\underset{h\to a}{\mathrm{Lt}}\, f(x) = f(a)$

$\therefore$ $f(x)$ continuous at $x = a$, But 'a' is arbitrary.

Tan x is continuous for all x in the interval $0 \le x < \frac{\pi}{2}$.

Example 20.49 : Examine the continuity of function $f(x) = \frac{x^2 - 4}{x+2}$ at $x = 2$.

Solution : We know that $(x^2 + 4)$ is continuous at $x = 2$.

Also $(x + 2)$ is continous at $x = 2$.

$$\underset{x\to 2}{\mathrm{Lt}}\, \frac{x^2-4}{x+2} = \underset{x\to 2}{\mathrm{Lt}}\, \frac{(x+2)(x-2)}{x+2}$$

$$= \underset{x\to 2}{\mathrm{Lt}}\,(x-2)$$

$$= 2 - 2 = 0 \quad \text{...(i)}$$

Also $f(2) = \frac{2^2 - 4}{2+2} = \frac{0}{4} = 0$ ...(ii)

$\underset{x \to 2}{\text{Lt}} f(x) = f(2)$. Thus $f(x)$ is continuous at $x = 2$.

If $x^2 - 4$ and $x + 2$ are two continuous functions at $x = 2$.

Then $\frac{x^2 - 4}{x+2}$ is also continous.

Example 20.50 : If $f(x) = \begin{cases} \frac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$. Find whether $f(x)$ is continous at $x = 0$ OR Not.

Solution : Here $f(x) = \begin{cases} \frac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

Left hand limit $= \underset{x \to 0^-}{\text{Lt}} \frac{\sin 2x}{x}$

$$= \underset{h \to 0}{\text{Lt}} \frac{\sin 2(0-h)}{0-h} = \underset{h \to 0}{\text{Lt}} \frac{-\sin 2h}{-h}$$

$$= \underset{h \to 0}{\text{Lt}} \left(\frac{\sin 2h}{2h} \times \frac{2}{1} \right) = 1 \times 2 = 2 \quad \text{...(i)}$$

Right hand limit $= \underset{x \to 0^+}{\text{Lt}} \frac{\sin 2x}{x}$

$$= \underset{h \to 0}{\text{Lt}} \frac{\sin 2(0+h)}{0+h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{\sin 2h}{2h} \times \frac{2}{1}$$

$$= 1 \times 2 = 2 \quad \text{...(iii)}$$

Also $f(0) = 2$

From (i) & (iii)

$$\underset{x \to 0}{\text{Lt}} f(x) = 2 = f(0)$$

$f(x)$ is continuous at $x = 0$.

Example 20.51 : If $f(x)=\dfrac{x^2-1}{x-1}$ for $x \neq 1$ and $f(x)$ = 2 when x = 1 show that the function $f(x)$ is continuous at x = 1.

Solution : Here $f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1\\ 2 & x=1\end{cases}$

LHL $\underset{x\to 1^-}{\text{Lt}}\, f(x) = \underset{h\to 0}{\text{Lt}}\, f(x-h)$

$$= \underset{h\to 0}{\text{Lt}}\, \frac{(1-h)^2-1}{(1-h)-1} = \underset{h\to 0}{\text{Lt}}\, \frac{1-2h+h^2-1}{-h}$$

$$= \underset{h\to 0}{\text{Lt}}\, \frac{h(h-2)}{-h}$$

$$= \underset{h\to 0}{\text{Lt}}\, -(h-2)$$

$$= 2 \qquad \text{...(i)}$$

RHL $\underset{x\to 1^+}{\text{Lt}}\, f(x) = \underset{h\to 0}{\text{Lt}}\, f(1+h)$

$$= \underset{h\to 0}{\text{Lt}}\, \frac{(1+h)^2-1}{(1+h)-1} = \underset{h\to 0}{\text{Lt}}\, \frac{1-2h+h^2-1}{-h}$$

$$= \underset{h\to 0}{\text{Lt}}\, \frac{h(h+2)}{h}$$

$$= 2 \qquad \text{...(ii)}$$

Also $f(1)$ = 2 (Given) ...(iii)

From (i) & (iii)

$$\underset{x\to 1}{\text{Lt}}\, f(x) = f(1)$$

Thus $f(x)$ continuous at x = 1.

Example 20.52 : Check the continuity of the following function at 2.

$$f(x) = \begin{cases} \frac{1}{2}(x^2-4) & \text{if } 0 < x < 2 \\ 0 & \text{if } x = 2 \\ 2-8x^{-3} & \text{if } x > 2 \end{cases}$$

$f(x)$ continuous at $x = 2$

if $\underset{x \to 2^-}{\text{Lt}} f(x) = \underset{x \to 2^+}{\text{Lt}} f(x) = 2$

Solution : LHL $\underset{x \to 2^-}{\text{Lt}} f(x)$

$$= \underset{x \to 2^-}{\text{Lt}} \frac{1}{2}(x^2-4)$$

Replace x by $2 - h$; $h \to 0$

$$= \underset{h \to 0}{\text{Lt}} \frac{1}{2}\left[(2-h)^2-4\right] = \underset{h \to 0}{\text{Lt}} \frac{1}{2}(4-4) = 0$$

RHL $\underset{x \to 2^+}{\text{Lt}} f(x)$

$$= \underset{x \to 2^+}{\text{Lt}} (2-8x^{-3})$$

Replace x by $2 + h$; $h \to 0$

$$= \underset{h \to 0}{\text{Lt}} 2-8(2+h)^{-3} = 2-8(+2)^{-3}$$

$$= \underset{h \to 0}{\text{Lt}} 2-8(2+h)^{-3} = 2$$

$f(2) = 0$

LHL $\neq$ RHL

Hence function is discontinuous at $x = 2$.

Example 20.53 : Show that $f(x) = \begin{cases} \cos ax - \cos bx & \text{if } x \neq 0 \\ \frac{1}{2}(b^2-a^2) & \text{if } x = 0 \end{cases}$

Where a and b are real constants, is continuous at O.

Solution : $\underset{x\to 0}{\text{Lt}} \dfrac{\cos ax - \cos bx}{x^2}$

$$= \underset{x\to 0}{\text{Lt}} \frac{2\sin\left(\frac{ax+bx}{2}\right)\sin\left(\frac{bx-ax}{2}\right)}{x^2}$$

$$= 2\ .\ \underset{x\to 0}{\text{Lt}} \frac{\sin(a+b)\frac{x}{2}}{x}\ \underset{x\to 0}{\text{Lt}} \frac{\sin(b-a)\frac{x}{2}}{x}$$

$$= \underset{x\to 0}{\text{Lt}}\ 2.\frac{\sin(b+a)\frac{x}{2}}{(b+a)\frac{x}{2}} \times \frac{(b+a)}{2} \times \underset{x\to 0}{\text{Lt}} \frac{\sin(b-a)\frac{x}{2}}{(b-a)\frac{x}{2}} \times \frac{(b+a)}{2}$$

$$= \underset{x\to 0}{\text{Lt}}\ 2.\frac{\sin(b-a)\frac{x}{2}}{(b-a)\frac{x}{2}} \times \left(\frac{b-a}{2}\right)\left(\frac{b+a}{2}\right)$$

$$= 2\ .\left(\frac{b+a}{2}\right).\left(\frac{b-a}{2}\right)$$

$$= \frac{1}{2}\left[b^2 - a^2\right]$$

Also $f(0) = \frac{1}{2}\left(b^2 - a^2\right)$

$f(x)$ is continuous at $x = 0$

Example 20.54 : Check the continuity of given by

$$f(x) = \begin{cases} 4 - x^2 & \text{if} \quad x \le 0 \\ x - 5 & \text{if} \quad 0 < x \le 1 \\ 4x^2 - 9 & \text{if} \quad 1 < x < 2 \\ 3x + 4 & \text{if} \quad x \ge 2 \end{cases}$$

at the points 0, 1 and 2.

Solution : LHL $\underset{x\to 0^-}{\text{Lt}} (4 - x^2)$

$$= \underset{h\to 0}{\text{Lt}}\ 4 - (0-h)^2 = \underset{h\to 0}{\text{Lt}}\ 4 = 4$$

RHL $\underset{x\to 0^+}{\text{Lt}}\ x - 5$ Replace x by 0 th

$$\underset{h\to 0}{\text{Lt}}\ 0+h-5 = -5$$

LHL $\neq$ RHL at $x = 0$

Hence function is discontinous at $x = 0$.

LHL $\underset{x\to 1^-}{\text{Lt}}\ x-5$ Replace x by $1 - h$.

$$\underset{h\to 0}{\text{Lt}}\ (1-h)-5 = -4$$

RHL $\underset{x\to 1^+}{\text{Lt}}\ 4x^2-9$

$$= \underset{h\to 0}{\text{Lt}}\ 4(1+h)^2-9 = -5$$

LHL $\neq$ RHL $f(x)$ is discontunuous at $x = 1$

RHL = $\underset{x\to 2^+}{\text{Lt}}\ 3x+5$

$$= \underset{h\to 0}{\text{Lt}}\ 3(2+h)+4$$

$$= 3(2 + 0) + 4$$

$$= 10$$

LHL = $\underset{h\to 2^-}{\text{Lt}}\ (4x^2-9)$

$$= \underset{h\to 0}{\text{Lt}}\ 4(2-h)^2-9 = 7$$

$\therefore$ LHL $\neq$ RHL

$\therefore$ $f(x)$ is discontinuous at $x = 2$

Example 20.55 : Find real constants a, b so that the function f given by

$$f(x) = \begin{cases} \sin x & \text{if} \quad x \le 0 \\ x^2+a & \text{if} \quad 0 < x \le 1 \\ bx+3 & \text{if} \quad 0 \le x \le 3 \\ -3 & \text{if} \quad x > 3 \end{cases}$$

is continuous on **R**.

Solution : If $f(x)$ is continuous at $x = 0$

$$\underset{x\to 0^-}{\text{Lt}} f(x) = \underset{x\to 0^+}{\text{Lt}} f(x) = 0$$

$$\underset{x\to 0^-}{\text{Lt}} \sin x = \underset{x\to 0^+}{\text{Lt}} x^2 + a = f(0)$$

$$0 = a = f(0)$$

$$\boxed{a = 0}$$

If $f(x)$ is continuous at $x = 3$

$$\underset{x\to 3^-}{\text{Lt}} (bx+3) = \underset{x\to 3^+}{\text{Lt}} -3 = f(3)$$

$$\underset{h\to 0}{\text{Lt}}\ b(3-h)+3 = -3$$

$$3b+3=-3 \Rightarrow 3b=-6$$

$$\boxed{b=-2}$$

Example 20.56 : Show that f given by $f(x) = \dfrac{x-|x|}{x}$ $(x \neq 0)$ is continuous on $\mathbf{R} - \{0\}$.

Solution : $f(x) = \dfrac{x-|x|}{x}$

$$\underset{x\to 0^+}{\text{Lt}} f(x) = \underset{h\to 0^+}{\text{Lt}} \frac{x-|x|}{x}$$

$$= \underset{x\to 0^+}{\text{Lt}}\ 1 - \frac{|x|}{x}$$

$$= \underset{x\to 0^+}{\text{Lt}}\ 1-1 = 0$$

$$\underset{x\to 0^-}{\text{Lt}} f(x) = \underset{x\to 0^-}{\text{Lt}} \frac{x-|x|}{x}$$

$$= \underset{x\to 0^-}{\text{Lt}} \frac{x-(-x)}{x} = \underset{x\to 0}{\text{Lt}} \frac{x+x}{x}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{2x}{2} = 2$$

$$\therefore \underset{x\to 0^+}{\text{Lt}} f(x) \neq \underset{x\to 0^-}{\text{Lt}} f(x)$$

$\therefore f(x)$ is discontinue at $x = 0$.

Example 20.57 : Find a, b so that the function f defined by

$$f(x) = \begin{cases} ax^2+9x-5 & \text{if} \quad x<1 \\ b & \text{if} \quad x=1 \\ (x+3)(2x-a) & \text{if} \quad x>1 \end{cases}$$

is continuous on R.

Solution : $\underset{x\to1^-}{\text{Lt}} f(x) = \underset{x\to1^+}{\text{Lt}} ax^2+9x-5$

$= a(1)^2 + 9(1) - 5$

$= a + 4$

$\underset{x\to1^+}{\text{Lt}} f(x) = \underset{x\to1}{\text{Lt}} (x+3)(2x-a)$

$= (1 + 3)\ (2 - a) = 8 - 4a$

Since $f(x)$ is continuous

LHL = RHL

$a + 4 = 8 - 4a$

$5a = 4 \Rightarrow a = \dfrac{4}{5}$

Also $f(x)$ is continuous on **R**

$\underset{x\to1}{\text{Lt}} f(1) = f(1) = b$

$a(1)^2 + 9(1) - 5 = b$

$a + 4 = b$

$\dfrac{4}{5}+\dfrac{4}{1} = b$

$\Rightarrow b = \dfrac{24}{5}$.

EXERCISE 20.3

1. If $f(x)=\begin{cases}4x+3 & x\neq 2\\ 3x+5 & x=2\end{cases}$. Find whether the function f is continuous at $x = 2$.

2. Examine the continuity of $f(x)$ at $x = 1$ where

$$f(x)=\begin{cases}x^2 & x\leq 12\\ x+5 & x>1\end{cases}$$

3. Exaimine the continuity

$$f(x)=\begin{cases}\frac{1}{x}-x, & 0<x<\frac{1}{2}\\ \frac{1}{2}, & x=\frac{1}{2}\\ \frac{3}{2}-x, & \frac{1}{2}<x<1\end{cases} \text{ at } x=\frac{1}{2}$$

4. For waht value of K will the function

$$f(x)=\begin{cases}\frac{x^2-16}{x-4} & \text{if } x\neq 4\\ \text{K} & \text{if } x=4\end{cases}$$ be continuous at $x = 4$?

5. Find a so that f defined by

$$f(x)=\begin{cases}ax+3 & \text{if } x<3\\ 3-x+2x^2 & \text{if } x\geq 3\end{cases}$$ is continuous on **R**.

6. Check the continuity of f given by $f(x)$

$$f(x)=\begin{cases}\frac{(x^2-a)}{(x^2-2x-3)} & \text{if } 0<x<5 \text{ and } x\neq 3\\ 1.5 & \text{if } x=3 \text{ at the point 3}\end{cases}$$

SUPPORTIVE WEBSITES

- http : //www.wikipedia.org
- http:// math world . wolfram.com

PRACTICE EXERCISE

1. $\underset{x\to 0}{\text{Lt}} = \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$. Find the value

2. $\underset{x\to 0}{\text{Lt}} = \dfrac{(x+k)^4 - x^4}{k(k+2x)}$. Find the value

3. Find LHL, RHL of the following

   if $f(x) = \begin{cases} -2x+3 & \text{if } x \le 1 \\ 3x-5 & \text{if } x > 1 \end{cases}$ as $x \to 1$.

4. Evalute $\underset{x\to 0}{\text{Lt}} = \dfrac{\sin 7x}{2x}$

5. Evalute $\underset{x\to 0}{\text{Lt}} = \dfrac{\sin 2x + 3x}{2x + \sin 3x}$.

6. For what value of *k*, will the function

   $f(x) = \begin{cases} \dfrac{x^2-16}{x-4} & \text{if } x \ne 4 \\ k & \text{if } x = 4 \end{cases}$ be continuous at $x = 4$?

7. Show that the function $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x \ne 0 \\ 2 \end{cases}$ is continuous at $x = 0$.

8. Determine the value of '*a*' so that the function *f*(*x*) defined by

   $f(x) = \begin{cases} \dfrac{a\cos x}{\pi - 2x}, & \text{if } x \ne \dfrac{\pi}{2} \\ 5, & \text{if } x = \dfrac{\pi}{2} \end{cases}$ is continuous.

ANSWERS

EXERCISE 20.1

1. a) 17 b) 0 c) 8

2. a) $\dfrac{3}{2}$ b) 2 c) $\dfrac{1}{2}$ d) 4 e) $\dfrac{q}{b}$

3. a) 2a b) $\frac{1}{4}$ c) $\frac{-5}{6}$ d) 0

e) 0 f) $\frac{3}{4}$ g) $\frac{3}{2}$

4. a) $\frac{1}{2}$ b) 2 c) 3 d) $\frac{1}{2a}$ e) $\left(\frac{m}{n}\right)^2$

5. a) ∞ b) 11

EXERCISE 20.2

1. a) $\underset{x \to 3^+}{\text{Lt}} f(x) = 9,\ \underset{x \to 3^-}{\text{Lt}} f(x) = 5;$ $f(x)$ does not exists at $\underset{x \to 3}{\text{Lt}}$

   b) $\underset{x \to 2^-}{\text{Lt}} f(x) = 2,\ \underset{x \to 2^+}{\text{Lt}} f(x) = -1$ and $f(x)$ does not exist.

2. a) $2b \cos a$ b) $\frac{1}{2}(b^2 - a^2)$ c) $\frac{2m^2}{n^2}$

   d) $\sin a - a \cos a$

3. a) $\frac{1}{2}$ b) $\frac{1}{2}$ c) $\frac{1}{2\sqrt{2}}$ 4. $a = -2$

5. 2 6. $\frac{4}{9}$ 7. $\frac{a}{b}$ 8. $\frac{2}{3}$

9. 0 10. 2

EXERCISE 20.3

1. Continuous **2.** Dis continuous **3.** Dis continuous

4. K = 8 **5.** $a = 5$ **6.** Dis continuous at $x = 3$.

PRACTICE EXERCISE

1. 1 2. x^2 3. 1, –2 4. $\frac{7}{2}$

5. 1 6. k = 8 8. 10

DIFFERENTIATION

Chapter

21

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- Define derivative of function or derivative of f and is denoted by f'. The process of finding the derivative of a function is called differentiation.
- Interprete Geometrically the derivative of a function at a point.
- The derivative of a constant function on an interval is zero.
- Define the derivating of the function from the first principle.
- Define the function $y = x^n$ then $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$. This is known "Newton's" Power Formula or Power Rule.
- Define the derivative of the sum and difference of two functions.
- Define the derivative of the product of two functions.
- Define the derivative of the reciprocal of a function OR Quotient Rule.

PREREQUISITES

- Knowledge of function and their types, domain and range of a function.
- Formulae for trigonometric functions of sum, difference, multiple and sub-multiples of angles.

MODULE - V
Calculus

Notes

INTRODUCTION

The differential calculus was introduced some time during 1665 or 1666, when Isaac Newton First concieved the process we now know as differentiation (a mathematical process and it yields a result called derivative). Among the discoveries of Newton and Leibnitz are rules for find derivatives of Sums, Products and Quotients of Composite Functions together with many other results. In this lesson we define a derivative of function, give its geometrical and physical interpretations, discuss various laws of derivatives and introduce notion of second order derivative of a function.

Let $f(x)$ be differentiable or derivable funtion on $[a, b]$. Then corresponding to each point $c \in [a, b]$ we obtain a unique real number equal to the derivative $f'(c)$ or $f(x)$ at x = C. This correspondence between the points in $[a, b]$ and derivatives at these points defines a new real valued function with domain $[a, b]$ and range a subset of **R**. Set of real numbers, such that the image of x in $[a, b]$ is the value of the derivative of f at 'x' ie $f'(x)$ or D$f(x)$. This function is called the derivative or differentiation of $f(x)$ with respect to x or simply differentiation of $f(x)$ and is denoted by $f'(x)$ or D $f(x)$ or $\frac{d}{dx}\left(f(x)\right)$.

Thus $$\frac{d}{dx}\left(f(x)\right) = \underset{h\to 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h} \quad ...(i)$$

OR $$\frac{d}{dx}\left(f(x)\right) = \underset{h\to 0}{\text{Lt}} \frac{f(x-h)-f(x)}{-h} \quad ...(ii)$$

The differentiation of derivative of a function $f(x)$ is called the differential coefficient of $f(x)$. But we shall be using the words differentiation or derivative only.

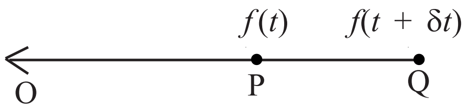
The symbols used like this

$$\frac{dy}{dx} = \underset{h\to 0}{\text{Lt}} \frac{f(x+h)-f(x)}{h}$$

Or $\frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

If $y = f(x)$, then $\frac{dy}{dx}$ is also denoted by y_1 or y'.

21.1 VELOCITY AS LIMIT



$OP = s = f(t)$

$OQ = OP + PQ = s + \delta s$

$= f(t + \delta t)$

The average velocity of the partticle in the interval δt is given by

$$= \frac{\text{Change in Distance}}{\text{Change in time}} = \frac{(s+\delta s) - s}{(t+\delta t) - t}$$

$$= \frac{f(t+\delta t) - f(t)}{\delta t}$$

Velocity at time $t = \underset{\delta t \to 0}{\text{Lt}} \frac{f(t+\delta t) - f(t)}{\delta t}$

it is denoted by $\frac{ds}{dt}$.

Example 21.1. The distance 'S' meters travelled in time 't' seconds by a car is given by the relation

$$S = 3t^2$$

Find the velocity of car at time $t = 4$ seconds.

Solution : Here $f(t) = s = 3t^2$

$f(t + \delta t) = s + \delta s = 3(t + \delta t)^2$

$$t = \underset{\delta t \to 0}{\text{Lt}} \frac{f(t+\delta t) - f(t)}{\delta t}$$

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{3(t+\delta t)^2 - 3t^2}{\delta t}$$

$$= \underset{\delta t \to 0}{\text{Lt}}\ 3(t^2 + \delta t^2 + 2t.\delta t) - 3t^2$$

$$= \underset{\delta t \to 0}{\text{Lt}} (6t + 3\delta t)$$

$= 6t.$

Velocity of car at $t = 4$ seconds

$= (6 \times 4)$ m/sec

$= 24$ m/sec.

Example 21.2 : Find the velocity of Particles moving along a straight line for the give time - distance relations at the indicated values of time t.

a) $s = 2 + 3t^2; \quad t = \frac{1}{3}$

Solution : $s = 2 + 3t^2$

$f(t) = s = 2 + 3t^2$

$f(t + \delta t) = s = s + \delta s = 2 + 3(t + \delta t)$

Velocity of car at any time

$$t = \underset{\delta t \to 0}{\text{Lt}} \frac{f(t+\delta t) - f(t)}{\delta t}$$

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{2 + 3(t+\delta t)^2 - (2 + 3t^2)}{\delta t}$$

$t = 3$

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{2 + 3\left[t^2 + 2t.\delta t + \delta t^2\right] - [2 + 3t^2]}{\delta t}$$

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{2+3t^2+6t.\delta t+3\delta t^2-2-3t^2}{\delta t}$$

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{6t\ \delta t}{\delta t}+\frac{3\ \delta t^2}{\delta t}$$

$$= 6t.$$

Velocity of car $t=\frac{1}{3}$ sec $\Rightarrow \left(6\times\frac{1}{3}\right)$ = 2 m/s.

Example 21.3 : S = δt – 7; t = 4

Solution : Here $f(t)$ = S = δt – 7 ; $f(t + \delta t)$ = 8(1 + δt) – 7

$$t = \underset{\delta t \to 0}{\text{Lt}} \frac{f(t+\delta t)-f(t)}{\delta t}$$

Velocity of car at any time

$$t = \underset{\delta t \to 0}{\text{Lt}} \frac{f(t+\delta t)-f(t)}{\delta t}$$

$$= \underset{\delta t \to 0}{\text{Lt}} \frac{8(t+\delta t)-7-[\delta t-7]}{\delta t}$$

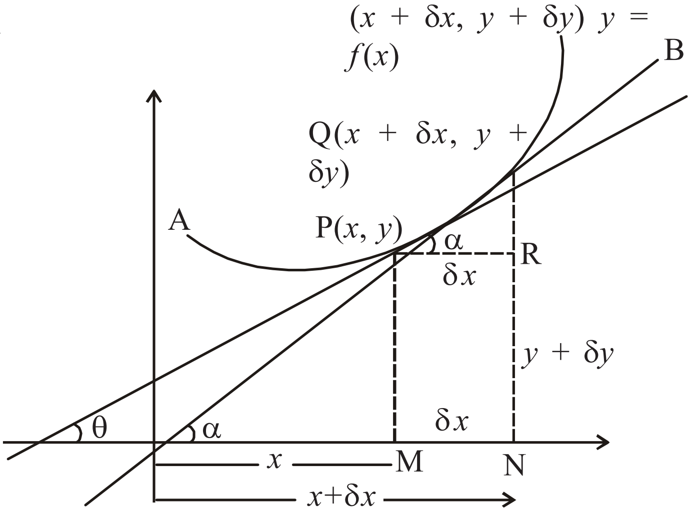
$$= \underset{\delta t \to 0}{\text{Lt}} \frac{8t+8\delta t-7-\delta t+7}{8t}$$

$$= \underset{\delta t \to 0}{\text{Lt}}\ 8$$

$$= 8$$

21.2 GEOMETRICAL INTERPRETATION OF $\frac{dy}{dx}$

Let P(x, y) be any pt on the graph $y = f(x)$. Let Q(x + δx, y + δy) be another point on the same curve in the neighbourhood of point P.



Draw PM, QN perpendicular to X-axis and PR is parallel to X-axis. Such that PR meets QN at R. Join QP and produce the secant line to any point S, QPS makes angle say α. With positive direction of X-axis. Draw PT tangent to the curve at the point P, making angle θ with the X-axis.

$\Delta QPR, \quad \angle QPR = \alpha$

$$\tan\alpha = \frac{QR}{PR} = \frac{QN - RN}{MN} = \frac{QN - PM}{ON - OM}$$

$$= \frac{(y+\delta y) - y}{(x+\delta x) - x} = \frac{\delta y}{\delta x}$$

$$\therefore \tan\alpha = \frac{\delta y}{\delta x}$$

$$\underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \alpha \to 0}}{\text{Lt}} \tan\alpha = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x}$$

Or $\quad \tan\theta = \dfrac{dy}{dx}$

$\frac{dy}{dx}$ of the function $y = f(x)$ at any, point P(x, y). On the curve represents the slope of 'GRADIENT' of the tangent at the point P.

This is called the Geometrical interpretation of $\frac{dy}{dx}$.

Corollary 1

If tangent to the curve at P is parallel to x-axis, then $\theta = 0°$ or $180°$, i.e., $\frac{dy}{dx} = \tan 0°$ or $\tan 180°$ i.e., $\frac{dy}{dx} = 0$.

That is tangent to the curve represented by $y = f(x)$ at P is parallel to x-axis.

Corollary 2

If tangent to the curve at P is perpendicular to x-axis, $\theta = 90°$ or $\frac{dy}{dx} = \tan 90° = \infty$.

That is, the tangent to the curve represented by $y = f(x)$ at P is parallel to y-axis.

Example 21.4 : Find the derivative

i) x^{10} ii) x^{50} iii) x^{91}

Solution :

i) $\frac{d}{dx}(x^{10}) = 10x^{10-1} = 10x^9$

ii) $\frac{d}{dx}(x^{50}) = 50x^{50-1} = 50x^{49}$

iii) $\frac{d}{dx}(x^{91}) = 91x^{91-1} = 91x^{90}$

Example 21.5 : Find derivative $y = \sqrt{x}$

Solution : $y = x^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(x)^{\frac{1}{2}-1} \times 1 = \frac{1}{2} \times (x)^{-\frac{1}{2}}$$

$$= \frac{1}{2} \times \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2} \times \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Example 20.6 : Find derivate of $\frac{1}{x}$.

Solution : $y = \frac{1}{x} \Rightarrow y = x^{-1}$

$$\frac{dy}{dx} = -1(x)^{-1-1} \times 1 = -1(x)^{-2} = \frac{-1}{x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{x^2}}$$

Example 21.7 : If $y = \sqrt[3]{x^2 + 5x - 7} = \left(x^2 + 5x + 7\right)^{\frac{1}{3}}$ find y'.

Solution : $y' = \frac{1}{3}\left[x^2 + 5x - 7\right]^{\frac{1}{3}-1}\left[2x + 5 + 0\right]$

$$y' = \frac{(2x+5)}{3}\left[x^2 + 5x + 7\right]^{\frac{1}{3}-1}$$

$$y' = \frac{2x+5}{3}\left[x^2 + 5x + 7\right]^{-\frac{2}{3}}$$

Example 21.8 : $y = (x^2 + 1)^2$ Find $\frac{dy}{dx}$.

Sol. $y = (x^2 + 1)(x^2 + 1)$

$y = f(x) \,.\, g(x)$

$y' = f'(x)g(x) + g'(x)f(x)$

$$\frac{dy}{dx} = (x^2+1)(2x) + (x^2+1)2x$$

$$= (x^2 + 1)\ (2x + 2x)$$

$$\frac{dy}{dx} = 4x(x^2+1)$$

Example 21.9 : $f(x) = 3\sqrt{x}$ Find $f'(x)$ using delta method.

Solution : $y = f(x) = 3\sqrt{x}$

$$y + \delta y = 3\sqrt{x+\delta x}$$

$$y + \delta y - y = 3\sqrt{x+\delta x} - 3\sqrt{x}$$

$$\delta y = 3\left(\sqrt{x+\delta x} - \sqrt{x}\right)$$

Rationalize the Nuemorator.

$$\delta y = \frac{3\left(\sqrt{x+\delta x} - \sqrt{x}\right)}{\sqrt{x+\delta x}+\sqrt{x}} \times \sqrt{x+\delta x}+\sqrt{x}$$

$$\delta y = \frac{3(x+\delta x - x)}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$\delta y = \frac{3\delta x}{\left(\sqrt{x+\delta x}+\sqrt{x}\right)} \qquad \therefore \frac{\delta y}{\delta x} = \frac{3}{\sqrt{x+\delta x}+\sqrt{x}}$$

$$\underset{\delta x\to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x\to 0}{\text{Lt}} \frac{3}{\left(\sqrt{x+\delta x}+\sqrt{x}\right)}$$

$$\frac{dy}{dx} = \frac{3}{\sqrt{x+0}+\sqrt{x}} = \frac{3}{2\sqrt{x}}$$

$$\therefore\ f'(x) = \frac{3}{2\sqrt{x}}$$

$$\therefore\ f'(2) = \frac{3}{2\sqrt{2}}$$

21.3 DERIVATIVE OF CONSTANT FUNCTION

Statement : The derivative of a constant is zero.

Proof : Let $y = c$ be a constant function. Then $y = c$ can be written as

Let $y = c \Rightarrow y = cx^0$ ...(i) $[\because x^0 = 1]$

$y + \delta y = c(x + \delta x)^0$ ...(ii) $[\because x^0 = 1]$

$(y + \delta y) - y = c\,(x + \delta x)^0 - cx^0$

$\delta y = c - c$ or $\delta y = 0$

$$\frac{\delta y}{\delta x} = \frac{0}{\delta x} = 0$$

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = 0 \quad \text{or} \quad \frac{dy}{dx} = 0$$

$\therefore$ Derivative of a constant quantity is zero.

Example 21.10 : $y = x^n$, find $\frac{dy}{dx}$.

Solution: Let $y = x^n$

$$y + \delta y = (x + \delta x)^n$$

$$(y + \delta y) - y = (x + \delta x)^n - x^n$$

$$\delta y = x^n\left(1 + \frac{\delta x}{x}\right)^n - x^n$$

$$= x^n\left[\left(1 + \frac{\delta x}{x}\right)^n - 1\right].$$

Expanding $\left(1 + \frac{\delta x}{x}\right)^n$ by Binomial theorem, we have

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \frac{dy}{dx} = x^n\left[\frac{n}{x} + 0 + 0 + ...\right]$$

$$\frac{dy}{dx} = \frac{nx^n}{x} = nx^{n-1}$$

$$\therefore \frac{d}{dx}(x^n) = nx^{n-1} \qquad [\because y = x^n]$$

This is known Newton's Power Formula

Or

Power Rule

21.4 DERIVATIVE OF A FUNCTION FROM FIRST PRINCIPLE

Recalling the definition of derivative of a function at a point, we have the following working rule for finding the derivative of a function from first principle:

Let $y = f\,x \Rightarrow y + \delta y = f(x + \delta x)$

$$\delta y = f(x + \delta x) - f(x)$$

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Lety $y = x^n$ .....(i)

For a small increment δx in x, let the corresponding increment in y be δy.

Then $y + \delta y = (x + \delta x)^n$ ...(ii)

Subtracing (i) from (ii) we have,

$$(y + \delta y) - y = f(x + \delta x)^n - x^n$$

$$\delta y = x^n \left(1 + \frac{\delta x}{x}\right)^n - x^n$$

$$= x^n\left[\left(1+\frac{\delta x}{x}\right)^n - 1\right]$$

Since $\frac{\delta x}{x} < 1$ as δx is a small quantity compared to x, we can expand $\left(1+\frac{\delta x}{x}\right)^n$ by Binomial theorem for any index.

Expanding $\left(1+\frac{\delta x}{x}\right)^n$ by Binomial theorem, we have

$$\delta y = x^n\left[1+n\left(\frac{\delta x}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{\delta x}{x}\right)^3+....-1\right]$$

$$= x^n(\delta x)\left[\frac{n}{x}+\frac{n(n-1)}{2!}\frac{\delta x}{x^2}+\frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3}+....\right]$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x} = x^n\left[\frac{n}{x}+\frac{n(n-1)}{2!}\frac{\delta x}{x^2}+\frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3}+....\right]$$

Proceeding to limit when $\delta x \to 0$, $(\delta x)^2$ and higher powers of δx will also tend to zero.

$$\therefore \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} x^n\left[\frac{n}{x}+\frac{n(n-1)}{2!}\frac{\delta x}{x^2}+\frac{n(n-1)(n-2)}{3!}\frac{(\delta x)^2}{x^3}+....\right]$$

or $$\operatorname*{lt}_{\delta x\to 0}\frac{\delta y}{\delta x} = \frac{dy}{dx} = x^n\left[\frac{n}{x}+0+0+....\right]$$

or $$\frac{dy}{dx} = x^n\cdot\frac{n}{x} = nx^{n-1}$$

or $$\frac{dy}{dx} = x^n\cdot\frac{n}{x} = nx^{n-1} \qquad [\because\ y = x^n]$$

This is known as Newton's Power Formula or Power Rule

Note: We can apply the above formula to find derivative of functions like x, x^2, x^3, ...

i.e. when n = 1,2,3,...

e.g. $$\frac{d}{dx}x = \frac{d}{dx}x^1 = 1x^{1-1} = 1x^0 = 1.1 = 1$$

$$\frac{d}{dx}x^2 = 2x^{2-1} = 2x$$

$$\frac{d}{dx}\left(x^2\right) = 3x^{3-1} = 3x^2$$, and so on.

Example 20.11 : $F(x) = x + \frac{1}{x}$ then find $f'(x)$ in the form of definition method.

Solution : $y = x + \frac{1}{x}$ say $\Rightarrow y + \delta y = (x + \delta x) + \frac{1}{x + \delta x}$

$$y + \delta y - y = x + \delta x + \frac{1}{x + \delta x} - \left(x + \frac{1}{x}\right)$$

$$= x + \delta(x) + \frac{1}{x + \delta x} - x - \frac{1}{x}$$

$$= \frac{\delta(x)}{1} + \frac{1}{x + \delta x} - \frac{1}{x}$$

$$= \frac{\delta x(x + \delta x)x + x - (x + \delta x)}{x(x + \delta x)}$$

$$= \frac{\delta x(x + \delta x)x + x - x - \delta x}{x(x + \delta x)}$$

$$\delta y = \frac{\left[\delta x(x + \delta x)x - 1\right]}{x(x + \delta x)}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{\delta x(x^2 + x\delta x - 1]}{x(x + \delta x)\delta x}$$

$$\frac{\delta y}{\delta x} = \frac{x^2 + x\delta x - 1}{x(x+\delta x)}$$

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{x^2 + x\delta x - 1}{x(x+\delta x)} = \frac{x^2 - 1}{x^2} = 1 - \frac{1}{x^2} \qquad \therefore \boxed{\frac{dy}{dx} = 1 - \frac{1}{x^2}}$$

21.5 ALGEBRA OF DERIVATIVES

Many functions arise as combinations of other functions. The combination could be sum, difference, product or quotient of functions. We also come across situations where a given function can be expressed as a function of a function.

In order to make derivative as an effective tool in such cases, we need to establish rules for finding derivatives of sum, difference, product, quotient and function of a function. These, in turn, will enable one to find derivatives of polynomials and algebraic (including rational) functions.

21.6 DERIVATIVES OF SUMAND DIFFERENCE OF FUNCTIONS

If $f(x)$ and $g(x)$ are both derivable functions and $h(x) = f(x) + g(x)$, then what is $h'(x)$?

Here $h(x) = f(x) + g(x)$

Let δx be the increment in x and δy be the correponding increment in y.

$$h(x + \delta x) = f(x + \delta x) + g(x + \delta x)$$

Hence $$h'(x) = \lim_{\delta x \to 0} \frac{[f(x+\delta x) + g(x+\delta x)][f(x) - g(x)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[f(x+\delta x) - f(x)] + [g(x+\delta x) - g(x)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x} + \frac{g(x+\delta x) - g(x)}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} + \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}$$

or $\quad h'(x) = f'(x) + g'(x)$

Thus we see that the *derivative of sum of two functions is sum of their derivatives*.

This is called the **SUM RULE**.

e.g. $\quad y = x^2 + x^3$

Then $\quad y' = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3)$

$$= 2x + 3x^2$$

Thus $y' = 2x + 3x^2$

This sum rule can easily give us the difference rule as well, because

if $\quad h(x) = f(x) - g(x)$

then $h(x) = f(x) - [g(x)]$

$\therefore \; h'(x) = f'(x) \, [-g'(x)]$

$$= f'(x) - g'(x)$$

i.e. *the derivative of difference of two functions is the difference of their derivatives*.

This is called **DIFFERENCE RULE.**

Thus we have

Sum rule : $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

Difference rule : $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

Derivatives of sum and difference of functions

$$h(x) = f(x) + g(x)$$

$$h(x + \delta x) = f(x + \delta x) + g(x + \delta x)$$

$$h'(x) = \underset{\delta x \to 0}{\text{Lt}} \frac{[f(x+\delta x)+g(x+\delta x)]-[f(x)+g(x)]}{\delta x}$$

we get

$h'(x) = f'(x) + g'(x)$ Sum Rule

$h'(x) = f'(x) - g'(x)$ Difference Rule

Example 21.12 : $y = x^2 + x^3$ Find $\frac{dy}{dx}$.

Solution : $y' = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3)$

$$y' = 2x + 3x^2$$

Example 21.13 : $y = 10t^2 + 20t^3$. Find $\frac{dy}{dt}$.

Solution : $\frac{dy}{dt} = 10(t^2)' + 20(t^3)'$

$$= 10(2t) + 20(3t^2)$$

$$\frac{dy}{dt} = 20t + 60t^2$$

Example 21.14 : $y = x^3 + \frac{1}{x^2} - \frac{1}{x}$ Find $\frac{dy}{dx}$.

Solution : $y = x^3 + x^{-2} - (x^{-1})$

$$\frac{dy}{dx} = 3x^2 + (-2)x^{-3} - (-1)x^{-2}$$

$$\frac{dy}{dx} = 3x^2 - 2x^{-3} + x^{-2}$$

$$\frac{dy}{dx} = 3x^2 - \frac{2}{x^3} + \frac{1}{x^2}$$

Example 21.15 : $y = x^3 + 3x^2 + 4x + 5, \quad x = 1$ Find $\frac{dy}{dx}$.

Solution : $\frac{dy}{dx} = \frac{d}{dx}\left[x^3 + 3x^2 + 4x + 5\right] = 3x^2 + 6x + 4$

$$\left.\frac{dy}{dx}\right]_{x=1} = 3(1)^2 + 6(1) + 4 = 13 .$$

Example 21.16 : $f(x) = \frac{x^8}{8} - \frac{x^6}{6} + \frac{x^4}{4} - 2$ then find $f'(x)$.

Sol: $f(x) = \frac{x^8}{8} - \frac{x^6}{6} + \frac{x^4}{4} - 2$

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$$

$$f'(x) = \frac{1}{8}\left(\frac{d}{dx}\right)x^8 - \frac{1}{6}\frac{d}{dx}(x^6) + \frac{1}{4}\frac{d}{dx}(x^4) - 2$$

$$f'(x) = \frac{1}{8} \times 8.x^{8-1} - \frac{1}{6}6x^{6-1} + \frac{1}{4}.4x^{4-1} - \frac{d}{dx}(2)$$

$$f'(x) = x^7 - x^5 + x^3$$

21.7 DERIVATIVE OF PRODUCT OF FUNCTIONS

You are all familiar with the four fundamental operations of Arithmetic : addition, subtraction, multiplication and division. Having dealt with the sum and the difference rules, we now consider the derivative of product of two functions.

Example 21.17 : Let $y = (x^2 + 1)^2$

$$y = (x^2 + 1)\ (x^2 + 1)$$

$$\frac{d}{dx}\left[f(x)g(x)\right] = \left[f(x)g'(x) + g(x)f'(x)\right]$$

OR $\boxed{\frac{d}{dx}(uv) = uv' + vu' \quad \text{Product Rule}}$

$$\frac{dy}{dx} = \left[(x^2+1)(x^2+1)' + (x^2+1)(x^2+1)'\right]$$

$$= 2x(x^2 + 1) + 2x(x^2 + 1)$$

$$\frac{dy}{dx} = 4x(x^2+1)$$

Remark : If $f(x)$, $g(x)$ and $h(x)$ are three given functions of x, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f(x)g(x)\frac{d}{dx}h(x) + g(x)h(x)\frac{d}{dx}f(x) +$$

$$h(x)f(x)\frac{d}{dx}g(x)$$

Example 21.18 : If $f(x) = \frac{x^2 - a}{a-2}$ $a \neq 2$. Find Derivative $f'(x)$.

Solution : $f(x) = \frac{x^2-a}{a-2} \Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x^2-a}{a-2}\right)$

$$= \frac{1}{a-2}\frac{d}{dx}(x^2-a)$$

$$= \frac{1}{a-2}(2x) = \frac{2x}{a-2}$$

$$f'(x) = \frac{2x}{a-2}$$

Example 21.19 : If $y = (5x - 3)^7$ Find y'

Solution : $y' = 7(5x-3)^{7-1}\ [5\times 1 - 0] = 7[5x-3]^6 \times 5$

$y' = 35(5x - 3)^6$.

Example 21.20 : $y = 5x^6(7x^2 + 4x)$ then find $\frac{dy}{dx}$.

Solution : Product of Two functions.

$$y = 5x^6\ (7x^2 + 4x)$$

$$y = f(x)\ g(x) \Rightarrow y' = f(x)g'(x) + g(x)f'(x)$$

$$\frac{dy}{dx} = 5x^6 . \frac{d}{dx}(7x^2 + 4x) + (7x^2 + 4x)\ \frac{d}{dx}(5x^6)$$

$$\frac{dy}{dx} = 5x^6\ (14x + 4) + (7x^2 + 4x)\,(5 \times 6x^5)$$

$$\frac{dy}{dx} = 70x^7 + 20x^6 + 210x^7 + 120x^6$$

$$\therefore\ \frac{dy}{dx} = 280x^7 + 140x^6$$

Example 21.21 : $f(x) = (x + 1)\ (-3x - 7)$ then find $f'(x)$.

Solution : Product of Two function.

$$f'(x) = f(x)g'(x) + g(x)f'(x) \text{ Product principle}$$

$$f'(x) = (x+1)\frac{d}{dx}(-3x-7) + (-3x-7)\frac{d}{dx}(x+1)$$

$$f'(x) = (x+1)(-3-0) - 3x - 7(1+0)$$

$$f'(x) = -3x - 3 - 3x - 7$$

$$f'(x) = -6x - 10 = -\ 2(3x + 5)$$

Example 21.22 : $f(x) = (x - 1)\ (x - 2)\ (x - 3)$ then find $f'(x)$.

Solution : Given function in the form of

$y = f(x)\ g(x)\ h(x)$ product form

$$y' = f(x)g(x)\frac{d}{dx}(hx) + f(x)h(x)\frac{d}{dx}(g(x)) + g(x)h(x)\frac{d}{dx}(f(x))$$

$$f'(x) = (x-1)(x-2)\frac{d}{dx}(x-3) + (x-1)(x-3)\frac{d}{dx}(x-2)$$
$$+(x-2)(x-3)\frac{d}{dx}(x-1)$$

$$f'(x) = (x-1)(x-2)(1) + (x-1)(x-3)(1) + (x-2)(x-3)(1)$$

$$f'(x) = x^2 - 2x - x + 2 + x^2 - 3x - x + 3 + x^2 - 3x - 2x + 6$$

$$f'(x) = 3x^2 - 12x + 11.$$

Example 21.23 : $f(x) = x(x-3)(x^2+x)$ Find $f'(x)$.

Solution : It is in the form of

$$f'(x) = f(x)g(x)\frac{d}{dx}(hx) + f(x)h(x)\frac{d}{dx}(gx) + f(x)g(x)\frac{d}{dx}f(x)$$

Now we can solve it

$$f(x) = x(x-3)(x^2+x)$$

$$f'(x) = x(x-3)\frac{d}{dx}(x^2+x) + x(x^2+x)\frac{d}{dx}(x-3) + (x-3)(x^2+x)\frac{d}{dx}x$$

$$f'(x) = (x^2-3x)(2x+1) + (x^3+x^2)(1) + (x^3+x^2-3x^2-3x)1$$

$$f'(x) = 2x^3 - 6x^2 + x^2 - 3x + 2x^3 - 3x^2 + 2x^2 - 3x$$

$$f'(x) = 4x^3 - 6x^2 - 6x.$$

21.8 QUOTIENT RULE

You have learnt sum Rule, Difference Rule and Product Rule to find derivative of a function expressed respectively as either the sum or difference or product of two functions. Let us now take a step further and learn the

"Quotient Rule for finding derivative of a function which is the quotient of two functions.

Let $g(x) = \frac{1}{r(x)}$ $\quad (r(x) \neq 0)$

$$g(x) = \frac{1}{r(x)}$$

$$g'(x) = \underset{\delta x \to 0}{\text{Lt}} \left[\frac{\frac{1}{r(x+\delta x)} - \frac{1}{r(x)}}{\delta x} \right]$$

We get

$$= -r'(x) \cdot \frac{1}{[r(x)]^2} = -\frac{r'(x)}{[r(x)]^2}$$

Consider any two functions $f(x)$ and $g(x)$ such that

$$Q(x) = \frac{f(x)}{g(x)}$$

$$Q(x) = f(x) \cdot \frac{1}{g(x)}$$

Using product rule

$$Q'(x) = f(x) \cdot \frac{1}{g(x)'} + \frac{1}{g(x)} \cdot f'(x)$$

$$= \frac{g(x)\ f'(x) - f(x)g'(x)}{\left[g(x)]^2 \right]}$$

$$Q'(x) \quad \text{Or} \quad \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{(v)^2} \quad \text{Quotient Rule}}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}.$$

Example 21.24 : Find $f'(x)$ if $f(x) = \frac{4x+3}{2x-1}, x \neq \frac{1}{2}$

Solution : Here is Quotient Method $\frac{u}{v}$ method.

$$f(x) = u.v \Rightarrow f'(x) = \frac{vu' - uv'}{v^2}$$

$$f(x) = \frac{4x+3}{2x-1} = \frac{u}{v}$$

$$f'(x) = \frac{(2x-1)(4x+3)' - (4x+3)(2x-1)'}{(2x-1)^2}$$

$$f'(x) = \frac{-10}{(2x-1)^2}$$

Example 21.25 : $f(x) = \frac{4x+3}{(2x-1)}, \ x \neq \frac{1}{2}$ then find $f'(x)$.

Solution : Here given function is Quotient form.

$$y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y = \frac{4x+3}{2x-1}$$

$$\frac{dy}{dx} = \frac{(2x-1)\frac{d}{dx}(4x+3) - (4x+3)\frac{d}{dx}(2x-1)}{(2x-1)^2}$$

$$\frac{dy}{dx} = \frac{(2x-1)(4) - (4x+3)(2)}{(2x-1)^2}$$

$$\frac{dy}{dx} = \frac{8x-4-8x-6}{(2x-1)^2} = \frac{-10}{(2x-1)^2}.$$

Example 21.26 : $f(x) = \frac{\sqrt{x}}{x^3+4}$ then find $f'(x)$.

Solution : Here it is in the form of Quotient

$$y = \frac{f(x)}{g(x)} \Rightarrow y' = \frac{g(x)f'(x) - f(x)g'(x)}{(gx)^2}$$

$$y = \frac{\sqrt{x}}{x^3+4}$$

$$\frac{dy}{dx} = \frac{(x^2+4)\frac{d}{dx}(\sqrt{x}) - (\sqrt{x})\frac{d}{dx}(x^3+4)}{(x^3+4)^2}$$

$$\frac{dy}{dx} = \frac{x^2+4\times\frac{1}{2\sqrt{x}} - (\sqrt{x})\ (3x^2)}{(x^3+4)^2}$$

$$\frac{dy}{dx} = \frac{\frac{x^3+4}{2\sqrt{x}} - \frac{3x^2(\sqrt{x})}{1}}{(x^3+4)^2}$$

$$\frac{dy}{dx} = \frac{x^3+4-6x^2(x)}{2\sqrt{x}(x^3+4)^2} = \frac{x^3-6x^3+4}{2\sqrt{x}(x^3+4)^2}$$

$$\frac{dy}{dx} = \frac{4-5x^3}{2\sqrt{x}(x^3+4)^2}$$

21.9 CHAIN RULE

Earlier, we have come across functions of the type $\sqrt{x^4+8x^2+1}$. This function can not be expened as a SUM, Difference, product, or a Quotient of two functions, there the techniques developed so far do not help us find the derivtive of such a function. Thus we need to develop a rule to find the derivative of such a function.

Let δt be the increment in t and δy, the corresponding increment in y.

Then $\delta y \to 0$ as $\delta t \to 0$

$$\therefore \frac{dy}{dt} = \lim_{\delta x \to 0} \frac{\delta y}{\delta t}$$

Similarly t is a function of x.

$$\therefore \delta t \to 0 \text{ as } \delta x \to 0$$

$$\therefore \frac{dt}{dx} = \lim_{\delta x \to 0} \frac{\delta t}{\delta x}$$

Here y is a function of t and t is a function of x. Therefore $\delta y \to 0$ as $\delta x \to 0$

From (i) and (ii), we get

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \left[\lim_{\delta x \to 0} \frac{\delta y}{\delta t}\right]\left[\lim_{\delta x \to 0} \frac{\delta t}{\delta x}\right]$$

$$= \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Thus $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

This is called the **Chain Rule.**

Example 21.27 : Let us write : $y = \sqrt{x^4 + 8x^2 + 1}$

or $y = \sqrt{t}$ where $t = x^4 + 8x^2 + 1$

$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$ This is called Chain Rule.

$y = \sqrt{x^4 + 8x^2 + 1}$; $y = \sqrt{t}$ where $t = x^4 + 8x^2 + 1$

$$\frac{dy}{dt} = \frac{1}{2\sqrt{t}} \text{ and } \frac{dt}{dx} = 4x^3 + 16x \quad \therefore \frac{dy}{dx} = \frac{1}{2\sqrt{t}}(4x^3 + 16x)$$

$$\frac{dy}{dx} = \left(\frac{4x^3 + 16x}{2\sqrt{x^4 + 8x^2 + 1}}\right) = \frac{2x^3 + 8x}{\sqrt{x^4 + 8x^2 + 1}}.$$

Example 21.28 : $y = \sqrt{x^4 + 8x^2 + 1}$ then find $\frac{dy}{dx}$.

Solution : Here it is the form of $y = \sqrt{t}$ $t = x^4 + 8x^2 + 1$

$$y = \sqrt{t} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{t}},$$

$$t = x^4 + 8x^2 + 1$$

$$\frac{dt}{dx} = 4x^3 + 16x + 0 = 4x^3 + 16x$$

But $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{1}{2\sqrt{t}} \times (4x^3 + 16x)$$

$$\frac{dy}{dx} = \frac{4x^3 + 16x}{2\sqrt{x^4 + 8x^2 + 1}}$$

Example 21.29 : $y = at^2$, $t = \frac{x}{2a}$ then Find $\frac{dy}{dx}$.

Solution : $y = at^2 \Rightarrow \frac{dy}{dx} = 2at$ ...(i)

$$t = \frac{x}{2a} \Rightarrow \frac{dt}{dx} = \frac{1}{2a}$$ ...(ii)

From (i) & (ii)

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2at \times \frac{1}{2a} = t$$

$$\therefore \frac{dy}{dx} = t = \frac{x}{2a}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{x}{2a}}$$

Example 21.30 : $y=\sqrt{\frac{1+x}{1-x}}$ Find y' using chain pricipal.

Solution : $y = \sqrt{\frac{1+x}{1-x}} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right)$

$$= \frac{1}{2\sqrt{\frac{1+x}{1-x}}} \times \frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1}{2}\sqrt{\frac{1+x}{1-x}} \times \left[\frac{(1-x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1-x)}{(1-x)^2}\right]$$

$$= \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\left[\frac{1-x+1+x}{(1-x)^2}\right] = \frac{1}{2}\sqrt{\frac{1-x}{1+x}} \times \frac{2}{(1-x)^2}$$

$$\frac{dy}{dx} = \frac{1}{(\sqrt{1+x})\,(1-x)^{3/2}}$$

21.10 DERIVATIVES OF A FUNCTION OF SECOND ORDER

Given y is a function of x say $f(x)$.

If the derivative $\frac{dy}{dx}$ is a derivable functon of x.

Then the derivative of $\frac{dy}{dx}$ is known on the second derivative of $y=f(x)$ with respect to x and is denoted by $\frac{d^2y}{dx^2}$, Other symbols used for the second derivative of y are D^2, f'', y'' y_2 etc.

$$\therefore\ f''(x) = \underset{h\to 0}{\text{Lt}} \frac{f'(x+h)-f'(h)}{h}$$

OR $$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}.$$

Example 21.31 : Find the second order derivative

i) $y = x^2$. Find y''

Solution : $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow \frac{d^2y}{dx} = 2$

ii) $y = x^3 + 1$. Find y''

Solution : $\frac{dy}{dx} = 3x^2 + 0$

$$\frac{d^2y}{dx} = y'' = \frac{d}{dx}(3x^2) = 6x$$

iii) $y = \frac{x+1}{x-1}$ Then find y''.

$$y' = \frac{(x-1)(x+1)' - (x+1)(x-1)'}{(x-1)^2}$$

$$y' = \frac{(x-1)(1) - (x+1)1}{(x-1)^2} = \frac{x-1-x-1}{(x-1)^2}$$

$$y' = \frac{-2}{(x-1)^2}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{-2}{(x-1)^2}\right] = -2\left[(x-1)^2\right]^{-1}$$

$$= (-2)\ [-2]\ (x-1)^{-2-1} \times (1)$$

$$\frac{d^2y}{dx^2} = 4(x-1)^{-3} = \frac{4}{(x-1)^3}.$$

Example 21.32 : $y = (x^2+1)(x-1)$ then find y''.

Sol. $\frac{dy}{dx} = x^2 + 1\left(\frac{d}{dx}\right)(x-1) + (x-1)\frac{d}{dx}(x^2+1)$

$$\frac{dy}{dx} = (x^2+1)(1) + (x-1)(2x)$$

$$\frac{dy}{dx} = 3x^2 - 2x + 1$$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 - 2x + 1)$

$$\boxed{\frac{d^2y}{dx^2} = 6x - 2}$$

Example 21.33 : $y = \frac{x^2+1}{x+1}$ find $\frac{d^2y}{dx^2}$.

Solution : $y = \frac{x^2+1}{x+1}$ it is in the form of $\frac{f(x)}{g(x)}$

$$y' = \frac{f(x)}{g(x)} = \frac{f(x)\ f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(x+1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$y' = \frac{2x(x+1) - (x^2+1)}{(x+1)^2}$$

$$y' = \frac{2x^2 + 2x - x^2 - 1}{(x+1)^2}$$

$$y' = \frac{x^2 + 2x - 1}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{x^2 + 2x - 1}{(x+1)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{x^2 + 2x - 1}{(x+1)^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{(x+1)^2 . \frac{d}{dx}(x^2 + 2x - 1) - (x^2 + 2x - 1)\frac{d}{dx}(x+1)^2}{\left[(x+1)^2\right]^2}$$

$$\frac{d^2y}{dx^2} = \frac{(x+1)^2(2x+2)-(x^2+2x-1)2(x+1).1}{\left[(x+1)^2\right]^2}$$

$$\frac{d^2y}{dx^2} = \frac{2(x+1)\left[x^2+1+2x-x^2-2x+1\right]}{(x+1)^4}$$

$$\frac{d^2y}{dx^2} = \frac{4}{(x+1)^3}.$$

Importent Principles

1. $f'(x) = \underset{\delta x \to}{\text{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x} \quad \delta x > 0$

2. The derivative of constant is zero $\frac{dc}{dx} = 0$

3. Newton's Power Formula

   $\frac{d}{dx}(x^n) = nx^{n-1}$

4. $\frac{d}{dx}\left[C.f(x)\right] = C.f'(x)$

5. $\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$

6. $\frac{d}{dx}\left[f(x).g(x)\right] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

7. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}$

8. $\frac{d}{dx}\left[f\{g(x)\}\right] = f'(x)g(x).\frac{d}{dx}\left[g(x)\right]$

9. $\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}.$

Example 21.34 : If $y = \frac{1}{x^2+3x+1}$. Find y'.

Solution : $y = \frac{1}{x^2+3x+1} \Rightarrow (x^2+3x+1)^{-1}$

$$y = (x^2+3x+1)^{-1}$$

$$y' = -1(x^2+3x+1)^{-1-1}\,[2x+3+0]$$

$$y' = -1(x^2+3x+1)^{-2}\,(2x+3) = \frac{-(2x+3)}{(x^2+3x+1)^2}$$

$$y' = \frac{-(2x+3)}{(x^2+3x+1)^2}.$$

EXERCISE 21.1

1. $y = \sqrt{x}$ then Find $\frac{dy}{dx}$
2. $y = 12$ then find $\frac{dy}{dx}$.
3. $y = 2x^3 - 3x^2$ then find $\frac{dy}{dx}$.
4. $y = x^3 + \frac{1}{x^2} - \frac{1}{x}$, $x \neq 0$ then find $\frac{dy}{dx}$.
5. $y = \sqrt{x} - \frac{1}{\sqrt{x}}$ then find $\frac{dy}{dx}$
6. $y = 16x + 2$ then find $y'(0)$, $y'(3)$, $y'(8)$
7. $y = \frac{ax+b}{cx+d}$, $x \neq \frac{-d}{c}$ find $\frac{dy}{dx}$
8. $y = x + \frac{1}{x}$, $x \neq 0$ find $\frac{dy}{dx}$

9. $y = x^2 + x^3$ then find $\frac{dy}{dx}$

10. $y = 10t^2 + 20t^3$ then find $\frac{dy}{dx}$

11. $S = 4.9t^2 + 2.4$ where $t = 1$, $t = 5$ then find $\frac{ds}{dt}$ at $t = 1$; $\frac{ds}{dt}$ at $t = 5$?

12. $y = x^3 + 3x^2 + 4x + 5$ the find $\frac{dy}{dx}$ at $x = 1$ value ?

13. $f(x) = \frac{2}{5}x^{2/3} - x^{\frac{-4}{5}} + \frac{3}{x^2}$ find $f'(x)$?

14. $y = (x - 1)(x - 2)$ find $\frac{dy}{dx}$

15. $y = \frac{3x-2}{x^2+x-1}$ find $\frac{dy}{dx}$.

16. $y = \frac{1}{\sqrt{7-3x^2}}$ find $\frac{dy}{dx}$

17. $y = x^3 + 1$ find $\frac{d^2y}{dx^2}$

18. $y = \sqrt{x^2+1}$ find $\frac{d^2y}{dx^2}$.

EXERCISE 21.2

1. Find the derivative of x^2 from the First Priciple.

2. Find the derivative of $\frac{1}{x}$ from the First Priciple.

3. $y = (2x + 3)(5x^2 - 7x + 1)$ find $\frac{dy}{dx}$.

4. $y = (x - 1)(x - 2)(x - 3)$ then find $\frac{dy}{dx}$.

5. Find $f'(x)$ if $f(x) = \frac{4x+3}{2-x}$

6. $f(x) = \frac{x}{x^2 + x + 1}$

7. $y = \sqrt[3]{x^2 + 5x - 7}$ Find $\frac{dy}{dx}$

8. $y = x + \sqrt{x^2 + 8}$ find $\frac{dy}{dx}$

9. $y = x^3 + 1$ then find $\frac{d^2y}{dx^2}$

10. $y = \sqrt{x^2 + 1}$ find $\frac{d^2y}{dx^2}$.

SUPPORTIVE WEBSITES

- http : //www.wikipedia.org
- http:// math world . wolfram.com

PRACTICE EXERCISE

1. The distance s meters travel in time 't' seconds by a car is given by the relation $s = t^2$ calculate.

   (a) the rate of change of distance with respect to time t.

   (b) the speed of car at time $t = 3$ seconds.

2. Find the derivatives of each of the following functions by the first principles

   (a) $2x^2 + 5$ (b) $x^3 + 3x^2 + 5$ (c) $(x - 1)^2$

3. Find the derivative of each of the following functions.

   (a) $f(x) = x^3 - 3x^2 + 5x - 8$

   (b) $f(x) = x + \frac{1}{x}$

   (c) $f(x) = \frac{x^2 - a}{a - 2}, \ a \neq 2$

   (d) $f(x) = \frac{3}{(x-1)^2} + \frac{10}{x^3}$

   (e) $f(x) = \frac{1}{(1+x)^4}$

   (f) $f(x) = \frac{(x+1)(x-2)}{\sqrt{x}}$

   (g) $f(x) = \frac{3x^2 + 4x + 5}{x}$

   (h) $f(x) = \frac{(x^3+1)(x-1)}{x^2}$

4. Use the chain rule find the derivative of the following

   (a) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ (b) $\sqrt{\frac{1+x}{1-x}}$

5. Find the second order derivative of the following

   (a) $\sqrt{x+1}$ (b) $x\sqrt{x-1}$

ANSWERS

EXERCISE 21.1

1. $\frac{1}{2\sqrt{x}}$

2. 0

3. $6x^2 - 6x$

4. $3x^2 - \frac{2}{x^3} + \frac{1}{x^2}$

5. $\frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}}$

6. 16, 16, 16

7. $\frac{ab - bc}{(cx + d)^2}$

8. $1 - \frac{1}{x^2}$

9. $2x + 3x^2$

10. $20t + 60t^2$

11. 9.8 ; 49

12. $3x^2 + 6x + 4$; 13

13. $\frac{4}{15}x^{-\frac{1}{3}} + \frac{4}{5}x^{\frac{-9}{5}} - 6x^3$

14. $2x - 3$

15. $\frac{-3x^2+4x-1}{(x^2+x+1)^2}$

16. $3x(7-3x^2)^{-3/2}$

17. $6x$

18. $\frac{x}{\sqrt{x^2+1}}$

EXERCISE 21.2

1. $2x$

2. $\frac{-1}{x^2}$

3. $30x^2+2x-19$

4. $30x^2-12x+11$

5. $\frac{-2}{(2x-1)^2}$

6. $\frac{1-x^2}{(x^2+x+1)^2}$

7. $\frac{1}{3}\left[x^2+5x-7\right]^{-2/3}(2x+5)$

8. $1+\frac{x}{\sqrt{x^2+8}}$

9. $6x$

10. $\frac{1}{\left(1+x^2\right)^{\frac{3}{2}}}$

PRACTICE EXERCISE

1. (a) $2t$ (b) 6 seconds

2. (a) $4x$ (b) $3x^2 + 6x +$ (c) $2(x - 1)$

3. (a) $3x^2 - 6x + 5$ (b) $1 - \frac{1}{x^2}$

   (c) $\frac{2x}{a-2}$ (d) $\frac{-6}{(x-1)^3} - \frac{30}{x^4}$

   (e) $\frac{-4x^3}{(1-x^4)^2}$ (f) $\frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}} + \frac{1}{x^{3/2}}$

   (g) $3 + \frac{5}{x^2}$ (h) $3x^2 - 2 - \frac{1}{x^2} + \frac{4}{x^3}$

4. (a) $1 - \frac{1}{x^2}$ (b) $\frac{1}{\sqrt{1+x}(1-x)^{3/2}}$

5. a) $-\frac{1}{4(x+1)^{3/2}}$ (b) $\frac{2+x-x^2}{4(x-1)^{1/2}}$

Chapter 22

DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- Find the derivative of trigonometric functions from the first principle.
- Find the derivative of Inverse trigonometric functions from first principle.
- Apply product, quotient and chian rule in finding the derivatives of trigonometric and Inverse trigonometric functions and
- Find second order derivative of function.
- Knowledge of trigonometric ratios as functions of angles.
- Standard Limitss of Trigonometric functions namely.
- i) $\underset{x \to 0}{\text{Lt}} \sin x = 0$ ii) $\underset{x \to 0}{\text{Lt}} \frac{\sin x}{x} = 1$

  iii) $\underset{x \to 0}{\text{Lt}} \cos x = 1$ iv) $\underset{x \to 0}{\text{Lt}} \frac{\tan x}{x} = 1$
- Definition of derivatives and rules of finding derivaives of functions.

PREREQUISITES

- Relations, functions, Trigonometric functions, exponential functions and logarithemic function

INTRODUCTION

Trigonometry is the Branch of Mathematics that has made itself indispensable geometry, functions - harmonic and simple and otherwise. Just can not be processed without encouraging trigonometric functions. Further within the specific limit, trigonometric functions give us the inverse on well.

The question now arises : Are all the rules of finding the derivatives studied by us so far appliacable to trigonometric functions ?

This is what we propose to explore in this lesson and in the process, develop the formulae or results for finding the derivatives of trigonometric functions and their inverses. In all discussions involving the trignometric functions and their inverses, radian measure is used, unless otherwise specifically mentioned.

22.1 DERIVATIVE OF TRIGONOMETRIC FUNCTION FROM FIRST PRINCIPLE

Example 22.1 : Let $y = \sin x$ then $\frac{dy}{dx}$.

Solution : For small increment δx in x , Let the corresponding increment in y be δy

$$y + \delta y = \sin(x + \delta x)$$

$$\delta y = \sin(x + \delta x) - y = \sin(x + \delta x) - \sin x$$

it is in the form of $\boxed{\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}}$

$$\delta y = 2\cos\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x-x}{2}\right)$$

$$\delta y = 2\cos\left(x+\frac{\delta x}{2}\right)\ \sin\left(\frac{\delta x}{2}\right)$$

$$\frac{\delta y}{\delta x} = 2\cos\left(x+\frac{\delta x}{2}\right)\ \frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$\underset{\delta x \to 0}{\text{Lt}}\ \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}}\ \cos\left(x+\frac{\delta x}{2}\right)\ .\ \underset{\delta x \to 0}{\text{Lt}}\ \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

Then $\dfrac{dy}{dx} = \cos x$

$$\therefore\ \frac{d}{dx}(\sin x) = \cos x.$$

Example 22.2 : Let $y = \cos x$ find $\dfrac{dy}{dx}$.

Solution : $y = \cos x \quad \Rightarrow y+\delta y = \cos(x+\delta x)$

$$\delta y = \cos(x+\delta x) - y = \cos(x+\delta x) - \cos x$$

it is in the form of $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\ \sin\dfrac{C-D}{2}$

$$\frac{\delta y}{\delta x} = -2\sin\left(x+\frac{\delta x}{2}\right)\ \sin\frac{\delta x}{2}$$

$$\frac{\delta y}{\delta x} = -2\sin\left(x+\frac{\delta x}{2}\right)\ \frac{\sin\frac{\delta x}{2}}{\delta x}$$

$$\underset{\delta x \to 0}{\text{Lt}}\ \frac{\delta y}{\delta x} = -\underset{\delta x \to 0}{\text{Lt}}\ \sin\left(x+\frac{\delta x}{2}\right).\underset{\delta x \to 0}{\text{Lt}}\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$

$$= -\sin x$$

$\dfrac{dy}{dx} = -\sin x$ $\qquad$ $\boxed{\therefore\ \dfrac{d}{dx}(\cos x) = -\sin x}$

Example 22. 3 : Let $y = \tan x$ then find $\frac{dy}{dx}$.

Solution : $y = \tan x \quad \Rightarrow y + \delta y = \tan(x + \delta x)$

$$\delta y = \tan(x+\delta x) - y$$

$$\delta y = \tan(x+\delta x) - \tan x$$

$$\delta y = \tan(x+\delta x) - \tan x$$

$$\delta y = \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+\delta x)\cos x - \sin x \cos(x+\delta x)}{\cos x \ \cos(x+\delta x)}$$

$$= \frac{\sin\left[(x+\delta x) - x\right]}{\cos(x+\delta x)\cos x}$$

$$= \frac{\sin \delta x}{\cos(x+\delta x)\cos x}$$

$$\frac{\delta y}{\delta x} = \frac{\sin \delta x}{\delta x} \cdot \frac{1}{\cos(x+\delta x)\cos x}$$

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{\sin \delta y}{\delta x} \cdot \underset{\delta x \to 0}{\text{Lt}} \frac{1}{\cos(x+\delta x)\cos x}$$

$$= 1 . \frac{1}{\cos^2 x} = \sec^2 x.$$

$$\therefore \frac{dy}{dx} = \sec^2 x \Rightarrow \boxed{\therefore \frac{d}{dx}(\tan x) = \sec^2 x}$$

Similarity 1. $\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$

2. $\boxed{\frac{d}{dx}(\sec x) = \sec x \ \tan x}$

3. $\boxed{\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x}$

Example 22.4 : Let $y = \cot x$ then, find $\frac{dy}{dx}$.

Solution : $y + \delta y = \cot(x + \delta x) = \delta y = \cot(x + \delta x) - \cot x$.

$$\delta y = \frac{\cos(x+\delta x)}{\sin(x+\delta x)} - \frac{\cos x}{\sin x} = \frac{\cos(x+\delta x)\sin x - \cos x \sin(x+\delta x)}{\sin x\,(x+\delta x)}$$

$$\delta y = \sin(x - x - \delta x) \times \frac{1}{\sin x \, \sin(x+\delta x)}$$

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{-\sin \delta x}{\delta x} \times \frac{1}{\sin x \, \sin(x+\delta x)}$$

$$= -\frac{1}{\sin^2 x} = -\text{cosec}^2 x$$

Example 22.5 : Let $y = \tan 2x$, find $\frac{dy}{dx}$.

Solution : $y = \tan 2x \Rightarrow y + \delta y = \tan 2(x + \delta x)$

$$\delta y = \tan 2(x + \delta x) - \tan 2x$$

$$\delta y = \frac{\sin 2(x+\delta x)}{\cos 2(x+\delta x)} - \frac{\sin 2x}{\cos 2x}$$

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{\sin 2(x+\delta x)\cos 2x - \sin 2x \cos 2(x+\delta x)}{\delta x . \cos 2x \cos 2(x+\delta x)}$$

$$= \underset{\delta x \to 0}{\text{Lt}} 2 . \frac{\sin \delta x}{\delta x} \times \underset{\delta x \to 0}{\text{Lt}} \frac{1}{\cos 2x \cos 2(x+\delta x)}$$

$$= \underset{\delta x \to 0}{\text{Lt}} 2 . \frac{\sin \delta x}{\delta x} \times \underset{\delta x \to 0}{\text{Lt}} \frac{1}{\cos 2x \cos 2(x+\delta x)}$$

$\boxed{\underset{\theta \to 0}{\text{Lt}} \frac{\sin \theta}{\theta} = 1}$

$$\frac{dy}{dx} = 2.\frac{1}{\cos^2 2x} = 2\sec^2 2x$$

Example 22.6 : $y = \sec 3x$ then find $\frac{dy}{dx}$.

Solution : $y + \delta y = \sec 3(x + \delta x)$

$$\delta y = \sec 3(x+\delta x) - \sec 3x$$

$$\delta y = \frac{1}{\cos 3(x+\delta x)} - \frac{1}{\cos 3x}$$

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{\cos 3x - \cos 3(x+\delta x)}{\delta x . \cos 3x \cos 3(x+\delta x)}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{\cos 3x - \cos 3(x+\delta x)}{\delta x} . \underset{\delta x \to 0}{\text{Lt}} \frac{1}{\cos 3x \cos 3(x+\delta x)}$$

$$\cos \text{C} - \cos \text{D} = -2\sin\frac{\text{C}+\text{D}}{2}\sin\frac{\text{C}-\text{D}}{2}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{-2\sin\left(3x + \frac{3\delta x}{2}\right)\sin\left(\frac{3\delta x}{2}\right)}{\left(\frac{3\delta x}{2}\right)\frac{2}{3}} \times \underset{\delta x \to 0}{\text{Lt}} \frac{1}{\cos 3x \cos 3(x+\delta x)}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{-2\sin\left(3x + \frac{3\delta x}{2}\right)\left(-\sin\frac{3\delta x}{2}\right)}{\left(\frac{3\delta x}{2}\right)\frac{2}{3}} \times \underset{\delta x \to 0}{\text{Lt}} \frac{1}{\cos 3x \cos 3(x+\delta x)}$$

$$= \underset{\frac{3\delta x}{2} \to 0}{\text{Lt}} 2\sin\left(3x + \frac{3\delta x}{2}\right) \times \frac{3}{2} \times \underset{\frac{3\delta x}{2} \to 0}{\text{Lt}} \frac{\sin\frac{3\delta x}{2}}{\frac{3\delta x}{2}} \times \underset{\delta x \to 0}{\text{Lt}} \frac{1}{\cos 3x \cos 3(x+\delta x)}$$

$$= 2\sin 3x \times \frac{3}{2} \times 1 \times \frac{1}{\cos^2(3x)}$$

$$= \frac{3.\sin 3x}{\cos^2 3x}$$

$$\frac{dy}{dx} = \frac{3.\sin 3x}{\cos^2 3x} \Rightarrow \boxed{\frac{dy}{dx} = 3\tan 3x \sec^2 3x}$$

Example 22.7 : Find the derivative of $\sec^2 x$ from first principle.

Solution : Let $y = \sec^2 x$

$$y + \delta y = \sec^2(x + \delta x)$$

$$\delta y = \sec^2(x + \delta x) - \sec^2 x$$

$$\delta y = \frac{1}{\cos^2(x+\delta x)} - \frac{1}{\cos^2 x}$$

$$\delta y = \frac{\cos^2 x - \cos^2(x+\delta x)}{\cos^2 x \cos^2(x+\delta x)}$$

$$= \frac{\sin(x+\delta x+x)\ \sin(x+\delta x-x)}{\cos^2 x \cos^2(x+\delta x)}$$

$$\delta y = \frac{\sin(2x+\delta x)\ \sin\delta x}{\cos^2 x \cos^2(x+\delta x)}$$

$$\frac{\delta y}{\delta x} = \frac{\sin(2x+\delta x)\ \sin\delta x}{\cos^2 x \cos^2(x+\delta x)\delta x}$$

$$\underset{\delta x\to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x\to 0}{\text{Lt}} \frac{\sin(2x+\delta x)\ \sin\delta x}{\cos^2 x \cos^2(x+\delta x)\delta x}$$

$$= \underset{\delta x\to 0}{\text{Lt}} \frac{\sin(2x+\delta x)}{\cos^2 x \cos^2(x+\delta x)} \underset{\delta x\to 0}{\text{Lt}} \frac{\sin\delta x}{\delta x}$$

$$= \frac{\sin 2x}{\cos^2 x.\cos^2 x} \times 1$$

$$\frac{\delta y}{\delta x} = \frac{\sin 2x}{\cos^2 x \cos^2 x} = \frac{2\sin x\cos x}{\cos^2 x\cos^2 x} = 2\tan x\sec^2 x$$

$$\therefore \boxed{\frac{dy}{dx}(\sec^2 x) = 2\tan x \sec^2 x}$$

Important Principles

1. i) $\frac{d}{dx}(\sin x) = \cos x$ ii) $\frac{d}{dx}(\cos x) = -\sin x$

iii) $\frac{d}{dx}(\tan x) = \sec^2 x$ iv) $\frac{d}{dx}(\sec x) = \sec x \tan x$

v) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$ v) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

2. Differentiate with respect to the 'x'

i) $\frac{d}{dx}(\sin u) = \cos \frac{dy}{dx}$ ii) $\frac{d}{dx}(\cos x) = -\sin u \frac{du}{dx}$

iii) $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$ iv) $\frac{d}{dx}(\cot u) = -\text{cosec}^2 u \frac{du}{dx}$

v) $\frac{d}{dx}(\sec u) = \sec u \ \tan u \frac{du}{dx}$

vi) $\frac{d}{dx}(\text{cosec } u) = -\text{cosec } u \ \cot u \frac{du}{dx}$

3. i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ iv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

v) $\frac{d}{dx}(\sec^{-1} x) = \frac{-1}{|x| \sqrt{x^2-1}}$

vi) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{1}{|x| \sqrt{x^2-1}}$

4. i) 2 sin A cos B = sin(A + B) + sin (A − B)

ii) 2 cois A sin B = sin (A + B) − sin (A − B)

iii) 2 sinA sin B = cos(A − B) − cos (A + B)

iv) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

v) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

vi) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

vii) $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

viii) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

ix) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$.

22.2 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

You have learnt how we can find the derivative of a trigonometric function from first principle and also how to deal with these functions as a function of a function as shown in the alternative method. Now we consider some more examples of these derivatives.

Example 22.8 : $y = \tan\sqrt{x}$ then find $\frac{dy}{dx}$.

Solution : $y = \tan\sqrt{x} \Rightarrow \frac{dy}{dx} = \sec^2\sqrt{x}\,.\,\frac{d}{dx}\sqrt{x}$

$$\frac{dy}{dx} = \sec^2\sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$$

Example 22.9 : $y = \pi \cot 3x$ then find $\frac{dy}{dx}$.

Solution : $y = \pi \cot 3x$

$$\frac{dy}{dx} = -\pi\ \cot 3x\ \text{cosec}\, 3x\ \frac{d}{dx}(3x)$$

$$\frac{dy}{dx} = -3\pi\ \cot 3x\ \text{cosec}\, 3x$$

Example 22.10 : $y = x^4 \sin 2x$, find $\frac{dy}{dx}$.

Solution : $y = x^4 \sin 2x \Rightarrow \frac{dy}{dx} = x^4 \frac{d}{dx}(\sin 2x) + \sin 2x \frac{d}{dx}(x^4)$

$$\frac{dy}{dx} = x^4 (2\cos 2x) + 4\ x^3 \sin 2x$$

$$\frac{dy}{dx} = 2x^3 (x\cos 2x + 2\sin 2x)$$

Example 22.11 : $y = \frac{\sin x}{1+\cos x}$ then, find $\frac{dy}{dx}$

Solution : $y = \frac{\sin x}{1+\cos x} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}$

$$y = \tan\frac{x}{2}$$

$$\frac{dy}{dx} = \sec^2\frac{x}{2} \cdot \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}\sec^2\frac{x}{2}$$

Example 22.12 : $y = \frac{\sin x}{x}$ find $\frac{dy}{dx}$

Solution : $y = \frac{\sin x}{x}$

$$\frac{dy}{dx} = \frac{x\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(x)}{x^2}$$

$$\frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}$$

Example 22.13 : $y = \sqrt{\sin^3 x}$ then find $\frac{dy}{dx}$

Solution : $y = \sqrt{\sin^3 x} = (\sin^3 x)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(\sin^3 x)^{\frac{1}{2}-1}\left[\frac{d}{dx}(\sin^3 x)\right]$$

$$= \frac{1}{2}(\sin^3 x)^{-\frac{1}{2}}\ 3\sin^2 x.\cos x$$

$$= \frac{1}{2}\frac{1}{\sqrt{\sin^3 x}} \times\ 3\sin^2 x.\cos x$$

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{\sin x}\ \cos x\ .$$

Example 22.14 : $y = \sqrt{\frac{1-\sin x}{1+\sin x}}$ then find $\frac{dy}{dx}$

Solution : $y = \sqrt{\frac{1-\sin x}{1+\sin x}}$

$$y = \sqrt{\frac{1-\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}}$$

$$y = \frac{1-\sin x}{\sqrt{1-\sin^2 x}} = \frac{1-\sin x}{\cos x}$$

$y = \sec x - \tan x$

diff w.r.t 'x' both sides

$$\frac{dy}{dx} = \sec x\tan x - \sec^2 x = \frac{1}{\cos x}.\frac{\sin x}{\cos x} - \frac{1}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} - \frac{1}{\cos^2 x} = \frac{\sin x - 1}{\cos^2 x} = \frac{\sin x - 1}{1-\sin^2 x}\qquad \boxed{\frac{dy}{dx} = \frac{-1}{1+\sin x}}$$

Example 22.15 : $x = a\cos^3\theta;\ y = a\sin^3\theta$ then find $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Solution : $x = a\cos^3\theta;\ y = a\sin^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\ .\ \sin\theta; \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\ .\ \cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} \text{ (using chain rule)}$$

$$\frac{dy}{dx} = 3a\sin^2\theta \cos\theta \times \frac{-1}{3a\cos^2\theta \sin\theta}$$

$$\boxed{\frac{dy}{dx} = -\tan\theta}$$

Example 22.16 : $y = \dfrac{\cos x}{\sin x + \cos x}$ then find $\dfrac{dy}{dx}$.

Solution : $y = \dfrac{\cos x}{\sin x + \cos x} \Rightarrow \dfrac{f(x)}{g(x)} \Rightarrow \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$$\frac{dy}{dx} = \frac{(\sin x + \cos x)(-\sin x) - \cos x(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$\frac{dy}{dx} = \frac{-(\sin^2 x + \cos^2 x)}{(\sin x + \cos x)^2} = \frac{-1}{(\sin x + \cos x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{\sin^2 x + \cos^2 x + 2\sin x \cos x} = \frac{-1}{1 + \sin 2x}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{-1}{1 + \sin 2x}}$$

Example 22.17 : $y = \sin^m x \cos^n x$ then find $\dfrac{dy}{dx}$.

Solution : $y = \sin^m x \cos^n x$ use product rule

$$y' = (\sin^m x)(\cos^n x)' + (\cos^n x)(\sin^m x)'$$

$$y' = (\sin^m x)\left[n\cos^{n-1} x(-\sin x)\right] + \cos^n x[m\sin^{m-1} x.\cos x]$$

$$y' = -n\sin^{m+1} x\cos^{n-1} x + m\cos^{n+1} x\sin^{m-1} x$$

$$y' = m\cos^{n+1} x\sin^{m-1} x - n\sin^{m+1} x + \cos^{n-1} x$$

Example 22.18 : If $x = 2\cos t + \cos 2t + 1$; $y = 2\sin t + \sin 2t$ find $\frac{dy}{dx}$

Solution : $x = 2\cos t + \cos 2t + 1 \Rightarrow \frac{dx}{dt} = -2(\sin t + \sin 2t)$

$$y = 2\sin t + \sin 2t \Rightarrow \frac{dy}{dt} = 2\cos t + 2\cos 2t$$

$$\frac{dy}{dx} = \left(\frac{dy/dt}{dx/dt}\right) = \frac{2(\cos t + \cos 2t)}{-2(\sin t + \sin 2t)}$$

$$\frac{dy}{dx} = -\frac{(\cos t + \cos 2t)}{\sin t + \sin 2t}$$

Example 22.19 : If $x = 2e^{-t}$; $y = 4e^{-t}$ then find $\frac{dy}{dx}$

Solution : $\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} \Rightarrow x = 2e^{-t} \Rightarrow \frac{dx}{dt} = -2e^{-t}$.

$$y = 4e^{t} \Rightarrow \frac{dy}{dx} = 4e^{t} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t}{-2e^{-t}} = -2e^{2t}.$$

$$\boxed{\therefore \frac{dy}{dx} = -2e^{2t}}$$

Example 22.20 : If $\sin y = x \sin(a + y)$ then find $\frac{dy}{dx}$

Solution : Given $\sin y = x \sin(a + y)$ ...(i) on diff w.r.t. 'x' we get

$$\cos y \frac{dy}{dx} = \sin(a+y).1 + x\cos(a+y)\frac{dy}{dx}$$

$$[\cos y - x\cos(a+y)]\frac{dy}{dx} = \sin(a+y)$$

$$\frac{dy}{dx} = \frac{\sin(a+y)}{[\cos y - x\cos(a+y)]}$$

From (i) $x = \frac{\sin y}{\sin(a+y)}$

$$\frac{dy}{dx} = \frac{\sin(a+y)}{\cos y - \frac{\sin y}{\sin(x+y)} \cdot \cos(x+y)}$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\cos y \sin(a+y) - \sin y \cos(a+y)}$$

$\therefore$ sin AcosB − cos A sin B = sin(A − B)

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin(a+y-y)} = \frac{\sin^2(a+y)}{\sin a}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}}$$

Example 22.21 : If $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\infty}}}$ then show that $(2y-1)\frac{dy}{dx} = 1$

Solution : $y = \sqrt{x+y} \Rightarrow y^2 = x + y$

diff w.r.t 'x'

$$2y \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = 1$$

$$\boxed{\therefore (2y-1)\frac{dy}{dx} = 1}$$

Example 22.22 : $x = 3\cot t - 2\cot^3 t$; $y = 3\sin t - 2\sin^3 t$ then find $\frac{dy}{dx}$.

Solution : $y = 3 \sin t - 2 \sin^3 t$

$$\frac{dy}{dt} = 3\cos t - 2 \times 3\sin^2 t . \cos t$$

$$\frac{dy}{dt} = 3\cos t - 6\sin^2 t \cos t$$

$$\frac{dy}{dt} = 3\cos t\,(1 - 2\sin^2 t) \qquad ...(1)$$

$x = 3 \cos t - 2 \cos^3 t$

$$\frac{dx}{dt} = -3\sin t - 6\cos^2 t\,(-\sin t)$$

$$\frac{dx}{dt} = -3\sin t\,(1-2\cos^2 t) \qquad ...(2)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3\cos t\,(1-2\sin^2 t)}{-3\sin t(1-2\cos^2 t)}$$

$$\frac{dy}{dx} = \frac{-\cot t\,(1-2\sin^2 t)}{1-2(1-\sin^2 t)}$$

$$\frac{dy}{dx} = \frac{-\cot t\,(1-2\sin^2 t)}{-(1-2\sin^2 t)}$$

$$\boxed{\frac{dy}{dx} = \cot t}$$

Example 22.23 : $x = a\left[\frac{1-t^2}{1+t^2}\right]$, $y = \frac{2bt}{1+t^2}$ Find $\frac{dy}{dx}$.

Solution : $y = \frac{2bt}{1+t^2} \Rightarrow \frac{dy}{dt} = \frac{2b(1+t^2)-(2t)[2bt]}{(1+t^2)^2}$

$$\frac{dy}{dt} = \frac{2b[1+t^2-2t^2]}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{2b[1-t^2]}{(1+t^2)^2}$$

$$\therefore\ x = a\left[\frac{1-t^2}{1+t^2}\right]$$

$$\frac{dx}{dt} = a\left\{\frac{(1+t^2)(-2t)-(2t)(1-t^2)}{(1+t^2)^2}\right\}$$

$$\frac{dx}{dt} = a\left[(-2t)\frac{2}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = \frac{-4at}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at}$$

$$\frac{dy}{dx} = \frac{-b(1-t^2)}{2at}$$

Example 22.24 : If $\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$ then, show that $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

Solution : Given $\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$

Put $x = \sin\theta$, $y = \sin\alpha$

$$\sqrt{1-\sin^2\theta}+\sqrt{1-\sin^2\alpha} = a(\sin\theta-\sin\alpha)$$

$$\cos\theta + \cos\alpha = a(\sin\theta - \sin\alpha)$$

$$\therefore \cos C+\cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C-\sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$2\cos\frac{\theta+\alpha}{2}\cos\frac{\theta-\alpha}{2} = a\left[2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}\right]$$

$$\cos\frac{\theta-\alpha}{2} = a\sin\frac{\theta-\alpha}{2}$$

$$\frac{1}{a} = \tan\frac{\theta-\alpha}{2}$$

$$\Rightarrow \frac{\theta-\alpha}{2} = \tan^{-1}\left(\frac{1}{a}\right)$$

$$\theta - \alpha = 2\tan^{-1}\left(\frac{1}{a}\right)$$

$$\alpha = \theta - 2\tan^{-1}\left(\frac{1}{a}\right)$$

$$\sin^{-1} y = \sin^{-1} x - 2\tan^{-1}\left(\frac{1}{a}\right)$$

diff w.r.t 'x'

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}}$$

22.3 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS FROM FIRST PRINCIPLE

We know find dervatives of standard inverse Trigonometric functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ by first principle.

Example 22.25 : $y = \sin^{-1}x$. Find $\frac{dy}{dx}$ through first principle.

Solution : $y = \sin^{-1}x$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$y = \sin^{-1}x \Rightarrow x = \sin y$ so $x + \delta x = \sin(y + \delta y)$

As $\delta x \to 0$, $\delta y \to 0$

$$x + \delta x = \sin(y + \delta y)$$

$$\delta x = \sin(y + \delta y) - x \Rightarrow \sin(y + \delta y) - \sin y$$

$$\delta x = \sin(y + \delta y) - \sin y$$

dividing both by δx.

$$1 = \frac{\sin(y+\delta y)-\sin y}{\delta x}$$

$$1 = \underset{\delta x \to 0}{\text{Lt}} \frac{\sin(y+\delta y)-\sin y}{\delta x} \cdot \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} \quad [\because \delta y \to 0 \text{ when } \delta x \to 0]$$

it is in the form of sin C − sin D = $2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

$$1 = \left[\underset{\delta x \to 0}{\text{Lt}}\, 2\cos\left(y+\frac{1}{2}\delta y\right)\sin\left(\frac{1}{2}\delta y\right)\right]. \frac{dy}{dx}$$

$$1 = \cos y.\frac{dy}{dx} \qquad \cos y = \sqrt{1-\sin^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{dy}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}}$$

Example 22.26 : Show that $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$.

Solution : $y = \cos^{-1} x \Rightarrow \cos y = x \Rightarrow x + \delta x = \cos(y+\delta y)$

As $\delta x \to 0$, $\delta y \to 0$

$$x + \delta x = \cos(y + \delta y) \Rightarrow \delta x = \cos(y + \delta y) - x$$

$$\delta x = \cos(y + \delta y) - \cos y$$

$$1 = \frac{\cos(y+\delta y)-\cos y}{\delta x} \quad \text{divide by both } \delta x.$$

$$1 = \frac{\cos(y+\delta y)-\cos y}{\delta y} . \frac{\delta y}{\delta x}$$

$$1 = \underset{\delta x \to 0}{\text{Lt}} \frac{\cos(y+\delta y)-\cos y}{\delta y}\ \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} \quad [\therefore \delta y \to 0 \text{ when } \delta x \to 0]$$

$$\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$$

$$1 = \underset{\delta x \to 0}{\text{Lt}} \left[\frac{2\sin\left(\frac{y+\delta y+y}{2}\right)\sin\left(\frac{y-(\delta y+y)}{2}\right)}{\delta y} \right] \cdot \frac{dy}{dx}$$

$$1 = \underset{\delta x \to 0}{\text{Lt}} - \left[\frac{2\sin\left(y+\frac{1}{2}\delta y\right)\sin\frac{\delta y}{2}}{\delta y} \right] \cdot \frac{dy}{dx}$$

$$1 = -(\cos y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-1}{\cos y}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\sin^2 y}} = \frac{-1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{dy}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}}$$

Example 22.27 : Show that $\frac{d}{dx}(\text{Tan}^{-1}x) = \frac{1}{1+x^2}$.

Solution : $y = \tan^{-1} x \Rightarrow$ then $x = \tan y \Rightarrow x + \delta x = \tan(y + \delta y)$

As $\delta x \to 0, \ \delta y \to 0$

$$\delta x = \tan(y + \delta y) - \tan y$$

Divide by δx both sides.

$$1 = \frac{\tan(y+\delta y)-\tan y}{\delta x} \times \frac{\delta y}{\delta y}$$

$$1 = \underset{\delta y \to 0}{\text{Lt}} \frac{\tan(y+\delta y)-\tan y}{\delta y} \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} \quad [\because \delta y \to 0 \text{ when } \delta x \to 0]$$

$$1 = \underset{\delta y \to 0}{\text{Lt}} \left[\frac{\frac{\sin(y+\delta y)}{\cos(y+\delta y)} - \frac{\sin y}{\cos y}}{\delta y} \right] \cdot \frac{dy}{dx}$$

$$1 = \frac{dy}{dx} \cdot \underset{\delta y \to 0}{\text{Lt}} \frac{\sin(y+\delta y)\cos y - \cos(y+\delta y)\sin y}{\delta y \cdot \cos(y+\delta y)\cos y}$$

$$1 = \frac{dy}{dx} \cdot \underset{\delta y \to 0}{\text{Lt}} \frac{\sin(y+\delta y - y)}{\delta y \cdot \cos(y+\delta y)\cos y}$$

$$1 = \frac{dy}{dx} \cdot \underset{\delta y \to 0}{\text{Lt}} \left[\frac{\sin \delta y}{\delta y} \cdot \frac{1}{\cos(y+\delta y)\cos y} \right]$$

$$1 = \frac{dy}{dx} \cdot \frac{1}{\cos^2 y} = \frac{dy}{dx} \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Similarly $\frac{d}{dx}(\cot^{-1} x) = \frac{1}{1+x^2}$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

22.4 DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

In the previous section, we have learnt to find derivatives of inverse trignometric functions by first principle. Now we learn to find derivatives of inverse trigonometric functions by alternative methods. We start with standard inverse trignometric functions $\sin^{-1} x$, $\cos^{-1} x$, ...

Example 22.28 : $y = \sin^{-1} x$ Find $\frac{dy}{dx}$

Sol. $y = \sin^{-1} x \Rightarrow x = \sin y$

diff w.r.t. 'x' $\frac{dy}{dx} = \cos y = \sqrt{1-\sin^2 y}$

$$\frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

Similarly $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$.

Example 22.29 : $y = \tan^{-1}x$ then find $\frac{dy}{dx}$

Solution : $y = \tan^{-1}x \Rightarrow x = \tan y$

diff w.r.t 'x'

$$\frac{dx}{dy} = \sec^2 y \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\frac{dx}{dy} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\frac{d}{dy}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Similarly $\frac{d}{dy}(\cot^{-1} x) = \frac{-1}{1+x^2}$.

Example 22.30 : $y = \sec^{-1}x$ then find $\frac{dy}{dx}$

Solution : $x = \sec y$ diff w.r.t 'x' $\frac{dx}{dy} = \sec y \tan y$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\left[\pm\sqrt{\sec^2 y - 1}\right]}$$

$$\frac{dy}{dx} = \frac{1}{\pm\sec y\sqrt{\sec^2 y - 1}} = \frac{1}{|x|\sqrt{\sec^2 y - 1}} = \frac{1}{|x|\sqrt{x^2 - 1}}$$

Similar $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$.

Example 22.31 : $y = \cos^{-1}x$ find $\frac{d^2y}{dx^2}$

Solution : Let $y = \cos^{-1}x$ differentiating w.r.t 'x' both sides.

we get

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} = \frac{-1}{(1-x^2)^{\frac{1}{2}}} = -(1-x^2)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -(1-x^2)^{-\frac{1}{2}}$$

differentiating w.r.t 'x' both sides we get

$$\frac{d^2y}{dx^2} = -\left[\frac{-1}{2}(1-x^2)^{-\frac{1}{2}-1}\left[0-2x\right]\right]$$

$$\frac{d^2y}{dx^2} = -\left[\frac{-1}{2}(1-x^2)^{-\frac{3}{2}}(-2x)\right]$$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{x}{(1-x^2)^{\frac{3}{2}}}}$$

Example 22.32 : If $y = \sin^{-1}x$, show that $(1 - x^2)y^2 - xy_1 = 0$

where y_2 and y_1 represents denoted the second order and first order derivatives of y w.r.t. 'x'.

Solution : Let $y = \sin^{-1}x \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ squaring both sides

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{1-x^2} \Rightarrow (1-x^2)y_1^2 = 0 \qquad ...(1)$$

diff w.r.t 'x' both sides again

$$(1-x^2), 2y_1\frac{d}{dx}(y_1) + (-2x)y_1^2 = 0$$

$$(1 - x^2)\ 2y_1y_2 - 2x\,y_1^2 = 0$$

$$2y_1[(1 - x^2) - y_2 - xy_1] = 0$$

$$(1 - x^2)\ y_2 - xy_1 = 0.$$

Example 22.33 : $y = \cos^{-1}(4x^3 - 3x)$ then find $\frac{dy}{dx}$

Solution : $y = \cos^{-1}(4x^3 - 3x)$

Let $x\cos\theta$ say $\theta = \cos^{-1} x$

$$y = \cos^{-1}[4\cos^3\theta - 3\cos\theta] = \cos^{-1}[\cos 3\theta]$$

$$y = 3\theta = 3\cos^{-1} x \Rightarrow \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}$$

Example 22.34 : $y = \sin^{-1}\left(\frac{3x-1}{4}\right)$ find $\frac{dy}{dx}$

Solution : $y = \sin^{-1}\left(\frac{3x-1}{4}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{3x-1}{4}\right)^2}}\left(\frac{3.1}{4} - 0\right)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{16-(3x-1)^2}{16}}} \times \frac{3}{4}$$

$$= \frac{4}{\sqrt{16 - 9x^2 - 1 + 6x}}\ \frac{3}{4}$$

$$\frac{dy}{dx}=\frac{3}{\sqrt{15+6x-9x^2}}$$

Example 22.35 : $y=\sin h^{-1}\left(\frac{1-x}{1+x}\right)$ find $\frac{dy}{dx}$.

Solution : Let $u=\frac{1-x}{1+x}$

$$\frac{du}{dx}=\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$$

$$\frac{du}{dx}=\frac{-2}{(1+x)^2} \qquad ...(1)$$

$$y=\sinh^{-1}(u) \quad \frac{dy}{du}=\frac{1}{\sqrt{1+4^2}}$$

$$\frac{dy}{du}=\frac{1}{\sqrt{\frac{1}{1}+\frac{(1-x)^2}{(1+x)^2}}}=\frac{1}{\sqrt{\frac{(1+x)^2+(1-x)^2}{(1+x)^2}}}$$

$$\frac{dy}{du}=\frac{1+x}{\sqrt{2+(1+x^2)}}=\frac{1+x}{\sqrt{2}\,(\sqrt{1+x^2})}$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1+x}{\sqrt{2}(\sqrt{1+x^2})}\times\frac{-2}{(1+x^2)}$$

$$\frac{dy}{dx}=\frac{-\sqrt{2}}{(1+x)\sqrt{1+x^2}}.$$

Example 22.36 : $y=\tan^{-1}\left[\frac{3a^2x-x^3}{a(a^2-3x^2)}\right]$ find $\frac{dy}{dx}$.

Solution : $x=a\tan\theta \Rightarrow \theta=\tan^{-1}\left(\frac{x}{a}\right)$

$$y = \tan^{-1}\left[\frac{3a^2(a\tan\theta) - a^3\tan^3\theta}{a(a^2 - 3a^2\tan\theta)}\right]$$

$$y = \tan^{-1}\left[\frac{a^3[3\tan\theta - \tan^3\theta]}{a^3(1-\tan^2\theta)}\right] \qquad \because \left[\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-\tan^2\theta}\right]$$

$$y = \tan^{-1}[\tan 3\theta]$$

$$y = 3\theta \Rightarrow 3.\tan^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx} = 3 \,.\, \frac{1}{1+\left(\frac{x}{a}\right)^2} \times \frac{1}{a}$$

$$\boxed{\frac{dy}{dx} = \frac{3a}{a^2+x^2}}$$

Example 22.37 : $y = \sin^{-1}\left(\frac{2x}{1-x^2}\right)$ then find $\frac{dy}{dx}$.

Solution : $x = \tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$

$$y = \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$y = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$y = 2\theta \Rightarrow 2.\tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \frac{dy}{dx} = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}$$

Example 22.38 : $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$. Find $\frac{dy}{dx}$

Solution : $x = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right)$

$$y = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right] = \tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]$$

$$y = \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$$

$$y = \tan^{-1}\left[\frac{2\sin^2\theta/2}{2\sin\theta/2\,\cos\theta/2}\right]$$

$$y = \tan^{-1}\left[\tan\frac{\theta}{2}\right] = \frac{\theta}{2}$$

$$y = \frac{1}{2}[\theta] = \frac{1}{2}\left[\tan^{-1}\frac{x}{a}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{1+x^2} = \frac{1}{2(1+x^2)}$$

$$\boxed{\therefore\ \frac{dy}{dx} = \frac{1}{2(1+x^2)}}$$

Example 22.39 : $y = \tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right)$ then find $\dfrac{dy}{dx}$.

$$\therefore\ 1-\cos x = 2\sin^2\frac{x}{2}$$

$$1+\cos x = 2\cos^2\frac{x}{2}$$

Solution : $y = \tan^{-1}\left(\sqrt{\dfrac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right) = \tan^{-1}\left[\tan\dfrac{x}{2}\right]$

$$y = \frac{x}{2} = \frac{1}{2}[x]$$

diff w.r.t 'x'

$$\boxed{\frac{dy}{dx} = \frac{1}{2}}$$

Example 22.40 : $f(x) = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$; $g(x) = \sqrt{1-x^2}$ differentiative w.r.t. $g'(x)$.

Solution : Let $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ and $u = \sqrt{1-x^2}$

Put $x = \cos\alpha \Rightarrow y = \sec^{-1}\left(\frac{1}{2\cos^2\alpha - 1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\alpha}\right)$

$$y = \sec^{-1}(\sec 2\alpha) = 2\alpha$$

$$y = 2\alpha \Rightarrow \frac{dy}{du} = 2.1 = 2\frac{dy}{d\alpha} = 23 \qquad ...(1)$$

$$u = \sqrt{1-x^2} = \sqrt{1-\cos^2\alpha} = \sin\alpha$$

$$\frac{du}{d\alpha} = \cos\alpha \qquad ...(2)$$

From 1, 2

$$\frac{dy}{du} = \frac{dy}{d\alpha} \times \frac{d\alpha}{du} = 2.\frac{1}{\cos\alpha}$$

$$\frac{dy}{du} = \frac{2}{\cos\alpha} = \frac{2}{x} \qquad \boxed{\therefore \cos\alpha = x}$$

$$\boxed{\therefore \frac{dy}{dx} = 2.\frac{1}{x}}$$

Example 22.41 : $f(x) = \tan^{-1}\left[\frac{2x}{1-x^2}\right]$; $g(x) = \sin^{-1}\left[\frac{2x}{1+x^2}\right]$

diff w.r.t. $g(x)$

Solution : Let $y = \tan^{-1}\left[\frac{2x}{1-x^2}\right]$ put $x = \tan\theta$

$$y = \tan^{-1}\left[\frac{2\tan\theta}{1-\tan^2\theta}\right] = \tan^{-1}[\tan 2\theta] = 2\theta$$

$$y = 2\theta \Rightarrow \frac{dy}{d\theta} = 2 \qquad ...(i)$$

$$Z = \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right] = \sin^{-1}[\sin 2\theta]$$

$$Z = 2\theta = 2.\theta \Rightarrow \frac{dz}{d\theta} = 2 \qquad ...(ii)$$

From (i), (ii)

$$\frac{dy}{d\theta} \cdot \frac{d\theta}{dz} = 2 \times \frac{1}{2} = 1$$

Example 22.42 : If $f(x) = 2\sqrt{x+1}\ \sin^{-1}x + 4\sqrt{1-x}$ Find $f'(x)$

Sol. $f(x) = 2\sqrt{x+1}\ \sin^{-1}x + 4\sqrt{1-x}$

$$f'(x) = \frac{d}{dx}\left(2\sqrt{x+1}\right)\sin^{-1}x + 2\sqrt{x+1} + \frac{d}{dx}(\sin^{-1}x) + \frac{4(-1)}{2\sqrt{1-x}}$$

$$= \frac{2}{2\sqrt{x+1}}\sin^{-1}x + \frac{2\sqrt{x-1}}{\sqrt{1-x^2}} - \frac{2}{\sqrt{1-x}}$$

$$= \frac{\sin^{-1}x}{\sqrt{1+x}} + \frac{2}{\sqrt{1-x}} - \frac{2}{\sqrt{1+x}}$$

$$\boxed{f'(x) = \frac{\sin^{-1}x}{\sqrt{x+1}}}$$

Example 22.43 : Show that derivative of the function $\tan^{-1}\left\{\frac{\cos x}{1+\sin x}\right\} = -\frac{1}{2}$

Solution : $y = \tan^{-1}\left\{\frac{\cos x}{1+\sin x}\right\} \Rightarrow y = \tan^{-1}\left\{\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\right\}$

$$y = \tan^{-1}\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}$$

$$y = \tan^{-1}\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} \Rightarrow y = \tan^{-1}\left[\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right]$$

$$y = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]$$

$\Rightarrow y = \frac{\pi}{2}-\frac{x}{2}$ on diff w.r.t 'x' we get

$\frac{dy}{dx} = -\frac{1}{2}$ $\quad\therefore$ Thus $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right] = \frac{-1}{2}$

Example 22.44 : The derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ with respect to $\sqrt{1+3x}$ at $x = -\frac{1}{3}$ is zero.

Solution : $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$$\therefore\ y = \sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right) \Rightarrow y = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$y = \sec^{-1}(\sec 2\theta)$

$y = 2\theta \Rightarrow y = 2\cos^{-1}x$

On differentiating w.r.t 'x' we get

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

Also let $z = \sqrt{1+3x}$

On differentiating w.r.t 'x' we get

$$\frac{dz}{dx} = \frac{1}{2}(1+3x)^{\frac{1}{2}-1}(3)$$

Now $$\frac{dy}{dz} = \frac{dy}{dx} \times \frac{dx}{dz} = \frac{-2}{\sqrt{1+x^2}} \cdot \frac{1}{\left(\frac{1}{2}\right)(1+3x)^{-\frac{1}{2}}(3)}$$

$$\frac{dy}{dz} = \frac{-4}{3}\sqrt{\frac{1+3x}{1-x^2}}$$

At $x = \dfrac{-1}{3}$

$$\frac{dy}{dz} = \frac{-1}{4}\sqrt{\frac{1-1}{1-\frac{1}{9}}}$$

$$\boxed{\frac{dy}{dz} = 0}$$

Example 22.45 : If $\sin^{-1}\left[\dfrac{b+a\sin x}{a+b\sin x}\right]$ then find $\dfrac{dy}{dx} = \dfrac{\sqrt{a^2-b^2}}{(a+b\sin x)}$

Solution : Let $y = \sin^{-1}\left[\dfrac{b+a\sin x}{a+b\sin x}\right]$

$$y' = \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{b+a\sin x}{a+b\sin x}\right)^2}} \times$$

$$\frac{(a+b\sin x)(a\cos x)-(b\cos x)(b+a\sin x)}{(a+b\sin x)^2}$$

$$\left[\therefore\ y=\sin^{-1}x\ \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}\times 1\right]$$

$$y'=\frac{a+b\sin x}{\sqrt{(a+b\sin x)^2-(b+a\sin x)^2}}\left[\frac{(a\cos x)(a+b\sin x)-b(\cos x)(b+a\sin x)}{(a+b\sin x)^2}\right]$$

$$y'=\frac{a+b\sin x}{\sqrt{(a+b\sin x)^2-(b+a\sin x)^2}}\left[\frac{(a\cos x)(a+b\sin x)-b(\cos x)(b+a\sin x)}{(a+b\sin x)^2}\right]$$

$$y'=\frac{1}{\sqrt{(a^2+2ab\sin x+b^2\sin^2 x)-(b^2+a^2\sin^2 x+2ab\sin x)}}$$

$$\times\frac{(a\cos x)(a+b\sin x)-(b\cos x)(b+a\sin x)}{(a+b\sin x)}$$

$$y'=\frac{1}{(a-b\sin x)}\frac{1}{\sqrt{(a^2-b^2)-(a^2-b^2)\sin^2 x}}\cos[a^2-b^2]$$

$$y'=\frac{(a^2-b^2)\cos x}{(a+b\sin x)\sqrt{(a^2-b^2)-(a^2-b^2)\sin^2 x}}$$

$$y'=\frac{(a^2-b^2)\cos x}{(a+b\sin x)\sqrt{(a^2-b^2)[1-\sin^2 x]}}$$

$$y'=\frac{(a^2-b^2)\cos x}{(a+b\sin x)}\times\frac{1}{(\sqrt{(a^2-b^2)}\cos x}$$

$$y'=\frac{\sqrt{(a^2-b^2)}}{(a+b\sin x)}$$

$$\therefore\ \frac{dy}{dx}=\frac{\sqrt{a^2-b^2}}{a+b\sin x}$$

Example 22.46 : If $y = \tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$ for $0 < |x| < 1$ find $\dfrac{dy}{dx}$.

Solution : Substituting $x^2 = \cos 2\theta$ we get

$$y = \tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right]$$

$$y = \tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{1-\cos^2\theta}}\right]$$

$$y = \tan^{-1}\left[\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right]$$

$$y = \tan^{-1}\left[\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right] \Rightarrow y = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$y = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$y = \frac{\pi}{4}+\theta$$

Therefore $y = \dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}(x^2)$

Hence $\dfrac{dy}{dx} = \dfrac{1}{2}\,\dfrac{(-1)}{\sqrt{1-x^4}}\,.\,2x$

$$\boxed{\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^4}}}$$

Example 22.47 : Differentiate $f(x)$ w.r.t $g(x)$

$$f(x) = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right],\ g(x) = \tan^{-1} x$$

Solution : $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ and

$$z = \tan^{-1}x \Rightarrow \tan z = x$$

$$y = \tan^{-1}\left[\frac{\sqrt{1+\tan^2 z}-1}{\tan z}\right] \qquad \therefore\ 1 + \tan^2\theta = \sec^2\theta.$$

$$y = \tan^{-1}\left[\frac{\sec z - 1}{\tan z}\right] = \tan^{-1}\left[\frac{\frac{1}{\cos z}-\frac{1}{1}}{\frac{\sin z}{\cos z}}\right]$$

$$y = \tan^{-1}\left[\frac{\frac{1-\cos z}{\cos z}}{\frac{\sin z}{\cos z}}\right] \Rightarrow y = \tan^{-1}\left[\frac{1-\cos z}{\sin z}\right]$$

$$y = \tan^{-1}\left[\frac{2\sin^2\frac{z}{2}}{2\sin\frac{z}{2}\cos\frac{z}{2}}\right] \qquad \left[\begin{array}{l}\therefore \cos\theta = 2\sin^2\frac{\theta}{2} \\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\end{array}\right]$$

$$y = \tan^{-1}\left[\tan\frac{z}{2}\right] = \frac{z}{2}$$

$$y = \frac{z}{2}$$

$$\Rightarrow \boxed{\frac{dy}{dz} = \frac{1}{2}}$$

Example 22.48 :

If $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)+\tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right] - \tan^{-1}\left[\frac{4x-4x^3}{1-6x^2+x^4}\right]$ then show that

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

Solution : $x = \tan\theta$

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)+\tan^{-1}\left[\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right]$$

$$- \tan^{-1}\left[\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}\right]$$

$$y = \tan^{-1}(\tan 2\theta)+\tan^{-1}(\tan 3\theta)-\tan^{-1}(\tan 4\theta)$$

$$y = 2\theta + 3\theta - 4\theta$$

$$y = \theta$$

$$y = \tan^{-1}x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}x\right)$$

$$\boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$$

22.5 SECOND ORDER DERIVATIVES

We know that the second order derivative of a function is the derivative of the first derivative of that function. In this section, we shall find the second order derivatives of Trigonometric and inverse Trigonometric functions. In the process, we shall be using product rule, quotient rule and chain rule.

Example 20.49 : Find the second order derivative $y = \sin x$

Solution : Let $y = \sin x$ diff w.r.t 'x' $\frac{dy}{dx} = \cos x$

differentiative both sides w.r.t. 'x'

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cos x) = -\sin x .$$

$$\boxed{\frac{d^2y}{dx^2} = -\sin x}$$

Example 20.50 : Let $y = x \cos x$ find $\frac{d^2y}{dx^2}$.

Solution : $y = x \cos x$

diff w.r.t 'x'

$$\frac{dy}{dx} = x(-\sin x + \cos x)$$

$$\frac{dy}{dx} = -x\sin x + \cos x$$

diff w.r.t 'x' again both sides.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\sin x.x + \cos x)$$

$$= -(x \,.\, \cos x + \sin x \,.\, 1) + (-\sin x)$$

$$\frac{d^2y}{dx^2} = -x\cos x - \sin x - \sin x$$

$$\therefore \frac{d^2y}{dx^2} = -[x\cos x + 2\sin x]$$

Example 22.51 : $y = \cos^{-1} x^2$ then find $\frac{dy}{dx}$

Solution : $y = \cos^{-1} x^2 \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-(x^2)^2}} \times \frac{d}{dx}(x^2)$

$$y = \frac{-1}{\sqrt{1-x^4}} \times 2x \qquad \left(\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dx} = \frac{-2x}{\sqrt{1-x^4}}$$

Example 22.52 : $y = \tan^{-1}(\text{cosec } x - \cot x)$ then find $\frac{dy}{dx}$

Solution : $y = \tan^{-1}(\text{cosec } x - \cot x)$, say $\left[\because \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}\right]$

$$\frac{dy}{dx} = \frac{1}{1+(\text{cosec } x - \cot x)^2} \times \frac{d}{dx}(\text{cosec } x - \cot x)$$

$$\frac{dy}{dx} = \frac{1}{1+\text{cosec}^2 x + \cot^2 x - 2\text{cosec } x \cot x} \times (-\text{cosec } x \cot x + \text{cosec}^2 x)$$

$$\frac{dy}{dx} = \frac{2}{2\text{cosec}^2 x - 2\text{cosec } x \cot x} \times \text{cosec } x\ (\text{cosec } x - \cot x)$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{2\text{cosec } x(\text{cosec } x - \cot x)} \times 2\text{cosec } x\ (\text{cosec } x - \cot x)\right]$$

$$\because 1+\tan^2 x = \sec^2 x$$

$$1+\cot^2 x = \text{cosec}^2 x$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2}}.$$

Example 22.53 : $y = \cos^3 x$ then find $\frac{dy^2}{dx^2}$

Solution : $y = \cos^3 x$

$$\therefore \cos 3x = 4\cos^3 x - 3\cos x \Rightarrow \cos^3 x = \frac{1}{4}[\cos 3x + 3\cos x]$$

$$y = \frac{1}{4}(\cos 3x + 3\cos x)$$

$$\frac{dy}{dx} = \frac{1}{4}(-\sin 3x.3 + 3(-\sin x))$$

$$\frac{dy}{dx} = \frac{1}{4}[-3\sin 3x - 3\sin x]$$

$$\therefore \frac{dy}{dx} = \frac{1}{4}[-3(\cos 3x) \,.\, 3 - 3\cos x]$$

$$\frac{d^2y}{dx^2} = \frac{-1}{4}[9\cos 3x + 3\cos x]$$

Example 22.54 : $y = \sin^4 x$ then find $\frac{d^2y}{dx^2}$

Solution : $y = \sin^4 x \Rightarrow (\sin^2 x)^2 = \left[\frac{1-\cos 2x}{2}\right]^2$

$$y = \frac{1+\cos^2 2x - 2\cos 2x}{4}$$

$$y = \frac{1}{4}\left[\frac{1+\cos 4x}{2} + \frac{1}{1} - \frac{2\cos 2x}{1}\right]$$

$$y = \frac{1}{4}\left[\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right]$$

$$\frac{dy}{dx} = 0 + \frac{-4}{8}\sin 4x + \frac{4}{4}\sin 2x$$

$$\frac{dy}{dx} = \frac{-4}{8}\sin 4x + \frac{4}{4}\sin 2x$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-16}{8}\cos 4x + \frac{8}{4}\cos 2x}$$

Example 22.55 : Prove that if $ax^2 + 2hxy + by^2 = 1$ then $\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx+by)^2}$

Sol. Given $ax^2 + 2hxy + by^2 = 1$

diff w.r.t. 'x'

$$2ax+2h\left[x.\frac{dy}{dx}+y.1\right]+2by\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{-(ax+hy)}{(hx+by)}$$

$$\therefore\ \frac{d^2y}{dx^2}=\left[\frac{(hx+by)\left[a+h\frac{dy}{dx}\right]-\left[(ax+hy)h+b\frac{dy}{dx}\right]}{(hx+hy)^2}\right] \quad ...(i)$$

substituting $\frac{dy}{dx}$ value in (1)

$$\boxed{\frac{d^2y}{dx^2}=\frac{h^2-ab}{(hx+by)^2}}$$

Example 22.56 : If $y=ax^{n+1}+b^{-n}x$ then $x^2y_2=n(n+1)y$.

Sol. $y=ax^{n+1}+b^{-n}x$

$$\frac{dy}{dx}=a.(n+1)x^{n+1-1}.1+b^{-n}x^{-n-1}$$

$$\frac{dy}{dx}=(n+1)a\ x^{n}-n\,b\,x^{-n-1}$$

$$\frac{d^2y}{dx^2}=(n+1)a\ (n)x^{n-1}-n\,b\left[(-(n+1)^{(-(n+1)-1}x.1\right]$$

$$\frac{d^2y}{dx^2}=n(n+1)a\,x^{n-1}+n(n+1)\,bx^{-n-2}$$

$$\frac{d^2y}{dx^2}=n(n+1)\left[a\,x^{n-1}+bx^{-n-2}\right]$$

multiply with x^2 both sides

$$\frac{x^2d^2y}{dx^2}=n(n+1)\left[a\,x^{n-1}\times x^2+bx^{-n-2}\times x^2\right]$$

$$\frac{x^2 d^2 y}{dx^2} = n(n+1)\left[a x^{n+1} + bx^{-n}\right]$$

$$\therefore \boxed{x^2 y_2 = n(n+1)y}$$

Example 22.57 : If $y = ae^{nx} + be^{-nx}$ then $y'' = n^2 y$

Sol. $y = ae^{nx} + be^{-nx}$

$$\frac{dy}{dx} = nae^{nx} - bne^{-nx}$$

$$\frac{d^2 y}{dx^2} = n\left[a.n^{nx} + b\ n^2\ e^{-nx}\right]$$

$$\frac{d^2 y}{dx^2} = n^2 ae^{nx} + n^2 be^{-nx}$$

$$\frac{d^2 y}{dx^2} = n^2\left[ae^{nx} + be^{-nx}\right] = n^2 y.$$

Example 22.58 : If $x = \cos\theta,\ y = \sin 5\theta,$ then show that

$$(1-x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} = -25y.$$

Solution : $\dfrac{dx}{d\theta} = -\sin\theta;\ \dfrac{dy}{d\theta} = \cos 5\theta.5$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5\cos 5\theta}{-\sin\theta} \qquad ...(i)$$

Now differentiating w.r.t 'x' we get

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(\frac{5\cos 5\theta}{-\sin\theta}\right)\left(\frac{-1}{\sin\theta}\right) \qquad \boxed{\frac{u}{v} = \frac{vu' - uv'}{v^2}}$$

$$\frac{d^2y}{dx^2} = \left[\frac{\sin\theta\sin 5\theta.25 + 5\cos 5\theta\cos\theta}{\sin^2\theta}\right] \cdot \left[\frac{-1}{\sin\theta}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-25\sin 5\theta}{\sin^2\theta} - \frac{5\cos 5\theta\cos\theta}{\sin^3\theta}$$

Now $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = (1-\cos^2\theta)\left[\frac{-25\sin 5\theta}{\sin^2\theta} - \frac{5\cos 5\theta\cos\theta}{\sin^3\theta}\right]$

$$-\cos\theta\left[\frac{-5\cos 5\theta}{\sin^3\theta}\right]$$

$$= \sin^2\theta\left[\frac{-25\sin 5\theta}{\sin^2\theta} - \frac{5\cos 5\theta\cos\theta}{\sin^3\theta}\right] + \frac{5\cos\theta\cos 5\theta}{\sin\theta}$$

$$= -25\sin 5\theta - \frac{5\cos 5\theta\cos\theta}{\sin\theta} + \frac{5\cos\theta\cos 5\theta}{\sin\theta}$$

$$= -25\sin 5\theta$$

$$\boxed{(1-x^2)\frac{d^2y}{dx} - \frac{xdy}{dx} = -25y} \qquad (\therefore\ y = \cos 5\theta)$$

Example 22.59 : $y = x^2\tan^{-1}x$ then find second order Derivative

Solution : $y = x^2\tan^{-1}x$

diff w.r.t. 'x'

$$\frac{dy}{dx} = x^2.\frac{d}{dx}(\tan^{-1}x) + \tan^{-1}x.\frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = x^2.\frac{1}{1+x^2} + \tan^{-1}x\ .\ (2x)$$

$$\frac{dy}{dx} = \frac{x^2}{1+x^2} + 2x\tan^{-1}x$$

Again diff w.r.t 'x'

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} + 2\left[x.\frac{1}{1+x^2} + \tan^{-1}x.1\right]$$

$$\frac{d^2y}{dx^2} = \frac{2x}{(1+x^2)^2} + 2\left[\frac{x}{1+x^2} + \tan^{-1} x\right]$$

$$\frac{d^2y}{dx^2} = \frac{2x}{(1+x^2)^2} + \frac{2x}{1+x^2} + 2\tan^{-1} x$$

$$\frac{d^2y}{dx} = \frac{2x + 2x(1+x^2)}{(1+x^2)^2} + 2\tan^{-1} x$$

$$\frac{d^2y}{dx^2} = \frac{4x + 2x^3}{(1+x^2)^2} + 2\tan^{-1} x$$

$$\frac{d^2y}{dx^2} = \frac{2x(2+x^2)}{(1+x^2)^2} + 2\tan^{-1} x$$

Example 22.60 : $y = \frac{(\sin^{-1} x)^2}{2}$ Show that $(1 - x^2)\, y_2 - xy_1 = 1$

Solution : $y = \frac{(\sin^{-1} x)^2}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}.2(\sin^{-1} x)^{2-1} \times \frac{1}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = (\sin^{-1} x)^{-1} \times \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

$$y_1 = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

cross multiplication

$$(\sqrt{1-x^2})y_1 = \sin^{-1} x$$

squaring both sides

$$(1-x^2)y_1^2 = (\sin^{-1} x)^2$$

$$\therefore\ 2y = (\sin^{-1} x)^2$$

$\therefore (1-x^2)y_1^2 = 2y$

diff w.r.t x on both sides

$(1-x^2)2y_1y_2 + y_1^2(-2x) = 2y_1$

$(1-x^2)2y_1y_2 - 2xy_1^2 = 2y_1$

$\boxed{(1-x^2)y_2 - xy_1 = 1}$

Example 22.61 : $y = \cos(\cos x)$ Find $\dfrac{d^2y}{dx^2} - \cot x\dfrac{dy}{dx} + y\sin^2 x = 0$

Solution : $y = \cos\ (\cos\ x)$

$$\frac{dy}{dx} = -\sin(\cos x)(-\sin x)$$

$$\frac{dy}{dx} = \sin(\cos x).\sin x$$

diff w.r.t 'x'

$$\frac{d^2y}{dx^2} = \sin x\ \sin(\cos x)' + \sin(\cos x)(\sin x)'$$

$$\frac{d^2y}{dx^2} = \sin x\cos(\cos x)(-\sin x) + \sin(\cos x).\cos x$$

$$\frac{d^2y}{dx^2} = -\sin^2 x.y + \sin(\cos x).\cos x$$

But $\dfrac{dy}{dx} = \sin(\cos x).\sin x$

$$\sin(\cos x) = \frac{dy}{dx}\times\frac{1}{\sin x}$$

$$\frac{d^2y}{dx^2} = -\sin^2 x.y + \frac{dy}{dx}\times\frac{1}{\sin x}\times\cos x$$

$$\frac{d^2y}{dx^2} = -y\sin^2 x + \cot x\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -y\sin^2 x + \cot x\frac{dy}{dx}$$

$$\therefore\ \frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} + y\sin^2 x = 0$$

EXERCISE 22.1

1. $y = 2\sin^2 x$ find $\frac{dy}{dx}$
2. $y = \tan^2 x$ find $\frac{dy}{dx}$
3. $y = \tan\sqrt{x}$ find $\frac{dy}{dx}$.
4. $y = \cos^2 x$ Find $\frac{dy}{dx}$
5. $x = a(t + \sin t); y = a(1 - \cos t)$ Find $\frac{dy}{dx}$
6. If $y = \sqrt{\sin x + \sqrt{\sin x + \text{....to infinity}}}$ Find $\frac{dy}{dx}$
7. $y = 3\sin 4x$ then Find $\frac{dy}{dx}$
8. $y = \pi\cot 3x$ then Find $\frac{dy}{dx}$
9. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \text{.......}\infty}}}$ Find $\frac{dy}{dx}$
10. $y = \frac{\cos^{-1} x}{x}$ Find $\frac{dy}{dx}$.

EXERCISE 22.2

1. Find the derivative of tan x from the first principle.

2. Find the derivative of $\cot^2 x$ from the first principle.

3. $y = \dfrac{\sin x}{1+\cos x}$ then find $\dfrac{dy}{dx}$.

4. $y = \sqrt{\dfrac{1-\sin x}{1+\sin x}}$ then find $\dfrac{dy}{dx}$.

5. $\sin y = x(\sin a + y)$ prove that $\left[\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}\right]$

6. $y = \sqrt{\dfrac{(\sec x + \tan x)}{\sec x - \tan x}}$ then find $\dfrac{dy}{dx}$.

7. $y = \sin^{-1}\sqrt{x}$ find $\dfrac{dy}{dx}$

8. $y = \tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$ find $\dfrac{dy}{dx}$

9. $y = x \cos x$ then find $\dfrac{d^2y}{dx^2}$

10. $y = x + \tan x$ show that $\cos^2 x \dfrac{d^2y}{dx^2} - 2y + 2x = 0$

EXERCISE 22.3

1. $y = \cos^2 x$ find $\dfrac{dy}{dx}$ From the first principle.

2. $y = x^3$ Find $\dfrac{dy}{dx}$ from the first principle.

3. $y = \dfrac{1 - x\sqrt{x}}{1 + x\sqrt{x}}$ $x > 0$ find $\dfrac{dy}{dx}$

4. $y=\sin^{-1}\left(\frac{3x-1}{4}\right)$ Find $\frac{dy}{dx}$

5. $y=\cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$ Find $\frac{dy}{dx}$

6. $y=\frac{1-\cos 2x}{1+\cos 2x}$ Find $\frac{dy}{dx}$

7. If $f(x)=\sin^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}$ and $g(x)=\tan^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}$

Then $f'(x)=g'(x)$

8. $f(x)=(a^2-b^2)^{-\frac{1}{2}}\cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right)$

Then show that $f'(x)=(a+b\cos x)^{-1}$

9. $y=$ sin 2x sin 3x sin 4x find $\frac{d^2y}{dx^2}$

10. $y=\frac{x}{(x-1)^2(x-2)}$ find $\frac{d^2y}{dx^2}$.

11. If $ay^4=(x+b)^5$ then $5y\ \ y''=(y')^2$

$5y\ y_2=(y')^2$

SUPPORTIVE WEBSITES

- http : //www.wikipedia.org
- http:// math world . wolfram.com

PRACTICE EXERCISE

1. If $y=x^3\tan^2\frac{x}{2}$. Find $\frac{dy}{dx}$

2. If $y=\frac{5x}{\sqrt[3]{(1-x)^2}}+\cos^2(2x+1)$. Find $\frac{dy}{dx}$

3. If $y = \sec^{-1}\frac{\sqrt{x}+1}{\sqrt{x}-1} + \sin^{-1}\frac{\sqrt{x}-1}{\sqrt{x}+1}$. Then show that $\frac{dy}{dx} = 0$.

4. If $x = a\cos^3\theta$, then find $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

5. If $y = \sqrt{x+\sqrt{x+\sqrt{x+}}}$ ..... Find $\frac{dy}{dx}$

6. If $x = a\cos(\cos x)$, prove that $\frac{d^2y}{dx^2} - \cot x \cdot \frac{dy}{dx} + y\sin^2 x = 0$

7. If $y = \tan^{-1}x$. Show that $(1 + x)^2\, y_2 + 2xy_1 = 0$

8. If $y = (\cos^{-1}x)^2$, show that $(1 - x^2)\, y_2 - xy_1 - 2 = 0$

9. Show that the derivative of $\tan^{-1}\frac{2x}{1-x^2}$ w.r.t $\sin^{-1}\frac{2x}{1+x^2}$ is 1.

10. If $y = \tan^{-1}x$. Show that $(1 + x)^2\, y_2 + 2xy_1 = 0$

ANSWERS

EXERCISE 22.1

1. $2\sin 2x$

2. $2\tan x\sec^2 x$

3. $\frac{\sec^2\sqrt{x}}{2\sqrt{x}}$

4. $-\sin 2x$

5. $\tan\frac{t}{2}$

6. $\frac{\cos x}{2y-1}$

7. $12\cos 4x$

8. $-3\pi \text{cosec}^2 3x$

9. $\dfrac{\sec^2 x}{2y-1}$

10. $\dfrac{-1}{x\sqrt{1-x^2}} - \dfrac{\cos^{-1} x}{x^2}$

EXERCISE 22.2

1. $\sec x \tan x$

2. $-2x \text{ cosec } x^2$

3. $\dfrac{1}{2}\sec^2 \dfrac{x}{2}$

4. $\dfrac{1}{1+x}$

5. prove it

6. $\sec x (\sec x + \tan x)$

7. $\dfrac{1}{2\sqrt{x}\sqrt{1-x}}$

8. $-\dfrac{1}{2}$

9. $-(x\cos x + 2\sin x)$

EXERCISE 22.3

1. $-\sin 2x$

2. $3x^2$

3. $\dfrac{-3\sqrt{x}}{\left(1-x\sqrt{x}\right)^2}$

4. $\dfrac{3}{\sqrt{15+6x+9x^2}}$

5. $\dfrac{\sqrt{a^2-b^2}}{(a+b\cos x)}$

6. $2\tan\sec^2 x$

9. $\dfrac{1}{4}\left[81\sin 9x-25\sin 5x-9\sin 3x-\sin x\right]$

10. $\dfrac{-4}{(x-1)^3}+\dfrac{6}{(x-1)^4}-\dfrac{4}{(x-2)^3}$

PRACTICE EXERCISE

1. $x^3\tan\dfrac{x}{2}\sec^2\dfrac{x}{2}+3x^2\tan^2\dfrac{x}{2}$

2. $\dfrac{5(3-x)}{3(1-x)^{5/3}}-2\sin(4x+2)$

4. $\sec\theta$

5. $\dfrac{1}{2y-1}$

10. $\dfrac{2x}{\sqrt{1-x^4}}$

Chapter 23

DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

LEARNING OUTCOMES

After studying this lesson, student will be able to :

- Define the derivatives of exponential functions.
- Define the derivatives of logerithemic functions.
- Find the derivatives of exponential functions.
- Find the derivatives of logerithemic functions.
- Find the derivatives of functions expressed as a combination of algebraic, trigonometric exponential and logarithemic functions; and
- Find second order derivative of function.

PREREQUISITES

- Relations, functions, definition of derivative rules for finding derivatives of functions, exponential, logarithemic functions.

INTRODUCTION

We are aware that population generally grows but in some cases decay also. There are many other arears where growth and decay are continuous in nature. In the fields of Economics, Agriculture and Business can be cited, where growth and decay are continuous. Let us consider an example of bacteria growth. If there are 10,00,000 bacteria at present and say they are doubled in number after 10 hours, we are interested in knowing as to after how much time these bacteria will be 30,00,000 in number and so on.

Answers to the growth problem does not come from addition (repeated or otherwise), or multiplication by fixed number. In fact Mathematics has a tool known as exponential function that helps us to find growth and decay in such cases. Exponential function is inverse of Logarithmic function.

In this lesson, we propose to work with this tool and find the rules governing their derivatives.

Background Knowledge

Standard Limits:

- $\underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n = e$ OR $\underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{x}\right)^x = e$
- $\underset{n\to\infty}{\text{Lt}}(1+n)^{\frac{1}{h}} = e$ OR $\underset{n\to\infty}{\text{Lt}}(1+x)^{\frac{1}{x}} = e$
- $\underset{x\to\infty}{\text{Lt}}\frac{e^x-1}{x} = 1$
- $\underset{x\to\infty}{\text{Lt}}\frac{a^x-1}{x} = \log_e a$
- $\underset{h\to\infty}{\text{Lt}}\left(\frac{e^h-1}{h}\right) = 1$

23.1 DEFINITION OF DERIVATIVES AND RULES FOR FINDING DERIVATIVES OF FUNCTIONS

Derivative of Exponential Functions

Example 23.1 : Let $y = e^x$ be an exponential function ... (i)

..... $y + \delta y = e^{(x + \delta x)}$ (corresponding small increments) ... (ii)

From (i) and (ii) we have

$$\delta y = e^{(x+\delta x)} - y \Rightarrow \delta y = e^{(x+\delta x)} - e^x$$

Dividing both sides by δx and the limit as $\delta x \to 0$

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{e^x\left[e^{\delta x} - 1\right]}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}}\, e^x \,.\, \underset{\delta x \to 0}{\text{Lt}} \left(\frac{e^{\delta x} - 1}{\delta x}\right)$$

$$\frac{dy}{dx} = e^x.1 = e^x$$

$$\therefore \frac{d}{dx}(e^x) = e^x.\frac{d}{dx}(x) = e^x$$

Working rule $\frac{d}{dx}(e^x) = e^x.\frac{d}{dx}(x) = e^x$.

Example 23.2 : $y = e^{ax+b}$ Find $\frac{dy}{dx}$

Solution : $y = e^{ax+b} \Rightarrow y + \delta y = e^{a}(x + \delta x) + b$

$\delta x, \delta y$ are corresponding small increments

$$y + \delta y = e^{a(x+b)+b}$$

$$\delta y = e^{a(x+b)+b} - e^{ax+b}$$

$$\delta y = e^{ax+b}\left[e^{a\delta x} - 1\right]$$

$$\frac{\delta y}{\delta x} = a.e^{ax+b}\left[\frac{e^{a\delta x} - 1}{a\delta x}\right]$$ multiplying and dividing by 'a'

Taking $\underset{\delta x \to 0}{\text{Lt}}$, we have

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = a.e^{ax+b} . \underset{\delta x \to 0}{\text{Lt}}\left(\frac{e^{a\delta x} - 1}{a\delta x}\right) \qquad \left(\because \underset{h \to 0}{\text{Lt}} \frac{e^{n} - 1}{x} = 1\right)$$

$$\frac{dy}{dx} = a.e^{ax+b}.1$$

$$\boxed{\frac{dy}{dx} = a.e^{ax+b}}$$

$$\frac{dy}{dx} = \left(e^{ax+b}\right) = ae^{ax+b}$$

Example 23.3 : $y = e^{5x}$ find $\frac{dy}{dx}$.

Solution : $y = e^{5x}$ Let $5x = t$ say

$$y = e^{t} \qquad 5 . \frac{d}{dx}(x) = \frac{dt}{dx} \Rightarrow 5 = \frac{dt}{dx}$$

$$\frac{dy}{dt} = e^{t}$$

we know that $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = e^{t} . 5 = 5\, e^{5x}$

working rule $y = e^{5x} \Rightarrow \frac{dy}{dx} = 5.e^{5x}$

$$\boxed{\frac{dy}{dx} = 5e^{5x}}$$

Example 23.4 : $y = e^{\frac{-3x}{2}}$ then find $\frac{dy}{dx}$.

Solution : $\frac{dy}{dx} = e^{\frac{-3}{2}x} \frac{d}{dx}\left(\frac{-3}{2}x\right)$

$$\frac{dy}{dx} = \frac{-3}{2}e^{\frac{-3}{2}x}$$

Example 23.5 : $y = e^{x\cos x}$ then find $\frac{dy}{dx}$.

Solution : Let $y = e^{x\cos x} \Rightarrow \frac{dy}{dx} = e^{x\cos x}\frac{d}{dx}(x\cos x)$

$$\frac{dy}{dx} = e^{x\cos x}\left[x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}x\right]$$

$$\frac{dy}{dx} = e^{x\cos x}\left[x(-\sin x) + \cos x\right]$$

Example 23.6 : $y = \frac{1}{x}e^x$ then find $\frac{dy}{dx}$

Solution : $y = \frac{1}{x}e^x \Rightarrow \frac{dy}{dx} = \left[e^x.\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x}\frac{d}{dx}(e^x)\right]$

$$\frac{dy}{dx} = e^x\left(\frac{-1}{x^2}\right) + \frac{1}{x}e^x = e^x\left[\frac{1}{x} - \frac{1}{x^2}\right]$$

$$\frac{dy}{dx} = e^x\left(\frac{x-1}{x^2}\right) = \frac{e^x}{x^2}[x-1]$$

$$\therefore \frac{dy}{dx} = \frac{e^x}{x^2}[x-1].$$

Example 23.7 : Differentiate $f(x)$ w.r.t $g(x)$

$f(x)\ e^x,\ g(x)\sqrt{x}$

Solution : Let $y = e^x$ and $u = \sqrt{x}$

$$\frac{dy}{dx} = e^x \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{dy}{dx} \times \frac{xdx}{dt} = e^x \;.2\sqrt{x}$$

Example 23.8 : If $x = 2\ e^{-t}$; $y = 4e^t$ then cot $3x$ then find $\frac{dy}{dx}$.

Solution : $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \Rightarrow$

$$x = 2e^{-t} \Rightarrow \frac{dx}{dt} = -2e^{-t}$$

$$x = 4e^t \Rightarrow \frac{dy}{dt} = 4e^t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4e^{+t}}{-2e^{-t}} = -2e^{2t}$$

$$\boxed{\therefore \frac{dy}{dx} = -2e^{2t}}$$

Example 23.9 : $y = e^{\frac{-7}{2}x}$ then find $\frac{dy}{dx}$.

Solution : $y = e^{\frac{-7}{2}x} \Rightarrow \frac{d}{dx}\left(e^{\frac{-7}{2}x}\right) = e^{\frac{-7}{2}x}\left[\frac{d}{dx} - \frac{7}{2}x\right]$

$$= e^{\frac{-7}{2}x} \times \left(-\frac{7}{2}\right) \times 1 = \frac{-7}{2}\ e^{\frac{-7}{2}x}$$

$$\frac{dy}{dx} = \frac{7}{2}\ e^{\frac{-7}{2}x}$$

Example 23.10 : $y = e^{x^2+2x}$ then find $\frac{dy}{dx}$

Solution : $\frac{dy}{dx} = \frac{d}{d} e^{x^2} + \frac{d}{dx}(2x) = e^{x^2} \times 2x + 2 = 2e^{x^2} + 2$

$$\frac{dy}{dx} = 2(xe^{x^2} + 1)$$

Example 23.11 : $y = 5\sin x - 2e^x$ then find $\frac{dy}{dx}$.

Solution : $\frac{dy}{dx} = 5\frac{d}{dx}(\sin x) - 2\frac{d}{dx}(e^x)$

$= 5\cos x - 2e^x$.

Example 23.12 : $y = \frac{1}{3}e^x - 5e$

Solution : $\frac{dy}{dx} = \frac{1}{3}\frac{dy}{dx}(e^x) - 5\frac{d}{dx}(e)$

$\frac{dy}{dx} = \frac{1}{3}e^x - 0 \Rightarrow \boxed{\frac{dy}{dx} = \frac{e^x}{3}}$

Example 23.13 : $y = e^{\sqrt{x+1}}$ then find $\frac{dy}{dx}$.

Solution : $\frac{dy}{dx} = \frac{d}{dx}(e^{\sqrt{1+x}}) = e^{\sqrt{1+x}}\frac{d}{dx}(\sqrt{x+1})$

$= e^{\sqrt{1+x}} \times \frac{1}{2\sqrt{x+1}} = \frac{e^{\sqrt{1+x}}}{2\sqrt{x+1}}$

$$\boxed{\frac{dy}{dx} = \frac{e^{\sqrt{1+x}}}{2\sqrt{x+1}}}$$

Example 23.14 : $y = e^x \log x$ then find $\frac{dy}{dx}$

Solution : $\frac{dy}{dx} = \log x(e^x)^1 + e^x(\log x) = e^x(\log x)^1 + e^x.1$

$= e^x.\frac{1}{x} + \log x\, e^x$

$$\boxed{\frac{dy}{dx} = e^x\left(\frac{1}{x} + \log x\right)}$$

Example 23.15 : If $y = e^{a\sin^{-1}x}$ show that $\frac{dy}{dx} = \frac{ay}{\sqrt{1-x^2}}$

Solution : $y = e^{a\sin^{-1}x}$

$$\Rightarrow \frac{dy}{dx} = e^{a\sin^{-1}x} \times \frac{1}{\sqrt{1-x^2}} \cdot a \Rightarrow \frac{dy}{dx} = \frac{a \cdot e^{a\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\boxed{\frac{dy}{dx} = \frac{a \cdot y}{\sqrt{1-x^2}}} \qquad \because (y = e^{a\sin^{-1}x})$$

Example 23.16 : If $e^{x+y} = xy$ then find $\frac{dy}{dx}$

Solution : $\log e^{x+y} = \log xy \Rightarrow$ (x + y) log e = logx + log y

$x + y = \log x + \log y$

diff w.r.t. 'x'

$$1 + \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{y} \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left(1 - \frac{1}{y}\right) = \left(\frac{1}{x} - 1\right)$$

$$\frac{dy}{dx} = \frac{y(1-x)}{x(y-1)} = \frac{y(1-x)}{x(y-1)}$$

Example 23.17 : $y = e^{x\sin^2 x}$ then find $\frac{dy}{dx}$

Solution : $y = e^{x\sin^2 x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x\sin^2 x}\right) = e^{x\sin^2 x}\frac{d}{dx}(x\sin^2 x)$$

$$= e^{x\sin^2 x}\left[x.\frac{d}{dx}(\sin^2 x) + \sin^2 x\frac{d}{dx}(x)\right]$$

$$= e^{x\sin^2 x}\left[x.2\sin x\cos x + \sin^2 x\frac{d}{dx}(x)\right]$$

$$\frac{dy}{dx} = e^{x\sin^2 x}.\sin x\left[2\sin x\cos x + \sin^2 x.1\right].$$

Example 23.18 : $y = \dfrac{e^{2x}}{\sqrt{x^2+1}}$ then find $\dfrac{dy}{dx}$

Solution : $y = \dfrac{e^{2x}}{\sqrt{x^2+1}} = \dfrac{u}{v}$ $\qquad \boxed{\therefore \dfrac{vu' - uv'}{v^2}}$

$$\frac{dy}{dx} = \left[\frac{\sqrt{x^2+1}\,.\,\frac{d}{dx}(e^{2x}) - e^{2x}\frac{d}{dx}\left(\sqrt{x^2+1}\right)}{\left(\sqrt{x^2+1}\right)^2}\right]$$

$$\frac{dy}{dx} = \frac{\left[\dfrac{2e^{2x}\sqrt{x^2+1}}{1}\,.\,e^{2x}\,.\,\dfrac{1}{2}\,\dfrac{1}{\sqrt{x^2+1}}\times 2x\right]}{x^2+1}$$

$$\frac{dy}{dx} = \frac{2e^{2x}(x^2+1) - xe^{2x}}{(x^2+1)\left(\sqrt{x^2+1}\right)}$$

$$\frac{dy}{dx} = \frac{2e^{2x}(x^2+1) - xe^{2x}}{(x^2+1)^{3/2}}$$

$$\frac{dy}{dx} = \frac{e^{2x}(2x^2+2-x)}{(x^2+1)^{3/2}}$$

Example 23.19 : If $y = e^{-\frac{k}{2}x}(a\cos nx + b\sin nx)$ then

$$y + ky' + \left(n^2 + \frac{k^2}{4}\right)y = 0$$

Solution : $y = e^{-\frac{k}{2}x}(a\cos nx + b\sin nx)$

$$\frac{dy}{dx} = \frac{-k}{2}\,e^{\frac{-kx}{2}}.1\,(a\cos nx + b\sin nx) + e^{\frac{-kx}{2}}\,(-a\cos nx.n + b.n\sin nx)$$

$$y_1 = \frac{-k}{2}y + e^{\frac{-kx}{2}}(-a\sin nx.n + b\,n\cos nx)$$

$$y_2 = \frac{-k}{2}y_1 - e^{\frac{-kx}{2}}[-an\sin nx + b\,n\cos nx)$$

$$+\left[e^{\frac{-kx}{2}}(an^2\cos nx - b\,n^2\sin nx)\right]$$

$$y_2 + ky_1 + \left[n^2 + \frac{k^2}{4}\right]y = 0$$

Example 23.20 : If $y = x^x + e^{e^x}$ then find $\frac{dy}{dx}$

Sol. Let $y = x^x + e^{e^x}$

Consider $u = x^x$ and $y = e^{e^x}$

$u = x^x \Rightarrow \log u = x\log x$

$$\frac{1}{u}\frac{du}{dx} = \log x.1 + \frac{1}{x}.x$$

$$\frac{du}{dx} = u[\log x + 1] \Rightarrow x^x[1 + \log x]$$

$$\therefore \frac{du}{dx} = x^x[1 + \log x]$$

$$v = e^{e^x}$$

$\log v = e^x \log_e e \Rightarrow \log v = e^x.1$

$$\frac{1}{v}.\frac{dv}{dx} = e^x$$

$$\frac{dv}{dx} = v\,e^x = e^{e^x}(e^x)$$

$y = u + v$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{dy}{dx} = x^x\left[1+\log x\right]+e^{e^x}(e^x)$$

Example 23.21 : If $e^x \log y = \sin^{-1}x + \sin^{-1} y$ find $\frac{dy}{dx}$

Sol. $e^x \log y = \sin^{-1}x + \sin^{-1} y$

diff w.r.t. 'x' both sides, we get

$$e^x\left(\frac{1}{y}\frac{dy}{dx}\right) + e^x \log y = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+y^2}} \cdot \frac{dy}{dx}$$

$$\left(\frac{e^x}{y} - \frac{1}{\sqrt{1+y^2}}\right)\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - e^x \log y$$

$$\frac{dy}{dx} = y\sqrt{1-y^2}\left[\frac{1-e^x\sqrt{1-x^2}\log y}{\left[e^x\sqrt{1-y^2}-y\right]\sqrt{1-x^2}}\right]$$

$$\therefore \frac{dy}{dx} = \frac{y\sqrt{1-y^2}\left[1-e^x\sqrt{1-x^2}\log y\right]}{\left[e^x\sqrt{1-y^2}-y\right]\sqrt{1-x^2}}$$

23.2 DERIVATIVE OF LOGARITHMIC FUNCTIONS

Example 23.22 : $y = \log x$ find $\frac{dy}{dx}$ ...(i)

Sol: $y = \log x \Rightarrow y + \delta y = \log(x + \delta x)$ ...(ii)

δx and δy are corresponding small increments in x and y

$$y + \delta y = \log(x + \delta x)$$

$\delta y = \log(x + \delta x) - y$

$\delta y = \log(x + \delta x) - \log x \qquad \because \log m - \log n = \log \frac{m}{n}$

$\delta y = \log \frac{x+\delta x}{x}$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x} \log\left[1+\frac{\delta x}{x}\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{x} \cdot \frac{x}{\delta x} \log\left[1+\frac{\delta x}{x}\right]$$ multiply and divide by 'n'.

$$= \frac{1}{x} \cdot \log\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

Taking limits both sides as $\delta x \to 0$, we get

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \frac{1}{x} \underset{\delta x \to 0}{\text{Lt}} \log\left[1+\frac{\delta x}{x}\right]^{\frac{x}{\delta x}}$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \log\left[\underset{\delta x \to 0}{\text{Lt}} \left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \log e \qquad \because \underset{\delta x \to 0}{\text{Lt}} \left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} = e$$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

Example 23.23 : $y = \log(ax + b)$ find $\frac{dy}{dx}$

Sol: $y = \log(ax + b)$

$y + \delta y = \log(a(x + \delta x) + b)$

$\delta x, \delta y$ are corresponding small increments

$$\delta y = \log\,[a(x + \delta x) + b] - y$$

$$\delta y = \log\,[a(x + \delta x) + b] - \log\,(ax + b)$$

$$\delta y = \log\left[\frac{a(x+\delta x)+b}{\log(ax+b)}\right] \qquad \boxed{\because \log m - \log n = \log\frac{m}{n}}$$

$$\delta y = \log\left[1+\frac{a\delta x}{ax+b}\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\log\left[1+\frac{a\delta x}{ax+b}\right]$$

$$\frac{\delta y}{\delta x} = \frac{a}{ax+b}\times\frac{ax+b}{a}\cdot\frac{1}{\delta x}\log\left[1+\frac{a\delta x}{ax+b}\right]$$

$$= \frac{a}{ax+b}\log\left[1+\frac{a\delta x}{ax+b}\right]^{\frac{ax+b}{a\delta x}}$$

$$\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x} = \frac{a}{ax+b}\underset{\delta x\to 0}{\text{Lt}}\log\left[1+\frac{a\delta x}{ax+b}\right]^{\frac{ax+b}{a\delta x}}$$

$$\frac{dy}{dx} = \frac{a}{(ax+b)}\cdot\log e$$

$$\boxed{\therefore\ \frac{dy}{dx} = \frac{a}{ax+b}}$$

Working rule

$$\frac{d}{dx}\log(ax+b) = \frac{1}{ax+b}\cdot\frac{d}{dx}(ax+b)$$

$$= \frac{1}{ax+b}\cdot a = \frac{a}{ax+b}$$

$$\therefore\ \frac{d}{dx}\log(ax+b) = \frac{a}{ax+b}.$$

Example 23.24 : $y = \log x^5$ find $\frac{dy}{dx}$.

Solution : $y = \log x^5$ $\quad \boxed{\therefore \log m^n = n \log m}$

$\Rightarrow y = 5\log x \Rightarrow \boxed{\frac{dy}{dx} = 5 . \frac{1}{x}}$

Example 23.25 : $y = \log\sqrt{x}$ then find $\frac{dy}{dx}$.

Sol. $y = \log x^{\frac{1}{2}} \Rightarrow y = \frac{1}{2}\log x$

diff w.r.t 'x'

$\frac{dy}{dx} = \frac{1}{2} . \frac{1}{x} = \frac{1}{2x}$

Example 23.26 : $y = (\log x)^5$ then find $\frac{dy}{dx}$.

Sol. $y = (\log x)^5 \Rightarrow \frac{dy}{dx} = 5(\log x)^{5-4} . \frac{1}{x}$

$\boxed{\frac{dy}{dx} = \frac{5}{x}(\log x)^4}$

Example 23.27 : $y = x^3 \log x$, then find $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = \log x(3x^2) + x^3\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = 3x^2 \log x + x^2 \Rightarrow x^2(3\log x + 1)$

$\boxed{\therefore \frac{dy}{dx} = x^2(3\log x + 1)}$

Example 23.28 : $y = \log \tan x$, then find out $\frac{dy}{dx}$

Sol. $\frac{dy}{dx} = \frac{1}{\tan x} \times \sec^2 x = \frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x} = \frac{1}{\sin x \cos x}$

$$\therefore \frac{dy}{dx} = \operatorname{cosec} x . \sec x$$

Example 23.29 : $y = \log \cos x$ find $\frac{dy}{dx}$

Sol. $$\frac{dy}{dx} = \frac{1}{\cos x} \times -\sin x = -\tan x$$

$$\therefore \frac{dy}{dx} = -\tan x$$

Example 23.30 : $y = \log (\log x)$ then find $\frac{dy}{dx}$

Sol. $y = \log (\log x) \Rightarrow \frac{dy}{dx} = \frac{1}{\log x} \times \frac{1}{x} = \frac{1}{x \log x}$

$$\frac{dy}{dx} = \frac{1}{x \log x}$$

Example 23.31 : $y = \log [\sin \log x]$ then find $\frac{dy}{dx}$

Sol. $y = \log [\sin \log x] \Rightarrow \frac{dy}{dx} = \frac{1}{\sin(\log x)} \times \cos(\log x) \times \frac{1}{x}$

$$\frac{dy}{dx} = \frac{1}{x} \frac{\cos(\log x)}{\sin(\log x)} = \frac{\cot(\log x)}{x}$$

$$\frac{dy}{dx} = \frac{\cot(\log x)}{x}$$

Example 23.32 : $y = x^x$ then find $\frac{dy}{dx}$

Sol. $y = x^x \Rightarrow \log y = \log x^x$ take both side logrithum

diff w.r.t 'x' $\log y = x \log x$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \log x\cdot\frac{d}{dx}(x) + \frac{d}{dx}(\log x)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \log x + x.\frac{d}{dx}$$

$$\frac{dy}{dx} = y[\log x+1] = x^x[\log x+1]$$

$$\boxed{\frac{dy}{dx} = x^x[\log x+1]}$$

Example 23.33 : $y = \log(\sec x + \tan x)$ then find $\frac{dy}{dx}$

Sol. $y = \log(\sec x + \tan x)$

$$\frac{dy}{dx} = \frac{1}{\sec x+\tan x}[\sec x\tan x+\sec^2 x]$$

$$\frac{dy}{dx} = \frac{1}{\sec x+\tan x}\sec x(\sec x+\tan x)$$

$$\boxed{\frac{dy}{dx} = \sec x}$$

Example 23.34 : $y = (\tan x)^x$ find $\frac{dy}{dx}$

Sol. $y = (\tan x)^x$

$\log y = \log(\tan x)^x = x\log(\tan x)$

diff w.r.t 'x'

$$\boxed{\begin{aligned}(uv)' &= uv' + vu'\\ &= uv' + vu'\end{aligned}}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = x.\frac{d}{dx}(\log(\tan x)) + \log\tan x.\frac{d}{dx}(x)$$

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{\tan x}\times\sec^2 x + \log\tan x + 1$$

$$\frac{dy}{dx} = y\left[\frac{x}{\sin x \cos x} + \log \tan x\right]$$

$$\frac{dy}{dx} = (\tan x)^x\left[\frac{x}{\sin x \cos x} + \log \tan x\right]$$

Example 23.35 : Find the derivative $x^{x\sqrt{x}}$.

Sol. $y = x^{x\sqrt{x}} \Rightarrow x^{x^{\frac{3}{2}}}$

$$\log y = \log x^{x^{\frac{3}{2}}} = x^{\frac{3}{2}} \cdot \log x$$

$$\log y = x^{\frac{3}{2}} \cdot \log x \qquad (uv)' = uv' + vu'$$

diff w.r.t 'x'

$$\frac{1}{y} \cdot \frac{dy}{dx} = x^{\frac{3}{2}}\left(\frac{1}{x}\right) + \log x.\frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \left(x^{\frac{1}{2}} + \frac{3}{2}\log x\, x^{\frac{1}{2}}\right) = x^{\frac{1}{2}}\left(1 + \frac{3}{2}\log x\right)$$

$$\frac{dy}{dx} = y\sqrt{x}\left(1 + \frac{3}{2}\log x\right)$$

$$\frac{dy}{dx} = \sqrt{x}\; x^{x\sqrt{x}}\left(1 + \log x\sqrt{x}\right)$$

Example 23.36 : If $y = \log\left[\frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1}\right]$ then find $\frac{dy}{dx}$

Sol. diff on both sides w.r.t. 'x'

$$\frac{dy}{dx} = \left[\frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1}\right] \qquad \left(\frac{u}{v}\right)' = \left[\frac{vu' - uv'}{v^2}\right]$$

$$\frac{dy}{dx} = \left[\frac{\left(\sqrt{1+e^x}+1\right)\frac{d}{dx}\left[\sqrt{1+e^x}-1\right]-\left(\sqrt{1+e^x}-1\right)\frac{d}{dx}\left[\sqrt{1+e^x}+1\right]}{\left(\sqrt{1+e^x}+1\right)^2} \right]$$

$$\frac{dy}{dx} = \left[\frac{\left(\frac{\sqrt{1+e^x}}{1}+1\right)\frac{1}{2\sqrt{1+e^x}}\times e^x - \frac{\sqrt{1+e^x}-1}{1}\times\frac{1}{2\sqrt{1+e^x}}\times e^x}{\left(\sqrt{1+e^x}+1\right)^2} \right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+e^x}-1}\left[\frac{e^x\left(\sqrt{1+e^x}+1\right)-e^x\sqrt{1+e^x}+e^x}{2\sqrt{1+e^x}\left(\sqrt{1+e^x}+1\right)} \right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+e^x}-1}\left[\frac{e^x\sqrt{1+e^x}+e^x-e^x\sqrt{1+e^x}+e^x}{2\sqrt{1+e^x}\left(\sqrt{1+e^x}+1\right)} \right]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+e^x}-1}\left[\frac{e^x}{\sqrt{1+e^x}\left(\sqrt{1+e^x}-1\right)} \right]$$

$$= \frac{e^x}{\sqrt{1+e^x}\left(\sqrt{1+e^x}-1\right)} = \frac{1}{\sqrt{1+e^x}}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1+e^x}}}$$

Example 23.37 : If $y = x + 2\sqrt{1+e^x} - 2\log\left(1+\sqrt{1+e^x}\right)$ Find $\frac{dy}{dx}$

Sol. $\frac{dy}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}\left(2\sqrt{1+e^x}\right) - \frac{d}{dx}\left(2\log\left(1+\sqrt{1+e^x}\right)\right)$

$$= 1 + 2\left(\frac{1}{2\sqrt{1+e^x}}\right)e^x - 2\left(\frac{1}{1+\sqrt{1+e^x}}\right)\left(\frac{1}{2\sqrt{1+e^x}}\right)e^x$$

$$\frac{dy}{dx} = 1+2\left(\frac{1}{2\sqrt{1+e^x}}\right)e^x - \frac{e^x}{\left(1+\sqrt{1+e^x}\right)\left(\sqrt{1+e^x}\right)}$$

$$\frac{dy}{dx} = \frac{1}{1}+\frac{e^x}{\sqrt{1+e^x}} - \frac{e^x}{\left(1+\sqrt{1+e^x}\right)\sqrt{1+e^x}}$$

$$= 1+\frac{e^x + e^x\left(\sqrt{1+e^x} - e^x\right)}{\left(1+\sqrt{1+e^x}\right)\left(1+\sqrt{1+e^x}\right)}$$

$$= 1+\frac{e^x}{\sqrt{1+e^x}}\left(\frac{1-\sqrt{1+e^x}}{1-\sqrt{1+e^x}}\right)$$

$$= 1+\frac{e^x\left(1-\sqrt{1+e^x}\right)}{1-1-e^x}$$

$$= 1-1+\sqrt{1+e^x}$$

$$\boxed{\frac{dy}{dx} = \sqrt{1+e^x}}$$

Example 23.38 : If $x^{\log y} = \log x$ then show that $\frac{dy}{dx} = \left[\frac{1-\log x \log y}{(\log x)^2}\right]\frac{y}{x}$

Sol. $x^{\log y} = \log x \Rightarrow \log(x^{\log y}) = \log(\log x)$

$$\Rightarrow \log y \quad \log x = \log(\log x)$$

$$\frac{1}{y}\,.\,\log x\,.\,\frac{dy}{dx} + \log y\,.\,\frac{1}{x} = \frac{1}{\log x}\times\frac{1}{x}$$

$$\frac{\log x}{y}\,.\,\frac{dy}{dx} = \frac{1}{x}\left[\frac{1}{\log x} - \log y\right]$$

$$\frac{x}{y}\frac{dy}{dx} = \frac{1}{\log x}\left[\frac{1-\log x \ \log y}{\log x}\right]$$

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{1-\log x \log y}{(\log x)^2}\right]$$

Example 23.39 : If $x^y + y^x = a^b$ then find $\frac{dy}{dx} = -\left[\frac{yx^{y-1}+y^x\log y}{x^y\log x + xy^{x-1}}\right]$

Sol. $y_1 = x^y$ and $y_2 = y^x$

that $y_1 + y_2 = a^b$

$y_1 = x^y \qquad y_2 = y^x$

$\log y_1 = y \log x \ ; \quad \log y_2 = x \log y$

$$\frac{1}{y_1} \cdot \frac{dy_1}{dx} = y \cdot \frac{1}{x} + \log x . \frac{dy}{dx}$$

$$\frac{dy_1}{dx} = y_1\left[\frac{y}{x} + \log x \frac{dy}{dx}\right] = x^x\left[\frac{y}{x} + \log x \frac{dy}{dx}\right] \qquad ...(i)$$

Let $y_2 = y^x \Rightarrow \log \ y_2 = x \log y$

$$\frac{1}{y^2} \cdot \frac{dy_2}{dx} = \frac{1}{y} \cdot x . \frac{dy_1}{dx} + \log y$$

$$\frac{dy_2}{dx} = \left[y_2 \frac{x}{y} \cdot \frac{dy}{dx} + \log y\right]$$

$$= y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] \qquad ...(2)$$

$$y_1 + y_2 = a^b \Rightarrow \frac{dy_1}{dx} + \frac{dy_2}{dx} = 0$$

$$x^y\left[\frac{y}{x} + \log x + \frac{dy}{dx}\right] + y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = 0$$

$$y \cdot x^{y-1} + x^y \log x . \frac{dy}{dx} + x . y^{x-1}\frac{dy}{dx} + y^x \log y = 0$$

$$\frac{dy}{dx}\left(x^y \log x + x.y^{x-1}\right) = -\left(y.x^{y-1} + y^x \log y\right)$$

$$\frac{dy}{dx} = -\left[\frac{(y.x^{y-1} + y^x \log y)}{(x^y \log x + x.y^{x-1})}\right]$$

Example 23.40 : $y = x^{x^x}$ then Find $\frac{dy}{dx}$.

Sol. Let $y = x^{x^x} \Rightarrow$ taking log both sides

$$\log y = \log\left(x^{x^x}\right)$$

logy = x^x (log(x))

Again taking log on bothsides

$$\log(\log y) = \log\left[x^x \log x\right] \qquad \text{logmn = log m + log n}$$

$$\log(\log y) = \log x^x + \log(\log x)$$

$$\frac{1}{\log(y)} \cdot \frac{1}{y} \cdot \frac{dy}{dx} = x \log x + \log(\log x)$$

$$\frac{1}{\log y} \cdot \frac{1}{y} \cdot \frac{dy}{dx} = 1.\log x + \frac{1}{x} \cdot x + \frac{1}{\log x} \cdot \frac{1}{x}$$

$$\frac{1}{\log y} \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \log x + 1 \frac{1}{x(\log x)}$$

$$\frac{dy}{dx} = y \log y\left[1 + \log x + \frac{1}{x \log x}\right]$$

$$\frac{dy}{dx} = x^{x^x} \, . \log x^{x^x}\left[\log e + \log x + \frac{1}{x \log x}\right]$$

$$\frac{dy}{dx} = x^{x^x} \, . x^x \log x\left[\log e x + 1\frac{1}{x \log x}\right]$$

$$= x^{x^x} \; x^x \log x\left[\frac{x \log x \log e x}{x \log x} + 1\right]$$

$$\frac{dy}{dx} = x^{x^x + x - 1}\left[\frac{x \log x \log e x}{x \log x} + 1\right]$$

Example 23.41 : $y = (\sin x)^x + x^{\sin x}$

Sol: Let $y = (\sin)^x + x^{\sin x}$

Consider $u = (\sin\ x)^x$ and $v = x^{\sin\ x}$

Taking log on both sides

$\log u = x \log (\sin x)$

$$\frac{1}{u} . \frac{du}{dx} = x . \frac{1}{\sin x} . \cos x + \log(\sin x).1$$

$$\frac{1}{u} . \frac{du}{dx} = x \cot x + \log\ \sin x$$

$$\frac{du}{dx} = u[x \cot x + \log\ \sin x]$$

$$\frac{du}{dx} = (\sin^x x)[x \cot x + \log\ \sin x]$$

$v = x^{\sin x}$

$\log v = \sin x . \log x$

$$\frac{1}{v} . \frac{dv}{dx} = \sin x . \frac{1}{x} + \log x . \cos x$$

$$\frac{dv}{dx} = v\left[\frac{\sin x}{x} + \log x . \cos x\right]$$

$$\frac{dv}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + (\log x) \sin x\right]$$

$y = u + v$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{dy}{dx} = (\sin x)^x \left[x \cot x + \log(\sin x)\right] + x^{\sin x}\left[\frac{\sin x}{x} + (\log x) \cos x\right].$$

Example 23.42 : $y = x^x + (\cot x)^x$ then $\frac{dy}{dx}$ find.

Sol. $y = x^x + (\cot x)^x$

$u = x^x$ and $v = (\cot x)^x$

Let $u = x^x \Rightarrow$ taking log both sides

diff w.r.t 'x'

$$\frac{1}{u}\frac{du}{dx} = 1 \,.\, \log x + \frac{1}{x}\,.x$$

$$\frac{du}{dx} = u[\log x + 1] = x^x[\log x + 1]$$

$v = (\cot x)^x$

Talking log both sides

$$\frac{1}{v}\frac{dv}{dx} = x.\log(\cot x) + x.\frac{1}{\cot x}(-\text{cosec}^2 x)$$

$$= \log(\cot x) - x.\frac{\sin x}{\cos x} \times \frac{1}{\sin^2 x}$$

$$\frac{1}{v}\frac{dv}{dx} = \log(\cot x) - \frac{2x}{2\sin x \cos x}$$

$$\frac{dv}{dx} = v\left[\log(\cot x) - \frac{2x}{\sin 2x}\right]$$

$$\frac{dv}{dx} = (\cot x)^x\left[\log(\cot x) - \frac{2x}{\sin 2x}\right]$$

$y = u + v$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{dy}{dx} = x^x\,[1 + \log x] + (\cot x)^x\left\{\log(\cot x) - \frac{2x}{\sin 2x}\right\}.$$

Example 23.43 : $y = e^{-ax^2} \sin(x \log x)$

Sol. Let $y = e^{-ax^2} \sin(x \log x)$

diff w.r.t 'x'

$$\frac{dy}{dx} = e^{-ax^2}\left[\sin x(x\log x)\right]' + \sin(x\log x) . \left(e^{-ax^2}\right)'$$

$$\frac{dy}{dx} = e^{-ax^2} \cos(x\log x) . (x\log x)' + \sin(x\log x)\left(e^{-ax^2}\right)(-2ax)$$

$$\frac{dy}{dx} = e^{-ax^2} \cos(x\log x)\left[\log x + \frac{1}{x}.x\right] - \sin(\log x.x)\left(e^{-ax^2}\right)(2ax)$$

$$\frac{dy}{dx} = e^{-ax^2} \cos(x\log x)\left[1+\log x\right] - 2ax \sin(x\log x)\left(e^{-ax^2}\right)$$

$$= e^{-ax^2}\left[\cos(x\log x).\left(\log e + \log x\right) - 2ax\sin\left(x\log x\right)\right]$$

$$\frac{dy}{dx} = e^{-ax^2}\left[\cos(x\log x) \log ex - 2ax\sin\left(x\log x\right)\right]$$

Example 23.44 : $y = 20^{\log\tan x}$ then find $\frac{dy}{dx}$.

Sol. diff w.r.t 'x' $\frac{dy}{dx} = 20^{\log\tan x} . \log_e 20\left[\frac{1}{\tan x} . \sec^2 x\right]$

$$\frac{dy}{dx} = 20^{\log\tan x} . \log_e 20\left[\frac{\cos x}{\sin x} . \frac{1}{\cos^2 x}\right]$$

$$= 20^{\log\tan x} . \log_e 20\left[\frac{1}{\sin x\cos x}\right]$$

$$= 20^{\log(\tan x)} . \log_e 20\left[\frac{2}{2\sin x\cos x}\right]$$

$$\frac{dy}{dx} = 2.20^{\log(\tan x)} \log_e 20\left[\frac{1}{\sin x}\right]$$

$$\frac{dy}{dx} = 2.20^{\log(\tan x)} \log_e 20(\text{cosec } 2x)$$

Example 23.45 : If $y = \log\left\{\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}\right\} - \frac{1}{2}\tan^{-1}x$ then find $\frac{dy}{dx}$.

Sol. $y = \log\left\{\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}\right\} - \frac{1}{2}\tan^{-1}x$

$$\therefore \log\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} = \frac{1}{2}\tanh^{-1}x$$

$y = \frac{1}{2}\tanh^{-1}x - \frac{1}{2}\tan^{-1}x$

On differentiating w.r.t 'x' we get

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{1-x^2}\right) - \frac{1}{2}\left(\frac{1}{1+x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1+x^2-1+x^2}{1-x^4}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{2x^2}{1-x^4}\right] = \frac{x^2}{1-x^4}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{x^2}{1-x^4}}$$

Example 23.46 : If $y = (\sin x)^{\log x} + x^{\sin x}$ then find $\frac{dy}{dx}$

Sol. $y = (\sin x)^{\log x} + x^{\sin x}$

Consider $u = (\sin x)^{\log x}$

taking log on both sides

$\log u = \log(\sin x)^{\log x}$

$\log u = \log x \,.\, [\log(\sin x)]$ $\qquad uv = uv' + vu'$

diff w.r.t. 'x'

$$\frac{1}{u} \cdot \frac{du}{dx} = (\log x) \cdot \frac{1}{\sin x}(\cos x) + \frac{1}{x}\log(\sin x)$$

$$\frac{du}{dx} = u\left[(\log x)\cos x + \frac{\log(\sin x)}{x}\right]$$

consider $v = x^{\sin x}$

Taking log both sides

$$\log v = \log (x^{\sin x})$$

$$\Rightarrow \log v = \sin x \log x$$

diff w.r.t 'x'.

$$\frac{1}{v} \cdot \frac{dv}{dx} = (\cos x)\log x + \frac{1}{x}\sin x$$

$$\frac{dv}{dx} = v\left[(\cos x)\log x + \frac{\sin x}{x}\right]$$

$$\frac{dv}{dx} = x^{\sin x}\left[(\cos x)\log x + \frac{\sin x}{x}\right]$$

Now $y = u + v$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

So $$\frac{dy}{dx} = (\sin x)^{\log x}\left[(\log x)\cot x + \frac{\log(\sin x)}{x}\right]$$

$$+ x^{\sin x}\left\{\frac{\sin x}{x} + (\log x)\cos x\right\}$$

Example 23.47 : Differentiate $f(x)$ w.r.t. $g(x)$

$$f(x) = x^{\sin^{-1} x};\ g(x) = \sin^{-1} x$$

Sol. $y = x^{\sin^{-1} x}, \quad u = \sin^{-1} x$

$\log y = \log x^{\sin^{-1}x} = \sin^{-1}x \cdot \log x$

$\log y = \sin^{-1}x \cdot \log x$

diff w.r.t x

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \cdot \log x + \sin^{-1}x \cdot \frac{1}{x}$$

$$\frac{1}{y}\ \frac{dy}{dx} = \left[\frac{\log x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{x}\right]$$

$$\frac{d}{dx} = y\left[\frac{\log x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{x}\right]$$

$$\frac{dy}{dx} = x^{\sin^{-1}x}\left[\frac{\log x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{x}\right]$$

$$\frac{dy}{dx} = x^{\sin^{-1}x}\left[\frac{\log x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{x}\right] \qquad ...(1)$$

$$\text{U} = \sin^{-1}x \Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{dy}{dx} \times \frac{dx}{du}$$

$$\frac{dy}{dx} = x^{\sin^{-1}x}\left[\frac{\log x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{x}\right] \times \sqrt{1-x^2}$$

$$\therefore \frac{dy}{du} = x^{\sin^{-1}x}\left[\log x + \frac{\sqrt{1-x^2}\,\sin^{-1}x}{x}\right]$$

Example 23.48 : Find the derivative, if $y = (\log x)^x + (\sin^{-1}x)^{\sin x}$

Sol: $y = (\log x)x + (\sin^{-1})\sin x$

$y = u + v$

$u = (\log x)^x$ and $v = (\sin^{-1}x)^{\sin x}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$u = (\log x)^x$

diff w.r.t 'x'

$\log u = \log (\log x)^x$

$\log \ u = x \log (\log x)$

$$\frac{1}{u} \cdot \frac{du}{dx} = 1.\log(\log x) + x.\frac{1}{\log x} \cdot \frac{1}{x}$$

$$\frac{du}{dx} = u\left[\log(\log x) + \frac{1}{\log x}\right]$$

$$\frac{dy}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right] \qquad ...(1)$$

$v = (\sin^{-1} x)^{\sin x}$

$\log v = \sin x \log (\sin^{-1} x)$

diff w.r.t 'x'

$$\frac{d}{dx}(\log v) = \frac{d}{dx}\left[\sin x \log(\sin^{-1} x)\right]$$

$$\frac{1}{v} \cdot \frac{dv}{dx} = \sin x.\frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} + \cos x.\log(\sin^{-1} x)$$

$$\frac{dv}{dx} = v\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-n^r}} + \cos x.\log(\sin^{-1} x)\right]$$

$$= (\sin^{-1} x)^{\sin x}\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-n^r}} + \cos x.\log(\sin^{-1} x)\right] \qquad ...(ii)$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{dy}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$$

$$+ (\sin^{-1} x)^{\sin x}\left[\frac{\sin x}{\sin^{-1} x\sqrt{1-x^r}} + \cos x \log(\sin^{-1} x)\right]$$

Example 23.49 : Find $\frac{dy}{dx}$, If $y = (\cos x)^{(\cos x)^{(\cos x)....\infty}}$

Sol. We are given that

$$y = (\cos x)^{(\cos x)^{(\cos x)....\infty}}$$

$y = (\cos\ x)^y$

Taking both sides log

$\log y = y \log (\cos x)$

diff w.r.t 'x'

$$\frac{1}{y}.\frac{dy}{dx} = y.\frac{1}{\cos x}(-\sin x) + \log(\cos x).\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} - \log(\cos x)\frac{dy}{dx} = -y\tan x$$

$$[1 - y\log(\cos x)]\frac{dy}{dx} = -y^2 \tan x$$

$$\frac{dy}{dx} = \frac{-y^2 \tan x}{[1 - y\log(\cos x)}$$

Example 23.50 : $y = (\tan x)^{\cot x} + (\cot x)^x$ then find $\frac{dy}{dx}$

Sol. $y = (\tan x)^{\cot x} + (\cot x)^x$

$y_1 = (\tan x)^{\cot x}$, $y_2 = (\cot x)^x$ say

$$y = y_1 + y_2 \Rightarrow \frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx} \qquad ...(i)$$

$$y_1 = (\tan x)^{\cot x} \Rightarrow \log y_1 = \cot x \log(\tan x)$$

$$\frac{1}{y_1}\cdot\frac{dy_1}{dx} = \cot x.\frac{1}{\tan x}\cdot\sec^2 x + \log(\tan x)(-\text{cosec}^2 x)$$

$$\frac{1}{y_1}\cdot\frac{dy_1}{dx} = \text{cosec}^2 x - \text{cosec}^2 x\ \log(\tan x)$$

$$\frac{dy_1}{dx} = y_1[\text{cosec}^2 x(1-\log(\tan x)]$$

$$\therefore\ \frac{dy_1}{dx} = (\tan x)^{\cot x}\text{cosec}^2 x(1-\log(\tan x)) \qquad \text{...(ii)}$$

$$y_2 = (\cot x)^x \Rightarrow \log y_2 = x\log(\cot x)$$

$$\frac{1}{y_2}\cdot\frac{dy_2}{dx} = x.\frac{1}{\cot x}(-\text{cosec}^2 x) + \log(\cot x).1$$

$$\frac{dy_2}{dx} = y_2\left[\frac{-x\ \text{cosec}^2 x}{\cot x} + \log(\cot x)\right]$$

$$= (\cot x)^x\left[-\frac{x}{\sin x\cos x} + \log(\cot x)\right]$$

$$= (\cot x)^x\left[-x\ \text{cosec}^2 x\tan x + \log(\cot x)\right] \qquad \text{...(ii)}$$

Fron (i), (ii), (iii)

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$$

$$\frac{dy}{dx} = (\tan x)^{\cot x}\ \text{cosec}^2 x(1-\log\tan x)$$

$$+(\cot x)^x\left[-x\text{cosec}^2 x\tan x + \log\tan x\right]$$

Example 23.51 : $y = x^{\tan x} + (\sin x)^{\cos x}$. Then find $\frac{dy}{dx}$

Sol. $y_1 = x^{\tan x}$; $y_2 = (\sin x)^{\cos x}$ say

$\Rightarrow y = y_1 + y_2$

$\therefore\ y_1 = x^{\tan x}$ taking log both sides

$\log y_1 = \log x^{\tan x} \Rightarrow \log y_1 = \tan x \log x$

diff w.r.t 'x'

$$\frac{1}{y_1}\cdot\frac{dy_1}{dx} = \tan x.\frac{1}{x} + \log x\ \sec^2 x$$

$$\frac{dy_1}{dx} = y_1\left[\frac{\tan x}{x} + \sec^2 x\ \log x\right] \qquad \text{...(ii)}$$

$y_2 = (\sin x)^{\cos x}$ taking log both sides.

$\log y_2 = \log\ (\sin x)^{\cos x}$ diff. w.r.t 'x'

$$\frac{1}{y_2}\cdot\frac{dy_2}{dx} = \cos x.\frac{1}{\sin x}.\cos x - \log\sin x(-\sin x)$$

$$\frac{dy_2}{dx} = y_2\left[\frac{\cos^2 x}{\sin x} + \sin x\ \log\sin x\right]$$

$$\frac{dy_2}{dx} = (\sin x)^{\cos x}\left[\cos x\cot x - \sin x\ \log\sin x\right] \qquad \text{...(ii)}$$

From i, ii, iii

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$$

$$\frac{dy}{dx} = x^{\tan x}\left(\frac{\tan x}{x} + \sec^2 x\log x\right) + (\sin x)^{\cos x}\left[\cos x\cot x - \sin x\ \log\sin x\right]$$

23.3 DERIVATIVE OF LOGARITHMIC FUNCTION (CONTINUED)

We know that derivative of the function x^n w.r.t. 'x' is nx^{n-1} where n is constant when exponent is a variable, this rule is not applicable. In such cases we take logarithm of the function and then find its derivative.

Therefore, this process is useful, when the given function is of the type $[f(x)]^{g(x)}$. For example a^x, x^x etc...

Here $f(x)$ may be constant

Derivative of a^x w.r.t 'x'

Sol: Let $y = a^x$ $\quad a > 0$

taking log both sides

$\log y = \log a^x$ $\quad$ $\because \log m^n = n \log m$

$\log y = x \log a$

diff w.r.t 'x'

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log a \cdot \frac{d}{dx}(x)$$

$$\frac{1}{y} \frac{dy}{dx} = \log a$$

$$\Rightarrow \frac{dy}{dx} = y \log a$$

$$\Rightarrow \frac{dy}{dx} = a^x \log a$$

$$\therefore \frac{dy}{dx}(a^x) = a^x \log a \quad a > 0$$

23.4 SECOND ORDER DERIVATIVES

In the previous lesson we found the derivatives of second order of Trigonometric and inverse trigonometric functions by using the formulae for the derivaitves of trigonometric and Inverse trigonometric functions, various laws of derivatives including chain rule and power rule discussed earlier. In similar manner, we will discuss second order derivative of exponential and logarithmic functions.

Example 23.52 : $y = e^x$ then find $\frac{d^2y}{dx^2}$.

Sol. $y = e^x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(e^x) = e^x \Rightarrow \boxed{\frac{d^2y}{dx^2} = e^x}$$

Example 23.53 : If $x = \cos\theta + \theta\sin\theta,\ y = \sin\theta - \theta\cos\theta$ then find $\frac{d^2y}{dx^2}$.

Sol. $y = \sin\theta - \theta\cos\theta;\quad x = \cos\theta + \sin\theta$

On differentiating w.r.t 'θ' respectively, we get

$$\frac{dx}{d\theta} = -\sin\theta + \sin\theta.1 + \theta.\cos\theta$$

$$\frac{dx}{d\theta} = \theta.\cos\theta$$

$$\frac{dy}{d\theta} = \cos\theta - 1\cos\theta + \theta\sin\theta$$

$$\frac{dy}{d\theta} = \theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \theta\sin\theta\ \frac{1}{\theta\cos\theta} = \tan\theta$$

$$\frac{dy}{dx} = \tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\theta) = \frac{d}{d\theta}(\tan\theta)\ .\ \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \sec^2\theta \frac{1}{\theta\cos\theta}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^2\theta}{\theta}$$

Example 23.54 : $y = \dfrac{x^3}{(x+1)(x+2)}$ then find $\dfrac{d^2y}{dx^2}$

Sol. $y = \dfrac{x^3}{(x+1)(x+2)}$

$$\Rightarrow y = \frac{x^3}{x^2+3x+2}$$

Now $y = \dfrac{x^3}{x^2+3x+2} = (x-3) + \dfrac{7x+6}{(x+1)(x+2)}$

But $\dfrac{7x+6}{(x+1)(x+2)} = -\dfrac{1}{x+1} + \dfrac{-8}{-1(x+2)} = \dfrac{-1}{x+1} + \dfrac{8}{x+2}$

(According partial fractions)

$$y = \frac{x^3}{(x+1)(x+2)} = x - 3 - \frac{1}{(x+1)} + \frac{8}{(x+2)}$$

$$y = (x-3) - \frac{1}{(x+1)} + \frac{8}{(x+2)}$$

diff w.r.t 'x'

$$\frac{dy}{dx} = 1 - \frac{1(-1)}{(x+1)^2} + \frac{8(-1)}{(x+2)^2} \quad (1)$$

$$\frac{dy}{dx} = 1 + \frac{1}{(x+1)^2} - \frac{8}{(x+2)^2} \quad \text{...(i)}$$

Again diff w.r.t. 'x'

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0+\frac{-2}{(x+1)^3}+\frac{16}{(x+2)^3}$$

$$\frac{d^2y}{dx} = \frac{16}{(x+2)^3}-\frac{2}{(x+1)^3}.$$

Example 23.55 : If $\frac{x}{(x-1)^2(x-2)}$ then find $\frac{d^2y}{dx^2}$

Sol. $\frac{x}{(x-1)^2(x-2)} = \frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)}$

According Partial Fraction form

$$\frac{x}{(x-1)^2(x-2)} = \frac{A(x-1)(x-2)+B(x-2)+C(x-1)^2}{(x-1)^2(x-2)}$$

$x = A(x-1)(x-2)+B(x-2)+C(x-1)^2$

Put $x = 1$, $\Rightarrow$ $B = -1$

Put $x = 2$, $\Rightarrow$ $C = 2$

Put $x = 0$, $\Rightarrow$ $2A - 2B + C = 0$

$2A - 2(-1) + 2 = 0$

$2A = -4 \Rightarrow \boxed{A = -2}$

$$y = \frac{x}{(x-1)^2(x-2)} = \frac{-2}{(x-1)}+\frac{-1}{(x-2)^2}+\frac{2}{(x-2)}$$

$\therefore$ diff w.r.t 'x'

$$\frac{dy}{dx} = \frac{(-2)(-1)}{(x-1)^2}+\frac{(-1)(-2)}{(x-1)^3}+\frac{2(-1)}{(x-2)^2}$$

$$\frac{dy}{dx} = \frac{2}{(x-1)^2}+\frac{2}{(x-1)^3}-\frac{2}{(x-1)^2}$$

Again diff w.r.t 'x'

$$\frac{d^2y}{dx^2} = \frac{-4}{(x-1)^3} + \frac{6}{(x-1)^4} - \frac{4}{(x-2)^3}$$

EXERCISE 23.1

1. $y = e^{ax}$ then find $\frac{dy}{dx}$

2. $y = e^{\frac{3a}{2}}$ then find $\frac{dy}{dx}$

3. $y = \frac{1}{x} . e^x$ then find $\frac{dy}{dx}$.

4. $y = e^{7x+4}$ then Find $\frac{dy}{dx}$

5. $y = e^{\sqrt{2x}}$ then Find $\frac{dy}{dx}$

6. $y = (x - 1)e^x$ then Find $\frac{dy}{dx}$

7. $y = e^{x\sec^2 x}$ then Find $\frac{dy}{dx}$

8. $y = \log(\tan x)$ then Find $\frac{dy}{dx}$

9. If $y = \log[\cos(\log x)]$ Find $\frac{dy}{dx}$

10. $y = \frac{e^{x^2}}{\log x}$ Find $\frac{dy}{dx}$.

11. $y = \log(\log x)$ then find $\frac{dy}{dx}$.

12. $y = a^x$ then find $\frac{dy}{dx}$

13. $y = (\log)^{\sin x}$ then find $\frac{dy}{dx}$

14. $y = e^x$ then find $\frac{d^2y}{dx^2}$

15. $y = \frac{\log x}{x}$ then find $\frac{d^2y}{dx^2}$

EXERCISE 23.2

1. Find the derivative of $y = \frac{e^{2x}\cos x}{x\sin x}$

2. Find the derivative of $y = (\log x)^3$.

3. $y = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$ then find $\frac{dy}{dx}$.

4. $y = \log\sin(\log x)$ then find $\frac{dy}{dx}$.

5. $y = x^{(x^2+\sin x)}$ find $\frac{dy}{dx}$.

6. $y = (x)^{x^2} + (\log x)^{\log x}$ then find $\frac{dy}{dx}$.

7. $y = a\cos(\log x) + b\sin(\log x)$ show that

$$\frac{x^2 d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

8. $y = e^{\tan^{-1}x}$ prove that

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$

9. $y = (\log x)^{\tan x}$ then find $\dfrac{dy}{dx}$

10. Differntiatiate $f(x)$ w.r.t. $g(x)$

$f(x) = s^{\sin^{-1} x}$; $g(x) = \sin^{-1} x \cdot$

EXERCISE 23.3

1. If $x = a\left\{\cos\theta + \log\tan\left(\dfrac{\theta}{2}\right)\right\}$ and $y = a \sin\theta$ then find $\dfrac{dy}{dx}$.

2. If $y = \sin\left(\log_e x\right)$ then $\dfrac{x^2 d^2 y}{dx^2} + x\dfrac{dy}{dx}$ value.

3. If $u = \log(\sec x + \sec y + \sec z)$ then find $\Sigma \cot x \dfrac{dy}{dx}$

4. If $f(x) = (a^2 - b^2)^{-\frac{1}{2}} \quad \cos^{-1}\left[\dfrac{a\cos x + b}{a + b\cos x}\right]$ then find

$f'(x) = (a + b\cos x)^{-1}$

5. $y = \dfrac{x^3\sqrt{2+3x}}{(2+x)(1-x)}$ Find $\dfrac{dy}{dx}$

6. $y = 128 \sin^3 x \cos^4 x$ Find $\dfrac{d^2 y}{dx^2}$

7. If $y = a \cos x + (b + 2x) \sin x$ then $y'' + y = 4\cos x$

8. If $y = a\cos(\sin x) + b\sin(\sin x)$ then Find $y'' + (\tan x)y' + y\cos^2 x = 0$

9. If $y = e^{a\sin^{-1} x}$ then show that $(1 - x^2)\dfrac{d^2 y}{dx} - x\dfrac{dy}{dx} - a^2 y = 0$.

10. If $y = \left(x^x\right)^x$ then find $\dfrac{dy}{dx}$

SUPPORTIVE WEBSITES

- http : //www.wikipedia.org
- http:// math world . wolfram.com

PRACTICE EXERCISE

1. Find the derivative each of the following functions]

   (a) $\left(x^{x}\right)^{x}$ (b) $x^{\left(x^{x}\right)}$

2. Find $\frac{dy}{dx}$, if

   (a) $y = a^{x \log \sin x}$ (b) $y = (\sin x)^{\cos^{-1} x}$

   (c) $y = \left(1 + \frac{1}{x}\right)^{x^2}$ (d) $y = \log\left[e^{x}\left(\frac{x-4}{x+4}\right)^{3/4}\right]$

3. Find the derivative of function

   (a) $f(x) = [\sin^{-1} x]^2 \,.\, x^{\sin x}.e^{2x}$

   (b) $y = (\tan x]^{\log x} + \cos x$

   (c) $y = x^{\tan x} + (\sin x)^{\cos x}$

   (d) $y = \frac{x^4\sqrt{x+6}}{(3x+5)^2}$

   (e) $y = \frac{e^{x} + e^{-x}}{(e^{x} - e^{-x})}$

   (f) $y = 7^{x^2+2x}$

   (g) $y = x^2 e^{2x} \cos 3x$

MODULE - V
Calculus

Notes

(h) $y = \dfrac{2^x \cot x}{\sqrt{x}}$

(i) $y = x^x$ prove that $\dfrac{x.dy}{dx} = \dfrac{y^2}{1 - y\log x}$.

4. If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

5. If $y = e^{\tan^{-1} x}$, prove that $(1+x^2)\dfrac{d^2y}{dx^2} + (2x-1)\dfrac{dy}{dx} = 0$.

ANSWERS

EXERCISE 23.1

1. $a\ e^{ax}$

2. $\dfrac{-3}{2} e^{\frac{-3x}{2}}$

3. $\dfrac{e^x}{x^2}[x-1]$

4. $7e^{7x+4}$

5. $\sqrt{2}\ e^{\sqrt{2}x}$

6. $x \quad e^x$

7. $e^{x\sec^2 x}\ [\sec^2 x + 2x\sec^2 x \tan x]$

8. cosec x $\sec^2 x$

9. $-\dfrac{1}{x}\tan(\log x)$

10. $-\tan x$

11. $\dfrac{1}{x\log x}$

12. $a^x \log a$

13. $(\log)^{\sin x}\left[\cos x\log(\log x)\right]+\dfrac{\sin x}{x\log x}$ 14. e^x

15. $\dfrac{2\log x-3}{x^3}$

EXERCISE 23.2

1. $\dfrac{e^{2x}\left[(2x-1)\cot x-x\operatorname{cosec}^2 x\right]}{x^2}$ 2. $\dfrac{3}{x}[\log x]^2$

3. $\dfrac{\cot(\log x)}{x}$ 4. $-\tan x$

5. $x^{(x^2+\sin x)}\left[\dfrac{x^2+\sin x}{x}+(2x+\cos x)\log x\right]$

6. $(x)^{x^2}.x(1+2\log x)+(\log x)^{\log x}\left[\dfrac{1+\log(\log x)}{x}\right]$

9. $y\left[\dfrac{\tan x}{x\log x}+\log(\log x)\sec^2 x\right]$

10. $x^{\sin^{-1}x}\left[\log x+\sqrt{1-x^2}\,\dfrac{\sin^{-1}x}{x}\right]$

EXERCISE 23.3

1. $\tan\theta$ 2. $-y$

3. $\Sigma\cot x\dfrac{dy}{dx}=1$

5. $\dfrac{dy}{dx}=y\left[\dfrac{3}{x}+\dfrac{3}{2(2+3x)}-\dfrac{1}{2-x}+\dfrac{1}{1-x}\right]$

6. $-54\sin 3x-6\sin x+98\sin 7x+50\sin 5x$

10. $\dfrac{dy}{dx}=\left(x^x\right)^x[x+2x\log x]$

PRACTICE EXERCISE

1. (a) $\left(x^x\right)^x [x + 2x\log x]$ (b) $x^{(x)^x}[x^{x-1} + \log x(\log x + 1)]$

2. (a) $a^{x\log\sin x}[\log\sin x + x\cot x]\log a$

(b) $(\sin x)^{\cos^{-1}x}\left[\cos^{-1}x\cot x - \frac{\log\sin x}{\sqrt{1-x^2}}\right]$

(c) $\left(1+\frac{1}{x}\right)^{x^2}\left[2x\log\left(x+\frac{1}{x}\right) - 1 + \frac{1}{\frac{1}{x}}\right]$

(d) $1 + \frac{3}{4(x-4)} - \frac{3}{4(x+4)}$

3. (a) $(\sin^{-1}x)^2 . x^{\sin x}\, e^{2x}\left[\frac{2}{\sqrt{1-x^2}\sin^{-1}x} + \cos x\log x + \frac{\sin x}{x} + 2\right]$

(b) $(\tan x)^{\log x}\left[2\operatorname{cosec} x\log + \frac{1}{x}\tan x\right] + (\cos x)^{\sin x}$

$[-\sin x\tan x + \cos x\log(\cos x)]$

(c) $x^{\tan x}\left[\frac{\tan x}{x} + \sec^2 x\log x\right] + (\sin x)^{\cos x}[\cot x\cos x - \sin x\log(\sin x)]$

(d) $\frac{x^4\sqrt{x+6}}{(3x+5)^2}\left[\frac{4}{x} + \frac{1}{2(x+6)} - \frac{6}{3x+5}\right]$

(e) $\frac{-4e^{2x}}{(e^{2x}-1)^2}$

(f) $7^{x^2+2x}(2x+2)\log_e 7$

(g) $x^2e^{2x}\cos 3x\left\{\frac{2}{x} + 2 - 3\tan 3x\right\}$

(h) $\frac{2^x\cot x}{\sqrt{x}}\left[\log 2 - 2\operatorname{cosec} 2x - \frac{1}{2x}\right]$

Chapter 24

TANGENTS AND NORMALS

LEARNING OUTCOMES

After studying this chapter, student will be able to

- Compute slope of tangent and normal to a curve at a point
- Find equations of tangents and normals to a curve at a given points.
- Find length of tangent, length of normal subtangent and subnormal.
- State Rolle's and Lagrange's mean value Theorem.

PREREQUISITES

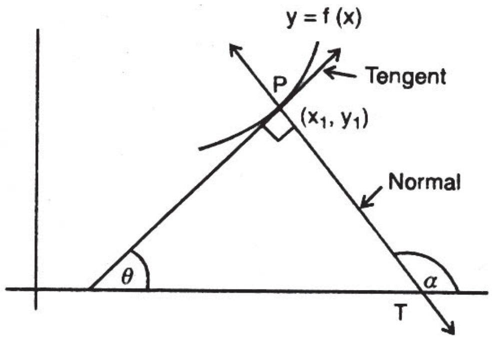
- Definition of tangent and normal to a curve, cordinate Geometry.

INTRODUCTION

Tangents and normals are the lines associated with curves. The tangent is a line touching the curve at a distinct point, and each of the points on the curve has a tangent. A line which is perpendicular to the tangent at the point of contact is called normal. In this chapter we will learn how to find the equations tangents, normal, sub tangent and sub normal for different curves.

24.1 SLOPE OF TANGENT AND NORMAL

Let y = f(x) be a continuous curve and let $P(x_1, y_1)$ be a point on it then the slope PT at $P(x_1, y_1)$ is given by



$$\left(\frac{dy}{dx}\right) \text{ at } (x_1, y_1) \quad \text{.... (1)}$$

and (i) is equal to tan θ

We know that a normal to a curve is a line perpendicular to the tangent at the point of contact

We know that $\alpha = \frac{\pi}{2} + \theta$

$$\Rightarrow \tan\alpha = \tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$$

$$= \frac{-1}{\tan\theta}$$

$$\therefore \text{ Slope of normal } = -\frac{1}{m} = -\frac{1}{\left(\frac{dy}{dx}\right)} \text{ at } (x_1, y_1) = -\frac{dx}{dy} \text{ at } (x_1, y_1)$$

Note

1. The tangent to a curve at any point will be parallel to x-axis if θ = 0 i.e., the derivative at the point will be zero.

   i.e., $\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$.

2. The tangent at a point to the curve $y = f(x)$ will be parallel to y-axis

   if $\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$.

Example 24.1 : Find the slope of tangent and normal to the curve

$$x^2 + x^3 + 3xy + y^2 = 5 \text{ at } (1, 1)$$

Solution : The equation of the curve is

$$x^2 + x^3 + 3xy + y^2 = 5 \qquad .....(i)$$

Differentialing (i),w.r.t. x, we get

$$2x + 3x^2 + 3\left[x\frac{dy}{dx} + y.1\right] + 2y\frac{dy}{dx} = 0$$

Substituting $x = 1$, $y = 1$, in (ii), we get

$$2x + 3x^2 + 3\left[x\frac{dy}{dx} + y.1\right] + 2y\frac{dy}{dx} = 0$$

or $\quad 5\frac{dy}{dx} = -8 \quad \Rightarrow \frac{dy}{dx} = -\frac{8}{5}$

$\therefore$ The slope of tangent to the curve at (1, 1) is $-\frac{8}{5}$

$\therefore$ The slope of normal to the curve at (1, 1) is $\frac{5}{8}$

Example 24.2 Show that the tangents to the curve $y = \frac{1}{6}\left[3x^5 + 2x^3 - 3x\right]$ at the points $x = \pm 3$ are parallel.

Solution : The equation of the curve is $y = \frac{3x^5 + 2x^3 - 3x}{6}$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{15x^4 + 6x^2 - 3}{6}$$

$$\left(\frac{dy}{dx}\right)_{x=3} \text{at} = \left[\frac{15(3)^4 + 6(3)^2 - 3}{6}\right]$$

$$= \frac{1}{6}[15 \times 9 \times 9 + 54 - 3]$$

$$= \frac{3}{6}[405 + 17] = 211$$

$$\left(\frac{dy}{dx}\right) \text{at } x = -3 = \frac{1}{6}[15(-3)^4 + 6(-3)^2 - 3] = 211$$

$\therefore$ The tangents to the curve at $x = \pm 3$ are parallel as the slopes at $x = \pm 3$ are equal.

Example 24.3 : The slope of the curve $6y^3 = px^2 + q$ at (2, –2) is $\frac{1}{6}$. Find the values of p and q.

Solution : The equation of the curve is

$$6y^3 = px^2 + q$$

Differentiating (i) w.r.t. x, we get

$$18y^2 \frac{dy}{dx} = p(2x) + 0$$

Putting $x = 2$, $y = -2$, we get

$$18(-2)^2 \frac{dy}{dx} = 2p(2) = 4p$$

$$\text{Slope} = \left.\frac{dy}{dx}\right|_{(2,-2)} = \frac{p}{18}$$

It is given equal to $\frac{1}{6}$

$$\therefore \quad \frac{1}{6} = \frac{p}{18} \Rightarrow p = 3$$

$\therefore$ The equation of curve becomes

$$6y^3 = 3x^2 + q$$

Also, the point (2, –2) lies on the curve

$$6(-2)^3 = 3(2)^2 + q$$

$6(-8) = 3(4) + q$

$-48 - 12 = q$

$q = -60$

$\therefore \quad p = 3, q = -60$

EXERCISE 24.1

1. Find the slopes of tangents and normals to each of the curves at the given points :

   (i) $y = x^3 - 2x$, $x = 2$ (ii) $x^2 + 3y + y^2 = 5$ at (1, 1)

   (iii) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at $\theta = \frac{\pi}{2}$

2. Find the values of p and q if the slope of the tangent to the curve $xy + px + qy = 2$ at (1, 1) is 2.

3. Find the points on the curve $x^2 + y^2 = 18$ at which the tangents are parallel to the line $x + y = 3$.

4. At what points on the curve $y = x^2 - 4x + 5$ is the tangent perpendiculat to the line $2y + x - 7 = 0$.

24.2 EQUATIONS OF TANGENT AND NORMAL TO A CURVE

We know that the equation of a line passing through a point (x_1, y_1) and with slope m is

$$y - y_1 = m(x - x_1)$$

As discussed in t he section before, the slope of tangent to the curve $y = f(x)$ at (x_1, y_1) is given by $\left(\frac{dy}{dx}\right)$ at (x_1, y_1) and that of normal is $\left(-\frac{dx}{dy}\right)$ at (x_1, y_1).

$\therefore$ Equation of tangent to the curve $y = f(x)$ at the point (x_1, y_1) is

$$y - y_1 = \frac{dx}{dy}(x - x_1)$$

And, the equation of normal to the curve $y = f(x)$ at the point (x_1, y_1) is

$$y - y_1 = \left(\frac{-1}{\frac{dy}{dx}}\right)(x - x_1)$$

Note

(i) Normal at (x_1, y_1) in parallel to y-axis

(ii) In case $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} \to \infty$ the tangent at (x_1, y_1) in parallel to y-axis and its equation in $x = x_1$. Normal at (x_1, y_1) in parallel $\hbar$ x-axis.s

Let us take some examples and illustrate

Example 24.4 : Find the equation of the tangent and normal to the circle $x^2 + y^2 = 25$ at the point (4, 3).

Solution : The equation of circle is

$$x^2 + y^2 = 25$$

Differentialing (1), w.r.t. x, we get

$$2x + 2y.\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\left.\frac{dy}{dx}\right|_{(4,3)} = \frac{-4}{3}$$

$\therefore$ Equation of tangent to the circle at (4, 3) is

$$y - 3 = -\frac{4}{3}(x - 4)$$

or $\quad 4(x-4)+3(y-3)=0 \quad$ or $\quad 4x+3y=25$

Also, slope of the normal $= \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(4,3)}} = \dfrac{3}{4}$

$\therefore$ Equation of the normal to the circle at (4,3) is

$$y-3=\frac{3}{4}(x-4)$$

or $\quad 4y-12=3x-12$

$\Rightarrow \quad 3x-4y=0$

$\therefore$ Equation of the tangent to the circle at (4, 3) is $4x+3y-25=0$,

Equation of the normal to the circle at (4, 3) is $3x-4y=0$

Example 24.5 : Find the equation of the tangent and normal to the curve $16x^2+9y^2=144$ at point (x_1, y_1) where $y_1>0$ and $x_1=2$

Solution : The equation of curve is

$$16x^2+9y^2=144$$

$$32x+18y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-16x}{9y}$$

$$\left.\frac{dy}{dx}\right|_{\left(2,\frac{4\sqrt{5}}{3}\right)}=\frac{-16(2)}{9\left(\frac{4\sqrt{5}}{3}\right)}=\frac{-8}{3\sqrt{5}}$$

$x_1=2$ and (x_1, y_1) lies on the curve

$\therefore \quad 16(2)^2+9(y)^2=144$

$$\Rightarrow \quad y^2=\frac{80}{9} \Rightarrow y=\pm\frac{4\sqrt{5}}{3}$$

As $\quad y_1>0 \Rightarrow y=\dfrac{4\sqrt{5}}{3}$

$\therefore$ Equation of the tangent to the curve at $\left(2\ ,\ \frac{4\sqrt{5}}{3}\right)$ is

$$y-\frac{4\sqrt{5}}{3}=\frac{-8}{3\sqrt{5}}(x-2)$$

or $\quad 3\sqrt{5}y-20=-8x+16$

or $\quad 8x+3\sqrt{5}y-36=0$

Also, equation of the normal to the curve at $\left(2,\frac{4}{3}\sqrt{5}\right)$ is

$$y-\frac{4\sqrt{5}}{3}=\frac{-1}{\left(\frac{-8}{3\sqrt{5}}\right)}(x-2)$$

$$y-\frac{4\sqrt{5}}{3}=\frac{3\sqrt{5}}{8}(x-2)$$

$$\frac{3y-4\sqrt{5}}{3}=\frac{3\sqrt{5}x-6\sqrt{5}}{8}$$

$$24y-32\sqrt{5}=9\sqrt{5}x-18\sqrt{5}$$

or $\quad 9\sqrt{5}x-24y+14\sqrt{5}=0$

Example 24.6 : Find the points on the curve $\frac{x^2}{9}-\frac{y^2}{16}=1$ at which the tangents are parallel to x-axis.

Solution : The equation of the curve is

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

Differentiating (i) w.r.t. x we get

$$\frac{2x}{9}-\frac{2y}{16}\frac{dy}{dx}=0$$

or $\frac{dy}{dx} = \frac{16x}{9y}$

For tangent to be parallel to x-axis, $\frac{dy}{dx} = 0$

$\Rightarrow \frac{16x}{9y} = 0 \quad \Rightarrow x = 0$

Putting x = 0 in (i), we get $y^2 = -16 \Rightarrow y = \pm 4i$

This implies that there are no real points at which the tangent to $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is parallel to x-axis.

Example 24.7 : Find the equation of all lines having slope – 4 that are tangents to the curve $y = \frac{1}{x-1}$

Solution : $y = \frac{1}{x-1}$ ...(i)

$\therefore \frac{dy}{dx} = \frac{-1}{(x-1)^2}$

It is given equal to – 4

$\therefore \frac{-1}{(x-1)^2} = -4$

$\Rightarrow (x-1)^2 = \frac{1}{4}$

$\Rightarrow x-1 = \pm\frac{1}{2} \quad \Rightarrow x = \frac{3}{2}, \frac{1}{2}$

Substituting $x = \frac{3}{2}$ in (i), we get

$$y = \frac{1}{\frac{3}{2}-1} = \frac{1}{\frac{1}{2}} = 2$$

Point $\left(\frac{3}{2}, 2\right)$

Substituting $x = \frac{1}{2}$ in (i), we get

$$y = \frac{1}{\frac{1}{2} - 1} = \frac{1}{-\frac{1}{2}} = -2$$

When $\quad x = \frac{1}{2},\ y = -2$

$\therefore$ This point are $\left(\frac{3}{2}, 2\right)$, $\left(\frac{1}{2}, -2\right)$

$\therefore$ The equation of tangents are

(a) $\quad y - 2 = -4\left(x - \frac{3}{2}\right)$

$\Rightarrow \quad y - 2 = -4x + 6 \quad$ or $\quad 4x + y = 8$

(b) $\quad y + 2 = -4\left(x - \frac{1}{2}\right)$

$\Rightarrow \quad y + 2 = -4x + 2 \quad$ or $\quad 4x + y = 0$

Example 24.8 : Find the equation of the normal to the curve $y = x^3$ at (2 , 8)

Solution : $y = x^3$

$$\Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \left.\frac{dy}{dx}\right|_{(2,8)} = 3(2)^2 = 12$$

$\therefore$ Slope of the normal $= -\frac{1}{12}$

$\therefore$ Equation of the normal is $y - 8 = -\frac{1}{12}(x - 2)$

or $\quad 12(y - 8) + (x - 2) = 0$

or $\quad x + 12y = 98$

EXERCISE 24.2

1. Find the equation of the tangent and normal at the indicated points :
   (i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0 , 5)
   (ii) $y = x^2$ at (1 , 1)
   (iii) $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.
2. Find the equation of the targent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).
3. Find the equation of the tangent to the hyperbola
   $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ at } (x_0, y_0).$$
4. Find the equation of normals to the curve $y = x^3+2x + 6$ which are parallel to the line $x + 14y + 4 + 0$.
5. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

24.3 LENGTHS OF TANGENT, NORMAL, SUBTANGENT AND SUBNORMAL

In this section we define the length of tangent normal, subtangent and subnormal and derive formulae to find these lengths.

Definition :

Suppose $P = (x_1, y_1)$ is a point on the curve $y = f(x)$. Let the tangent and normal to the curve at P meet the x-axis in L and G respectively. Let M be the foot of the perpendicular drawn from P on to the X-axis.

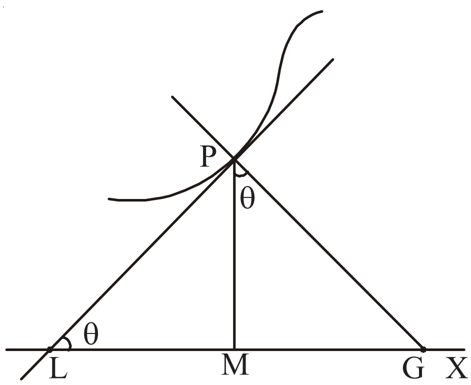
Then

(i) PL is called the length of the tangent.

(ii) PG is called the length of the normal.

(iii) LM is called the length of the subtangent.

(iv) MG is called the length of the subnormal.



In general it, $\theta \neq 0$ and $\theta \neq \frac{\pi}{2}$ we can find simple formulae for the above four lengths.

(i) Length of the tangent = PL $= \frac{PM}{\sin\theta}$

$$= \left|\frac{y_1}{\sin\theta}\right|$$

$$= \left|\frac{y_1}{\tan\theta\cos\theta}\right|$$

$$= \left|\frac{y_1\sec\theta}{\tan\theta}\right|$$

$$= \left|\frac{y_1\sqrt{1+\tan^2\theta}}{\tan\theta}\right|$$

$$= \left|\frac{y_1\sqrt{1+\left[\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}\right]^2}}{\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}}\right|$$

(ii) Length of the normal = PG = $PM\sec\theta$

$$= \left|y_1\sec\theta\right|$$

$$= \left|y_1\sqrt{1+\tan^2\theta}\right|$$

$$= \left|y_1\sqrt{1+\left[\left(\frac{dy}{dx}\right)_{(x_1,y_1)}\right]^2}\right|$$

(iii) Length of the subtangent = LM = $LM = \left|\frac{y_1}{\tan\theta}\right|$

$$= \left|\frac{y_1}{\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}}\right|$$

(iv) Length of the subnormal = MG = $|y_1 \tan\theta|$

$$= \left|y_1\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}\right|$$

In case of a general point (x, y) on a curve the above formulae can be remembered as

(i) Length of the tangent = $\left|\frac{y\sqrt{1+(y')^2}}{y'}\right|$

(ii) Length of normal = $\left|y\sqrt{1+(y')^2}\right|$

(iii) Length of subtangent = $\left|\frac{y}{y'}\right|$

(iv) Length of subnormal = $|y\ y'|$ $\left[y' = \frac{dy}{dx}\right]$

Example 24.9 : Show that the length of the subnormal at any point on the curve $y^2 = 4ax$ is a constant.

Solution : Differentiating $y^2 = 4ax$ with respect to x, we have $2y\,y' = 4a$

$$\Rightarrow \quad y' = \frac{4a}{2y} = \frac{2a}{y}$$

$$\Rightarrow \quad y\,y' = 2a$$

$\therefore$ The length of the subnormal at any point (x, y) on the curve $= |\,y\,y'\,| = |\,2a\,|$ a constant.

Example 24.10: Show that the length of the subntangent at any point on the curve $y = a^x$ $(a > 0)$ is a constant.

Solution : Differenting $y = a^x$ w.r.t. x, we have

$$y' = a^x \log a$$

∴ The length of the subtangent at any point (x, y) on the curve

$$= \left|\frac{y}{y'}\right| = \left|\frac{a^x}{a^x \log a}\right| = \frac{1}{\log a} = \text{constant.}$$

EXERCISE 24.3

1. Find the length of subtangent and subnormal at a point on the curve $y = b \sin \frac{x}{a}$.
2. Show that at any point (x, y) on the curve $y = be^{x/a}$ the length of the subtangent is a constant and the length of the subnormal is $\frac{y^2}{a}$.

24.4 ROLLE'S THEOREM

Let us now study an important theorem which reveals that between two points a and b on the graph of $y = f(x)$ with equal ordinates $f(a)$ and $f(b)$, there exists at least one point c such that the tangent at $[c, f(c)]$ is parallel to x-axis. (see Fig. 24.2).

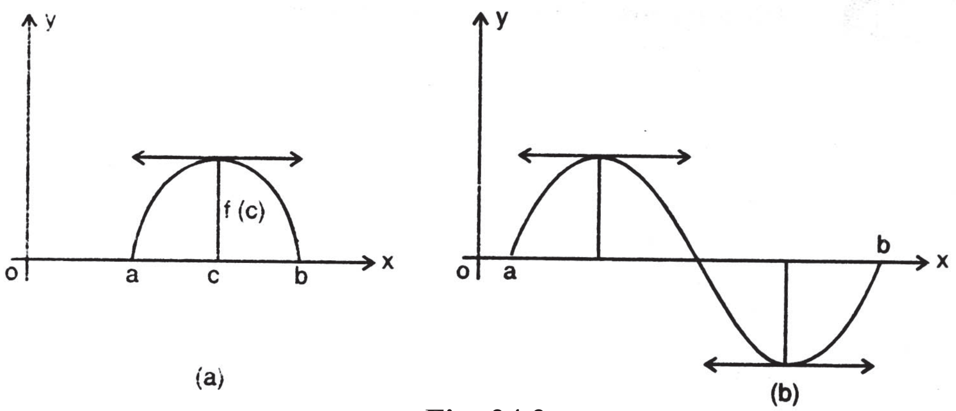


Fig. 24.2

24.4.1 Mathematical formulation of Rolle's Theorem

Let f be a real function defined in the closed interval $[a, b]$ such that

(i) f is continuous in the closed interval $[a, b]$

(ii) f is differentiable in the open inteval (a, b)

(iii) $f(a) = f(b)$

Then there is at least one point c in the open inteval (a, b) such that $f'(c) = 0$.

Remarks

(i) The remarks "at least one point" says that there can be more than one point $c \in (a, b)$ such that $f'(c) = 0$.

(ii) The condition of continuity of f on $[a, b]$ is essential and can not be relaxed

(iii) The condition of differentiability of f on (a, b) is also essential and can not be relaxed.

For example $f(x) = |x|, x \in [-1, 1]$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$ and Rolle's Theorem is valid for this

Let us take some examples

Example 24.11 Verify Rolle's for the function

$$f(x) = x(x - 1)(x - 2), x \in [0, 2]$$

Solution : $f(x) = x(x - 1)(x - 2)$

$$= x^3 - 3x + 2x$$

(i) $f(x)$ is a polynomial function and hence continuous in $[0, 2]$

(ii) $f(x)$ is differentiable on $(0, 2)$

(iii) Also $f(0) = 0$ and $f(2) = 0$

$\therefore \quad f(0) = f(2)$

$\therefore$ All the conditions of Rolle's theorem are satisfied.

Also, $f'(x) = 3x^3 - 6x + 2$

$\therefore\ f'(c) = 0$ gives $3c^2 - 6c + 2 = 0 \Rightarrow c = \dfrac{6 \pm \sqrt{36-24}}{6}$

$$\Rightarrow c = 1 \pm \frac{1}{\sqrt{3}}$$

We see that both the values of c lie in (0, 2)

Example 24.12 Discuss the applicability of Rolle's Theorem for

$$\sin x - \sin 2x,\ x \in [0, \pi]$$

(i) Is a sine function. It is continuous and differentiable on $(0, \pi)$

Again, we have

$f(0) = \sin 0 - \sin 2(0) = 0 - \sin 0 = 0 - 0 = 0$

$f(\pi) = \sin \pi - \sin 2\pi = 0 - 0 = 0$

$\Rightarrow \quad f(\pi) = f(0) = 0$

$\therefore$ All the conditions of Rolle's theorem are satisfied

$$f'(x) = \cos x - 2\cos x$$

Now $\quad f'(c) = 0$

$$\cos c - 2 \cos 2c = 0$$

$$\cos c - 2\,[2\cos^2 c - 1] = 0$$

$$4 \cos^2 c - \cos c - 2 = 0$$

$$\therefore \quad \cos c = \frac{1 \pm \sqrt{1+32}}{8}$$

$$= \frac{1+\sqrt{33}}{8}$$

As $\sqrt{33} < 6$

$$\therefore \quad \cos c < \frac{7}{8} = 0.875$$

which shows that c lies between 0 and π

EXERCISE 24.4

Verify Rolle's Theorem for each of the following functions :

(i) $f(x) = \dfrac{x^3}{3} - \dfrac{5x^2}{3} + 2x,\ x \in [0, 3]$

(ii) $f(x) = x^2 - 1,\ x \in [-1, 1]$

(iii) $f(x) = \sin x + \cos x - 1,\ x \in \left[0, \dfrac{\pi}{2}\right]$

(iv) $f(x) = (x^2 - 1)(x - 2),\ [-1, 2]$

24.5 LANGRANGE'S MEAN VALUE THEOREM

This theorem improves the result of Rolle's Theorem saying that it is not necessary that tangent may be parallel to x-axis. This theorem says that the tangent is parallel to the line joining the end points of the curve. In other words, this theorem says that there always exists a point on the graph, where the tangent is parallel to the line joining the end-points of the graph.

24.5.1 Mathematical Formulation of the Theorem

Let f be a real valued function defined on the closed interval $[a, b]$ such that

(a) f is continuous on $[a, b]$, and

(b) f is differentiable in (a, b)

(c) $f(b) \neq f(a)$

then there exists a point c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Remarks

When $f(b) = f(a)$, $f'(c) = 0$ and the theorem reduces to Rolle's Theorem

Let us consider some examples

Example 24.13 Verify Langrange's Mean value theorem for

$$f(x) = (x - 3)(x - 6)(x - 9) \text{ on } [3, 5]$$

Solution : $f(x) = (x - 3)(x - 6)(x - 9)$

$$= (x - 3)(x^2 - 15x + 54)$$

or $\quad f(x) = (x^3 - 18x^2 + 99x - 162)$

$f(x)$ is a polynomial function and hence continuous and differentiable in the given interval

Here, $f(3) = 0,\ f(5) = (2)(-1)(-4) = 8$

$$\therefore \quad f(3) \neq f(5)$$

$\therefore$ All the conditions of Mean value Theorem are satisfied

$$\therefore \quad f'(c) = \frac{f(5) - f(3)}{5-3} = \frac{8-0}{2} = 4$$

Now $\quad f'(x) = 3x^2 - 36x + 99$

$$\therefore \quad 3c^2 - 36c + 99 = 4 \text{ or } \quad 3c^2 - 36c + 95 = 0$$

$$\therefore \quad c = \frac{36 \pm \sqrt{1296 - 1140}}{6} = \frac{36 \pm 12.5}{6}$$

$$= 8.08 \text{ or } 3.9$$

$$c = 3.9 \in (3,5)$$

$\therefore$ Langranges mean value theorem is verified

Example 24.14 Find a point on the parabola $y = (x - 4)^2$ where the tangent is parallel to the chord joining (4, 0) and (5, 1)

Solution : Slope of the tangent to the given curve at any point is given by $(f'(x))$ at that point.

$$f'(x) = 2(x - 4)$$

Slope of the chord joining (4, 0) and (5, 1) is

$$\frac{1-0}{5-4} = 1 \qquad \left[\because m = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

$\therefore$ According to mean value theorem

$$2(x - 4) = 1 \quad \text{or} \quad x - 4 = \frac{1}{2}$$

$$x = \frac{9}{2}$$

which lies between 4 and 5

Now $y = (x - 4)^2$

When $x = \frac{9}{2},\ y = \left(\frac{9}{2} - 4\right)^2 = \frac{1}{4}$

The required points is $\left(\frac{9}{2}, \frac{1}{4}\right)$

EXERCISE 24.5

1. Check the applicability of Mean Value Theorem for each of the following functions :

   (i) $f(x) = 3x^2 - 4$ on $[2, 3]$ (ii) $f(x) = \log x$ on $[1, 2]$

   (iii) $f(x) = x + \frac{1}{x}$ on $[1, 3]$ (iv) $f(x) = x^3 - 2x^2 - x - 3$ on $[0, 1]$

2. Find a point on the parabola $y = (x + 3)^2$ where the tangent is parallel to the chord joining $(3, 0)$ and $(-4, 1)$

KEY WORDS

- The equation of tangent at (x_1, y_1) to the curve $y = f(x)$ is given by

$$y - y_1 = [f'(x)]_{\text{at}(x_1, y_1)} \{x - x_1\}$$

- The equation of normal at (x_1, y_1) to the curve $y = f(x)$ is given by

$$y - y_1 = \left[\frac{-1}{f'(x)}\right]_{(x_1, y_1)} (x - x_1)$$

- The equation of tangent to a curve $y = f(x)$ at (x_1, y_1) and parallel to x-axis is given by $y = y_1$ and parallel to y-axis is given by $x = x_1$.

- Suppose $p(x, y)$ in any point on the curve $y = f(x)$. Then

  (i) Length of the tangent = $\left|\dfrac{y\sqrt{1+(y')^2}}{y'}\right|$

  (ii) Length of normal = $\left|y\sqrt{1+(y')^2}\right|$

  (iii) Length of subtangent = $\left|\dfrac{y}{y'}\right|$

  (iv) Length of subnormal = $|yy'|$

- **Rolle's Theorem states :** If $f(x)$ is a function which is

  (i) continuous in the closed interval $[a, b]$

  (ii) differentiable in the open interval (a, b)

  (iii) $f(a) = f(b)$

  then there exists a point c in (a, b) such that $f'(c) = 0$

SUPPORTIVE WEB SITES

- *http://www.wikipedia.org*
- *http://mathworld.wolfram.com*

PRACTICE EXERCISE

1. Find the slopes of tangents and normals to each of the following curves at the indicated points :

   (i) $y = \sqrt{x}$ at $x = 9$

   (ii) $y = x^3 + x$ at $x = 2$

(iii) $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$ at $\theta = \dfrac{\pi}{2}$

(iv) $y = 2x^2 + \cos x$ at $x = 0$

(v) $xy = 6$ at $(1, 6)$

2. Find the equations of tangent and normal to the curve

$x = a\cos^3\theta,\ \ y = a\sin^3\theta$ at $\theta = \dfrac{\pi}{4}$

3. Find the point on the curve $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$ at which the tangents are parallel to y-axis.

4. Find the equation of the tangents to the curve $y = x^2 - 2x + 5$,

(i) which is parallel to the line $2x + y + 7 = 0$

(ii) which is perpendicular to the line $5(y - 3x) = 12$

5. Show that the tangents to the curve $y = 7x^3 + 11$at the points $x = 2$ and $x = -2$ are parallel.

6. Find the equation of normal at the point (am^2, am^3) to the curve $ay^2 = x^3$

7. Find length of tangent, length of normal, length of subtangent and sub-normal to the curve $y = x^3 + 1$ at the point (1, 2).

8. Verify Rolle's Theorem for each of the following functions:

(i) $f(x) = (x^2 - 1)(x - 2)$ on $[-1, 2]$

(ii) $f(x) = \dfrac{x(x-2)}{x-1}$ on $[0,2]$

(iii) $f(x) = \dfrac{8x^2}{3} - 2x,\ x \in \left[0, \dfrac{3}{4}\right]$

9. If Rolle's theorem holds for $f(x) = x^3 + bx^2 + ax$, [1, 3] with $c = 2 + \dfrac{1}{\sqrt{3}}$ find the values of a and b. •

10. Verify Mean Value Theorem for each of the following functions.

    (i) $f(x) = a\,x^3 + b\,x^2 + c\,x + d$ on $[0, 1]$

    (ii) $f(x) = \dfrac{1}{4x+1}$ on $[-1, 4]$

    (iii) $y = (x+3)^2$ on $[-4, 3]$

11. Find a point on the parabola $y = (x - 3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

ANSWERS

EXERCISE 24.1

1. (i) $10, \; -\dfrac{1}{10}$ (ii) $-\dfrac{2}{5}, \dfrac{5}{2}$ (iii) $1, -1$
2. $p = 5,\ q = -4$
3. $(3, 3),\ (-3, -3)$
4. $(3, 2)$

EXERCISE 24.2

1.

Tangent	**Normal**
(i) $y + 10x = 5$	$x - 10y + 50 = 0$
(ii) $2x - y = 1$	$x + 2y - 3 = 0$
(iii) $24x - y = 52$	$x + 24y = 483$

2. $\dfrac{x\,x_1}{a^2} + \dfrac{y\,y_1}{b^2} = 1$
3. $\dfrac{x\,x_0}{a^2} - \dfrac{y\,y_0}{b^2} = 1$
4. $x + 14y - 254 = 0,\ \ x + 14y + 86 = 0$

EXERCISE 24.3

1. Length of subtangent $\left|a\tan\frac{x}{a}\right|$

2. Length of subnormal $\left|\frac{b^2}{2a}\sin\frac{2x}{a}\right|$

EXERCISE 24.4

1. (i) $c = \frac{5 \pm \sqrt{7}}{3}$ (ii) $c = 0$

   (iii) $c = \frac{\pi}{4}$ (iv) $c = \frac{2 \pm \sqrt{7}}{3}$

EXERCISE 24.5

1. (i) $c = 2.5$ (ii) $c = \frac{1}{\log_e^2}$

   (iii) $c = \sqrt{3}$ (iv) $c = \frac{1}{3}$

2. $\left(-\frac{43}{14}, \frac{1}{196}\right)$

PRACTICE EXERCISE

1. (i) $\frac{1}{6}, -6$ (ii) $13, -\frac{1}{13}$

   (iii) $1, -1$

   (iv) 0, not defined (v) $-6, \frac{1}{6}$

2. $2\sqrt{2}\,(x + y) = a;\; x + y = 0$

3. (3, 0), (−3, 0)

4. (i) $2x + y - 5 = 0$

(ii) $12x + 36y = 155$

6. $2x + 3m\ y - am^2\ (2 + 3\ m^2) = 0$

7. Length or tangent : $\frac{2}{3}\sqrt{10}$

Length of normal $2\sqrt{10}$

Length of subtangent 2/3

Length of subnormal 6

8. $c = \dfrac{2 \pm \sqrt{7}}{3}$

(ii) At no real point

(iii) $c = \dfrac{3}{8}$

9. $a = 11, \quad b = -6$

10. (i) $c = \dfrac{1}{2}$

(ii) Not applicable

(iii) $c = -\dfrac{1}{2}$

11. $\left(\dfrac{7}{2}, \dfrac{1}{4}\right)$.

MAXIMA AND MINIMA

Chapter

25

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- define increasing and decreasing functions.
- find the stationary points of the given functions.
- find maxima and minima of a function.

PREREQUISITES

- Functions, Diffnition of a function, second derivative of a function.

INTRODUCTION

In this chapter using differentiation we find out intervals in which a given function is increasing or derceasing. We also show how differentiation can be used to find the maximum and minimum values of a function.

25.1 INCREASING AND DECREASING FUNCTIONS

You have already seen the common trends of an increasing or a decreasing function. Here we will try to establish the condition for a function to be an increasing or a decreasing.

Let a function $f(x)$ be defined over the closed interval $[a, b]$.

Let $x_1, x_2 \in [a,b]$ then the function $f(x)$ is said to be an increasing function in the given interval if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$. It is said to be strictly increasing if $f(x_2) > f(x_1)$ for all $x_2 > x_1$, $x_1, x_2 \in [a, b]$.

In Fig. 25.1, sin x increases from −1 to +1 as x increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

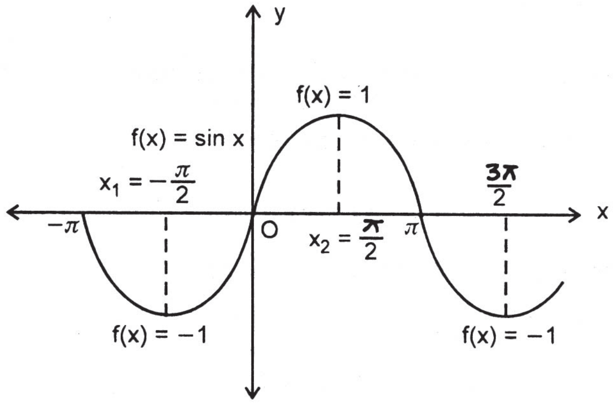


Fig. 25.1

Note : A function is said to be an increasing function in an interval if $f(x + h) > f(x)$ for all x belonging to the interval when h is positive.

A function $f(x)$ defined over the closed interval $[a, b]$ is said to be a decreasing function in the given interval, if $f(x_2) \leq f(x_1)$, whenever $x_2 > x_1$, $x_1, x_2 \in [a, b]$. It is said to be strictly decreasing if $f(x_1) > f(x_2)$ for all $x_2 > x_1$, $x_1, x_2 \in [a, b]$.

In Fig. 25.2, cos x decreases from 1 t o −1 as x increases from 0 to π.

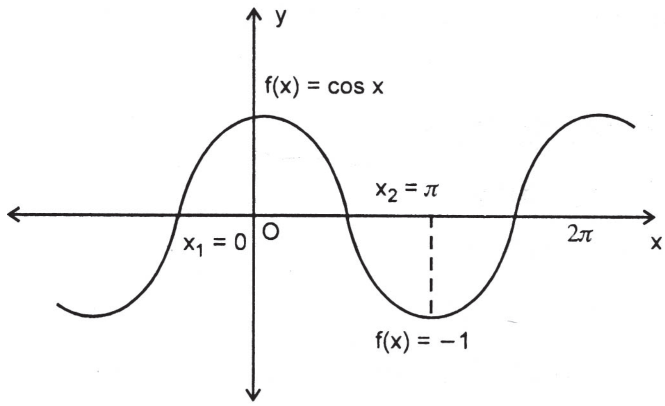


Fig. 25.2

Note : A function is said to be a decreasing in an internal if $f(x + h) + f(x)$ for all x belonging to the interval when h is positive.

25.2 MONOTONIC FUNCTIONS

Let x_1, x_2 be any two points such that $x_1 < x_2$ in the interval of definition of a function $f(x)$. Then a function $f(x)$ is said to be monotonic if it is either increasing or decreasing. It is said to be monotonically increasing if $f(x_2) > f(x_1)$ for all $x_2 > x_1$ belonging to the interval and monotonically decreasing if $f(x_1) > f(x_2)$.

Example 25.1 Prove that the function $f(x) = 4x + 7$ is monotonic for all values of $x \in R$.

Solution : Consider two values of x say $x_1, x_2 \in R$

such that $\quad x_2 > x_1 \quad$ .....(1)

Multiplying both sides of (1) by 4, we have $4x_2 > 4x_1$ .....(2)

Adding 7 to both sides of (2), to get

$$4x_2 + 7 > 4x_1 + 7$$

We have $\quad f(x_2) > f(x_1)$

Thus, we find $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

Hence the given function $f(x) = 4x_2 + 7$ is monotonic function. (monotonically increasing).

Example 25.2 Show that

$$f(x) = x^2$$

is a strictly decreasing function for all $x < 0$

Solution : Consider any two values of x say x_1, x_2 w

$$x_2 > x_1, \qquad x_1, x_2 < 0 \qquad \text{...(i)}$$

Order of the inequality reverses when it is multiplied by a negative number.

Now multiplying (i) by x_2, we have

$$x_2 \,.\, x_1 < x_1 \,.\, x_2$$

or $$x_2^2 < x_1 \; x_2 \qquad \text{...(ii)}$$

Now multiplying (i) by x_1, we have

$$x_1 \,.\, x_2 < x_1 \,.\, x_1$$

or $$x_1 \; x_2 < x_1^2 \qquad \text{...(iii)}$$

From (ii) and (iii), we have

$$x_2^2 < x_1 \, x_2 < x_1^2$$

or $$x_2^2 < x_1^2$$

$$f(x_2) < f(x_1) \qquad \text{...(iv)}$$

Thus, from (i) and (iv), we have for

$$x_2 > x_1$$

$$f(x_2) < f(x_1)$$

Hence, the given function is strictly decreasing for all $x < 0$.

EXERCISE 25.1

1. (a) Prove that the function

$$f(x) = 3x + 4$$

is monotonic increasing function for all values of $x \in$ R

(b) Show that the function

$$f(x) = 7 - 2x$$

is monotonically decreasing function for all values of $x \in$ R

(c) Prove that $f(x) = ax + b$ where a, b are constants and $a > 0$ is a strictly increasing function for all real values of x.

2. (a) Show that $f(x) = x^2$ is a strictly increasing function for all real $x > 0$.

(b) Prove that the function $f(x) = x^2 - 4$ is monotonically increasing for $x > 2$ and monotonically decreasing for $-2 < x < 2$ where $x \in R$

Theorem 1 : If $f(x)$ is an increasing function on an open interval $]a, b[$, then its derivative $f'(x)$ is positive at this point for all $x \in [a, b]$.

Proof : Let (x, y) or $[x, f(x)]$ be a point on the curve $y = f(x)$

For a positive δx, we have

$$x + \delta x > x$$

Now, function $f(x)$ is an increasing function

$$\therefore \quad f(x + \delta x) > f(x)$$

$$\text{or} \quad f(x + \delta x) - f(x) > 0$$

$$\text{or} \quad \frac{f(x+\delta x) - f(x)}{\delta x} > 0 \quad (\because \delta x > 0)$$

Taking δx as a small positive number and proceeding to limit, when $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} > 0$$

$$\text{or,} \quad f'(x) > 0$$

Thus, if $y = f(x)$ is an increasing function at a point, then $f'(x)$ is positive at that point.

Theorem 2 : If $f(x)$ is a decreasing function on an open interval $]a, b[$ then its derivative $f'(x)$ is negative at that point for all $x \in [a, b]$.

Proof : Let (x, y) or $[x, f(x)]$ be a point on the curve $y = f(x)$

For a positive δx we have

$$x + \delta x > x$$

Since the function is a decreasing function

$$\therefore \quad f(x + \delta x) < f(x)$$

or $\quad f(x+\delta x) - f(x) < 0$

Dividing by δx, we have $\dfrac{f(x+\delta x)-f(x)}{\delta x} < 0 \quad (\because \delta x > 0)$

or $$\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} < 0$$

or $\quad f'(x) < 0$

Thus, if $y = f(x)$ is a decreasing function at a point, then, $f'(x)$ is negative at that point.

Note : If $f(x)$ is derivable in the closed interval $[a, b]$, then $f(x)$ is

(i) increasing over $[a, b]$, if $f'(x) > 0$ in the open interval $]a, b[$

(ii) decreasing over $[a, b]$, if $f'(x) < 0$ in the open interval $]a, b[$.

25.3 RELATION BETWEEN THE SIGN OF THE DERIVATIVE AND MONOTONICITY OF FUNCTION

Consider a function whose curve is shown in the Fig. 25.3

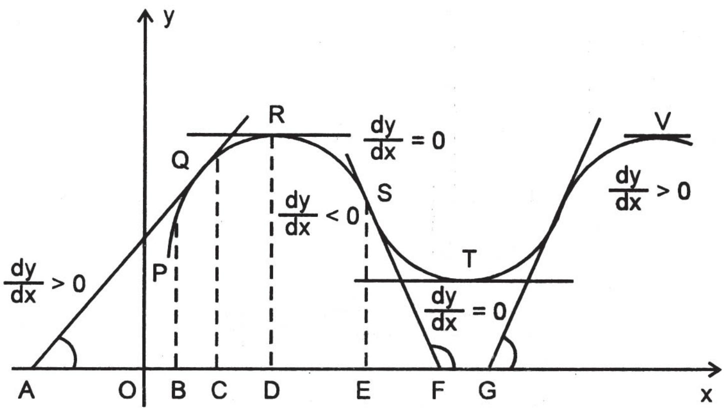


Fig. 25.3

We divide, our study of relation between sign of derivative of a function and its increasing or decreasing nature (monotonicity) into various parts as per Fig. 25.3

(i) P to R $\qquad$ (ii) R to T $\qquad$ (iii) T to V

(i) We observe that the ordinate (y-coordinate) for every succeeding point of the curve from P to R increases as also its x-coordinate. If (x_2, y_2) are the coordinates of a point that succeeds (x_1, y_1), then $x_2 > x_1$ yields $y_2 > y_1$ or $f(x_2) > f(x_1)$.

Also the tangent at every point of the curve between P and R makes acute angle with the positive direction of x-axis and thus the slope of the tangent at such points of the curve (except at R) is positive. At R where the ordinate is maximum the tangent is parallel to x-axis, as a result the slope of the tangent at R is zero.

We conclude for this part of the curve that

(a) The function is monotonically increasing from P to R

(b) The tangent at every point (except at R) makes an acute angle with positive direction of x-axis.

(c) The slope of tangent is positive i.e., $\frac{dy}{dx} > 0$ for all points of the curve for which y is increasing.

(d) The slope of tangent at R is zero i.e., $\frac{dy}{dx} = 0$ where y is maximum.

(ii) The ordinate for every point between R and T of the curve decreases though its x-coordinate increases. Thus, for any point $x_2 > x_1$ yields $y_2 < y_1$ or $f(x_2) < f(x_1)$.

Also the tangent at every point succeeding R along the curve makes obtuse angle with positive direction of x-axis. Consequently, the slope of the tangent is negative for all such points whose ordinate is decreasing. At T the ordinate attains minimum value and the tangent is parallel to x-axis and as a result the slope of the tangent at T is zero.

We now conclude :

(a) The function is monotonically decreasing from R to T.

(b) The tangent at every point, except at T, makes obtuse angle with positive direction of x-axis.

(c) The slope of the tangent is negative i.e., $\frac{dy}{dx} < 0$ for all points of the curve for which y is decreasing.

(d) The slope of the tangent at T is zero i.e., $\frac{dy}{dx} = 0$ where the ordinate is minimum.

(iii) Again, for every point from T to V

The ordinate is constantly increasing, the tangent at every point of the curve between T and V makes acute angle with positive direction of x-axis. As a result of which the slope of the tangent at each of such points of the curve is positive.

Conclusively,

$$\frac{dy}{dx} > 0$$

at all such points of the curve except at Tand V, where $\frac{dy}{dx} = 0$. The derivative $\frac{dy}{dx} < 0$ on onside, $\frac{dy}{dx} > 0$ on the other side of points R, T and V of the curve where $\frac{dy}{dx} = 0$.

Example 25.3 Find for what values of 'x', the function

$$f(x) = x^2 - 6x + 8$$

is increasing and for what values of x it is decreasing.

Solution : $f(x) = x^2 - 6x + 8$

$f'(x) = 2x - 6$

For $f(x)$ to be increasing, $f'(x) > 0$

i.e., $2x - 6 > 0$ or $2(x - 3) > 0$

or, $x - 3 > 0$ or $x > 3$

The function increases for $x > 3$.

For $f(x)$ to be decreasing,

$$f'(x) < 0$$

or, $2x - 6 < 0$ or $x - 3 < 0$

or, $x < 3$

Thus, the function decreases for $x < 3$

Example 25.4 Find the interval in which $f(x) = 2x^3 - 3x2 - 12x + 6$ is increasing or decreasing.

Solution : $f(x) = 2x^3 - 3x^2 - 12x + 6$

$$f'(x) = 6x^2 - 6x - 12$$

$$= 6(x^2 - x - 2)$$

$$= 6(x - 2)(x + 1)$$

For $f(x)$ to be increasing function of x,

$$f'(x) > 0$$

i.e., $6(x - 2)(x + 1) > 0$ or $(x - 2)(x + 1) > 0$

Since the product of two factors is positive, this implies either both are positive or both are negative.

Either $(x - 2) > 0$ and $(x + 1) > 0$ or $(x - 2) < 0$ $(x + 1) < 0$

i.e., $x > 2$, $x > -1$ i.e., $x < 2$ and $x < -1$

$x > 2$ implies $x > -1$ $x < -1$ implies $x < 2$.

$\therefore x > 2$ $\therefore x < -1$

Hence $f(x)$ is increasing for $x > 2$, $x < -1$

Now, for $f(x)$ to be decreasing,

$$f'(x) < 0$$

or, $6(x - 2)(x + 1) < 0$ or, $(x - 2)(x + 1) < 0$

Since the product of two factors is negative, only one of them can be negative, the other positive.

Therefore,

Either	or
$(x-2) > 0$ and $(x+1) < 0$	$(x-2) < 0$ and $(x+1) > 0$
i.e., $x > 2$, $x < -1$	i.e., $x < 2$, $x > -1$
There is no such possibility that $x > 2$ and at the same time	This can be put in this form
$x < -1$	$-1 > x < 2$

$\therefore$ The function is decreasing in $-1 < x < 2$

Example 25.5 Determine the intervals for which the function

$$f(x)=\frac{x}{x^2+1} \text{ is increasing or decreasing.}$$

Solution :

$$f'(x)=\frac{(x^2+1)\frac{dx}{dx}-x.\frac{d}{dx}(x^2+1)}{(x^2+1)}$$

$$=\frac{(x^2+1)-x.(2x)}{(x^2+1)^2}$$

$$=\frac{1-x^2}{(x^2+1)^2}$$

$$f'(x)=\frac{(1-x)(1+x)}{(x^2+1)^2}$$

As $(x^2+1)^2$ is positive for all real x.

Therefore, if $-1 < x < 0$, $(1-x)$ is positive and $(1+x)$ is positive, so $f'(x) > 0$

$\therefore$ If $0 < x < 1$, $(1-x)$ is positive and $(1+x)$ is positive $f'(x) > 0$

If $x < -1$, $(1-x)$ is positive and $(1+x)$ is negative, so $f'(x) < 0$

$x > 1$, $(1-x)$ is negative and $(1+x)$ is positive, so $f'(x) < 0$;

Thus we conclude that

the function is increasing for $-1 < x < 0$ and $0 < x < 1$

or, for $-1 < x < 1$

and the function is decreasing for $x < -1$ or $x > 1$.

Note : Points where $f'(x) = 0$ are critical points. Here, critical points are $x = -1,\ x = 1$

Example 25.6 Show that

(a) $f(x) = \cos x$ is decreasing in the interval $0 \le x \le \pi$

(b) $f(x) = x - \cos x$ is increasing for all x.

Solution : (a) $f(x) = \cos x$

$f'(x) = -\sin x$

$f(x)$ is decreasing

If $f'(x) < 0$

i.e., $-\sin x < 0$

i.e., $\sin x > 0$

$\sin x$ is positive in the first quadrant and in the second quadrant, therefore, $\sin x$ is positive in $0 \le x \le \pi$

$\therefore$ $f(x)$ is decreasing in $0 \le x \le \pi$

(b) $f(x) = x - \cos x$

$f'(x) = 1 - (-\sin x)$

$= 1 + \sin x$

Now, we know that the minimum value of $\sin x$ is -1 and its maximum; value is 1 i.e., $\sin x$ lies between -1 and 1 for all x,

i.e., $-1 \le \sin x \le 1$ or $1 - 1 \le 1 + \sin x \le 1 + 1$

or, $0 \le 1 + \sin x \le 2$

or, $0 \le f'(x) \le 2$

or, $\quad 0 \le f'(x)$

or $\quad f'(x) \ge 0$

$\therefore \quad f(x) = x - \cos x$ is increasing for all x.

EXERCISE 25.2

Find the intervals for which the followiong functions are increasing or decreasing.

1. (a) $f(x) = x^2 - 7x + 10$ (b) $f(x) = 3x^2 - 15x + 10$
2. (a) $f(x) = x^3 - 6x^2 - 36x + 7$ (b) $f(x) = x^3 - 9x^2 + 24x + 12$
3. (a) $y = -3x^2 - 12x + 8$ (b) $f(x) = 1 - 12x - 9x^2 - 2x^3$
4. (a) $y = \dfrac{x-2}{x+1}, x \ne -1$ (b) $y = \dfrac{x^2}{x-1}, x \ne -1$ (c) $y = \dfrac{x}{2} + \dfrac{2}{x}, x \ne 0$
5. (a) Prove that the function log sin x is decreasing in $\left[\dfrac{\pi}{2}, \pi\right]$.

   (b) Prove that the function cos x is increasing in the interval $[\pi, 2\pi]$

   (c) Find the intervals in which the function $\cos\left(2x + \dfrac{\pi}{4}\right), 0 \le x \le \pi$ is decreasing or increasing.

   Find also the points on the graph of the function at which the tangents are parallel to x-axis.

25.4 MAXIMUM AND MINIMUM VALUES OF A FUNCTION

We have seen the graph of a continuous function. It increases and decreases alternatively. If the value of a continious function increases upto a certain point then begins to decrease, then this point is called point of maximum and corresponding value at that point is called maximum value of the function. A stage comes when it again changes from decreasing to increasing . If the value of a continuous function decreases to a certain point and then begins to increase, then value at that point is called minimum value of the function and the point is called point of minimum.

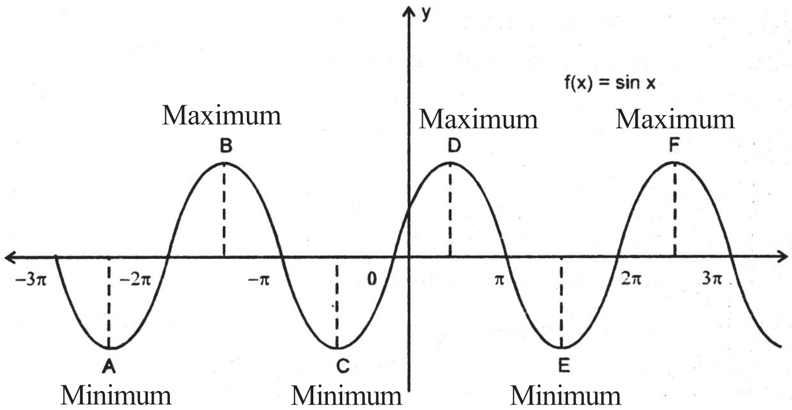


Fig. 25.4

Fig. 25.4 shows that a function may have more than one maximum or minimum values. So, for continuous function we have maximum (minimum) value in an interval and these values are not absolute maximum (minimum) of the function. For this reason, we sometimes call them as local maxima or local minima.

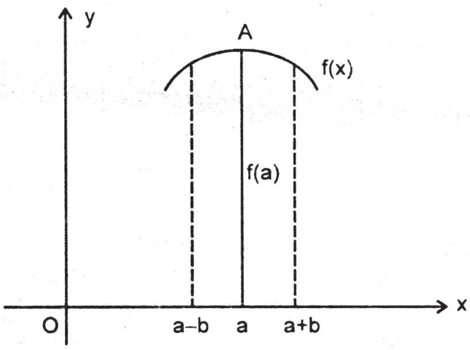


Fig. 25.5

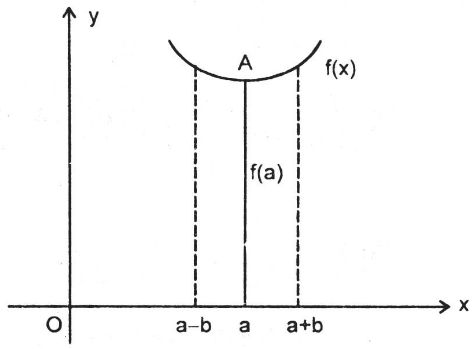


Fig. 25.6

A maximum (or local maximum) value of a function is the one which is greater than all other values on either side of the point in the immediate neighbourhood of the point.

A function $f(x)$ is said to have a minimum (or local minimum) at the point $x = a$ if $f(a) \geq f(a \pm b)$ where $a - b < a + b$.

for all sufficiently small positive b.

In Fig. 25.6, the function $f(x)$ has local minimum at the point $x = a$.

A minimum (or local miunimum) value of a function is the one which is less than all other values, on either side of the point in the immediate neighbourhood of the point.

Note : A neighbourhood of a point $x \in R$ is defined by open internal $]x - \in[$, when $\in > 0$.

25.5 CONDITIONS FOR MAXIMUM OR MINIMUM

We know that derivative of a function is positive when the function is increasing and the derivative is negative when the function is decreasing. We shall apply this result to find the condition for maximum or a function to have a minimum. Refer to Fig. 25.4, points B,D, F are points of maxima and points A,C,E are points of minima.

Now, on the left of B, the function is increasing and so $f'(x) > 0$, but on the right of B, the function is decreasing and, therefore, $f'(x) < 0$. This can be achieved only when $f'(x)$ becomes zero somewhere in betwen. We can rewrite this as follows :

A function $f(x)$ has a maximum value at a point if (i) $f'(x) = 0$ and (ii) $f'(x)$ changes sign from positive to negative in the neighbourhood of the point at which $f'(x) = 0$ (points taken from left to right).

Now, on the left of C (See Fig. 25.6), function is decreasing and $f'(x)$ therefore, is negative and on the right of C, $f(x)$ is increasing and so $f'(x)$ is positive. Once again $f'(x)$ will be zero before having positive values. We rewrite this as follows :

A function $f(x)$ has a minimum value at a point if (i) $f'(x) = 0$, and (ii) $f'(x)$ changes sign from negative to positive in the neighbourhood of the point at which $f'(x) = 0$.

We should note here that $f'(x) = 0$ is necessary condition and is not a sufficient condition for maxima or minima to exist. We can have a function which is increasing, then constant and then again increasing function. In this case, $f'(x)$ does not change sign. The value for which $f'(x) = 0$ is not a point of maxima or minima. Such point is called point of inflexion.

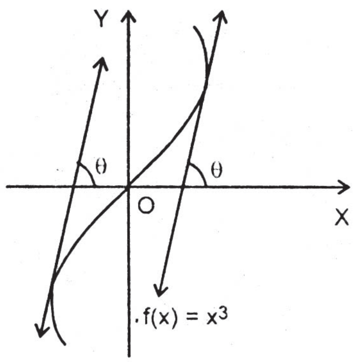


Fig. 25.7

For example, for the function $f'(x) = x^3$, $x = 0$ is the point of inflexion as $f'(x) = 3x^2$ does not change sign as x passes through 0. $f'(x)$ is positive on both sides of the value '0' (tangents make acute angles with x-axis) (See Fig. 25.7).

Hence $f'(x) = x^3$ has a point of inflexion at $x = 0$.

The points where $f'(x) = 0$ are called stationary points as the rate ofchange of the function is zero there. Thus points of maxima and minima are stationary points.

Remarks

The stationary points at which the function attains either local maximum or local minimum values are also called extreme points and both local maximum and local minimum values are called extreme values of $f(x)$. Thus a function attains an extreme value at $x = a$ if $f(a)$ is either a local maximum or a local minimum.

25.6 METHOD OF FINDING MAXIMA OR MINIMA

We have arrived at the method of finding the maxima or minima of a function. It is as follows :

(i) Find $f'(x)$

(ii) Put $f'(x) = 0$ and find stationary points

(iii) Consider the sign of $f'(x)$ in the neighbourhood of stationary points. If it changes sign from +ve to –ve, then $f(x)$ has maximum value at that point and if $f'(x)$ changes sign from –ve to +ve, then $f(x)$ has minimum value at that point.

(iv) If $f'(x)$ does not change sign in the neighbourhood of a point then it is a point of inflexion.

Example 25.7 Find the maximum (local maximum) and minimum (local minimum) points of the function $f(x) = x^3 - 3x^2 - 9x$

Solution : Here $f(x) = x^3 - 3x^2 - 9x$

$$f'(x) = 3x^2 - 6x - 9$$

Step I. Now $f'(x) = 0 \Rightarrow 3x^2 - 6x - 9 = 0$

or, $x^2 - 2x - 3 = 0$

or, $(x - 3)(x + 1) = 0$

or, $x = 3, -1.$

$\therefore$ Stationary points are $x = 3, x = -1.$

Step II. At $x = 3$

For $x < 3$ $f'(x) < 0$

and for $x > 3$ $f'(x) > 0.$

$\therefore$ $f'(x)$ changes sign from –ve to +ve in the neighbourhood of 3.

$\therefore$ $f(x)$ has minimum value at $x = 3.$

Step III : At $x = -1,$

For $x < -1,$ $f'(x) > 0$

abd for $x > -1,$ $f'(x) < 0.$

$\therefore$ $f'(x)$ changes sign from +ve to –ve in the neighbourhood of –1.

$\therefore$ has maximum value at $x = -1$

$\therefore$ $x = -1$ and $x = 3$ give us points of maxima and minima respectively. If we want to find maximum value (minimum value), then we have maximum value $= f(-1) = (-1)^3 - 3(-1)^2 - 9(-1)$

$= -1 - 3 + 9 = 5$

and minimum value $= f(3) = 3^3 - 3(3)^2 - 9(3) = -27.$

$\therefore$ $(-1, 5)$ and $(3, -27)$ are points of local maxima and local minima respectively..

Example 25.8 Find the local maximum and the local minimum of the function

$$f(x) = x^2 - 4x$$

Solution : $f(x) = x^2 - 4x$

$f'(x) = 2x - 4 = 2(x - 2)$

Putting $f'(x) = 0$ yields $2x - 4 = 0$ i.e., $x = 2$

We have to examine whether $x = 2$ is the point of local maximum or local minimum or neither maximum nor minimum.

Let us take $x = 1.9$ which is to the left of 2 and $x = 2.1$ which is to the right of 2 and find $f'(x)$ at these points.

$f'(1.9) = 2(1.9 - 2) < 0$

$f'(2.1) = 2(2.1 - 2) > 0$

Since $f'(x) < 0$ as we approach 2 from the left and $f'(x) > 0$ as we approach 2 from the right, therefore, there is a local minimum at $x = 2$.

We can put our findings for sign of derivatives of $f(x)$ in any tabular form including the one given below :

sign of $f'(x)$

point $x = 2$

left of 2	right 2
$f'(x) < 0$	$f'(x) > 0$

Local minimum

Example 25.9 Find all local maxima and local minima of the function

$$f(x) = 2x^3 - 3x^2 - 12x + 8$$

Solution : $\quad f(x) = 2x^3 - 3x^2 - 12x + 8$

$\therefore \quad f'(x) = 6x^2 - 6x - 12$

$\quad\quad = 6(x^2 - x - 2)$

$\therefore \quad f'(x) = 6(x + 1)(x - 2)$

Now solving $f'(x) = 0$ for x, we get

$$6(x + 1)(x - 2) = 0$$

$\Rightarrow \quad x = -1,\ x = 2.$

Thus $\quad x = -1,\ x = 2$ at $f'(x) = 0$

We examine whether these points are points of local maximum or local minimum or neither of them.

Consider the point $x = -1$

Let us take $x = -1.1$ which is to the left of -1 and $x = -0.9$ which is to the right of -1 and find $f'(x)$ at these points.

$f'(-1.1) = 6(-1.1 + 1)(-1.1 - 2)$, which is positive i.e., $f'(x) > 0$

$f'(-0.9) = 6(-0.9 + 1)(-0.9 - 2)$, which is positive i.e., $f'(x) < 0$

Thus, at $x = -1$, there is a local maximum.

Consider the point $x = 2$.

Now, let us take $x = 1.9$ which is to the left of $x = 2$ and $x = 2.1$ which is to the right of $x = 2$ and find $f'(x)$ at these points.

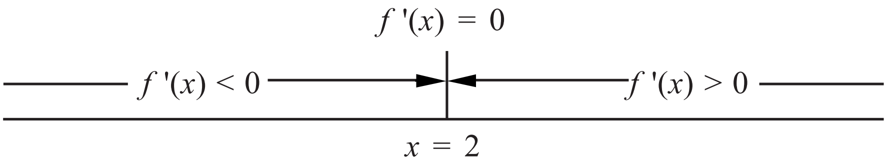
$f'(1.9) = 6(1.9 + 1)(1.9 - 2)$

$\quad = 6 \times$ (Positive number) $\times$ (negative number)

$\quad =$ a negative number

i.e., $\quad f'(1.9) < 0.$

and $\quad f'(2.1) = 6(2.1 + 1)(2.1 - 2)$, which is positive i.e., $f'(2.1) > 0$.



$\because$ $f'(x) < 0$ as we approach 2 from the left

and $f'(x) > 0$ as we approach 2 from the right.

$\therefore$ $x = 2$ is the point of local minimum

Thus $f(x)$ has local maximum at $x = -1$, maximum value of $f(x)$

$$f(-1) = -2 - 3 + 12 + 8 = 15$$

$f(x)$ has local minimum at $x = 2$, minimum value of $f(x)$ =

$$f(2) = 2(8) - 3(4) - 12(2) + 8 = -12.$$

Sign of $f'(x)$

Point $x = -1$			Point $x = 2$	
Left of -1	Right of -1		Left of 2	Right of 2
positive	negative		negative	positive
local maximum			local minimum	

Example 25.10 Find local maximum and local minimum, if any, of the following function

$$f(x) = \frac{x}{1+x^2}$$

Solution : $f(x) = \dfrac{x}{1+x^2}$

$$f'(x) = \frac{(1+x^2)(1) - (2x)(x)}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

For finding points of local maximum or local minimum, equate $f'(x)$ to 0

i.e.,

$\Rightarrow \quad 1 - x^2 = 0$

or $\quad (1 + x)(1 - x) = 0$ or $\quad x = 1, -1$

Consider the value $x = 1$

The sign of $f'(x)$ for values of x slightly less than 1 and slightly greater than 1 changes from positive to negative. Therefore there is a local maximum at $x = 1$, and the local maximum

$$\text{value} = \frac{1}{1+(1)^2} = \frac{1}{1+1} = \frac{1}{2}$$

Now consider $x = 1$

$f'(x)$ changes sign from negative to positive as x passes through –1, therefore, $f(x)$ has a local minimum at $x = -1$.

Thus, the local minimum value $= -\frac{1}{2}$

Example 25.11 Find the local maximum and local minimum, if any, for the function

$$f(x) = \sin x + \cos x, \quad 0 < x < \frac{\pi}{2}$$

Solution : We have $\quad f(x) = \sin x + \cos x$

$$f'(x) = \cos x - \sin x$$

For local maxima/minima, $f'(x) = 0$

$\therefore \quad \cos x - \sin x = 0$

or, $\quad \tan x = 1 \quad$ or, $\quad x = \frac{\pi}{4}$ in $0 < x < \frac{\pi}{2}$

At $\quad x = \frac{\pi}{4}$,

For $x < \frac{\pi}{4}$, $\cos x > \sin x$

$\therefore$ $f'(x) = \cos x - \sin x > 0$

For $x > \frac{\pi}{4}$, $\cos x - \sin x < 0$

$\therefore$ $f'(x) = \cos x - \sin x < 0$

$\therefore$ $f'(x)$ changes sign from positive to negative in the neighbourhood of $\frac{\pi}{4}$

$\therefore$ $x = \frac{\pi}{4}$ is a point of local maxima.

Maximum value $= f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

$\therefore$ Point of local maxima is $\left(\frac{\pi}{4}, \sqrt{2}\right)$.

EXERCISE 25.3

Find all points of local maxima and local minima of the following functions. Also, find the maxima and minima at such points.

1. $x^2 - 8x + 12$
2. $x^3 - 6x^2 + 9x + 15$
3. $2x^3 - 21x^2 + 36x - 20$
4. $x^4 - 62x^2 + 120x + 9$
5. $(x - 1)(x - 2)^2$
6. $\frac{x-1}{x^2+x+2}$

25.7 USE OF SECOND DERIVATIVE FOR DETERMINATION OF MAXIMUM AND MINIMUM VALUES OFA FUNCTION

We now give below another method of finding local maximum or minimum of a function whose second derivative exists. Various steps used are :

(i) Let the given function be denoted by $f(x)$.

(ii) Find $f'(x)$ and equate it to zero.

(iii) Solve $f'(x) = 0$ let one of its real roots be $x = a$.

(iv) Find its second derivative, $f''(x)$. For every real value 'a' of x obtained in step (iii), evaluate $f'(a)$. Then if

$f''(a) < 0$ then $x = a$ is a point of local maximum.

$f''(a) > 0$ then $x = a$ is a point of local minimum.

$f''(a) = 0$ then we use the sign of $f'(x)$ on the left of 'a' and on the right of 'a' to arrive at the result.

Example 25.12 Find the local minimum of the following function :

$$2x^3 - 21x^2 + 36x - 20$$

Solution : Let $f(x) = 2x^3 - 21x^2 + 36x - 20$

$$f'(x) = 6x^2 - 42x + 36$$

$$= 6(x^2 - 7x + 6)$$

$$= 6(x - 1)(x - 6)$$

For local maximum or min imum

$$f'(x) = 0$$

or $6(x - 1)(x - 6) = 0 \Rightarrow x = 1, 6$

$$f''(x) = \frac{d}{dx}\left[f'(x)\right]$$

$$= \frac{d}{dx}\left[6(x^2 - 7x + 6)\right]$$

$$= 12x - 42$$

$$= 6(2x - 7)$$

For $x = 1 \Rightarrow f''(1) = 6(2.1 - 7) = -30 < 0$

$x = 1$ is a point of local maximum.

and $f(1) = 2(1)^3 - 21(1)^2 + 36(1) - 20 = -3$ is a local maximum.

For $x = 6$

$f''(6) = 6(2.6 - 7) = 30 > 0$

$x = 6$ is a point of local minimum

and $f(6) = 2(6)^3 - 21(6)^2 + 36(6) - 20 = -128$ is a local minimum.

Example 25.13 Find local maxima and minima (if any) for the function

$$f(x) = \cos 4x; \ 0 < x < \frac{\pi}{2}$$

Solution : $f(x) = \cos 4x$

$\therefore \quad f'(x) = -4 \sin 4x$

Now $\quad f'(x) = 0 \Rightarrow -4\sin 4x = 0$

or, $\quad \sin 4x = 0 \quad$ or $\quad 4x = 0, \pi, 2\pi$

or, $\quad x = 0, \frac{\pi}{4}, \frac{\pi}{2}$

or, $\quad x = \frac{\pi}{4} \qquad \left[\because \ 0 < x < \frac{\pi}{2}\right]$

Now, $\quad f''(x) = -16\cos 4x$

at $\quad x = \frac{\pi}{4}, f''(x) = -16\cos\pi = -16(-1) = 16 > 0.$

$\therefore f(x)$ is minimum at $x = \frac{\pi}{4}$

Minimum value $f\left(\frac{\pi}{4}\right) = \cos\pi = -1$.

Example 25.14 : (a) Find the maximum value of $2x^3 - 24x + 107$ in the interval $[-3, -1]$

(b) Find the minimum value of the above function in the interval [1, 3]

Solution : Let $\qquad f(x) = 2x^3 - 24x + 107$

$$f'(x) = 6x^2 - 24.$$

For local maximum or minimum,

$$f'(x) = 0$$

i.e., $\qquad 6x^2 - 24 = 0 \qquad \Rightarrow \quad x = -2, 2.$

Out of two points obtained on solving $f'(x) = 0$, only -2 belong to the interval $[-3, -1]$. We shall, therefore, find maximum if any at $x = -2$ only.

Now, $\qquad f''(x) = 12x$

$\therefore \qquad f''(-2) = 12(-2) = -24.$

or $\qquad f''(-2) < 0$

which implies the function $f(x)$ has a maximum at $x = -2$

$\therefore$ Required maximum value $= 2(-2)^3 - 24(-2) + 107 = 139.$

Thus the point of maximum belonging to the given interval $[-3, -1]$ is -2 and, the maximum value of the function is 139.

(b) Now $f''(x) = 12x$

$\therefore \qquad f''(2) = 24 > 0, \qquad [\because$ 2 lies in [1, 3]]

which implies, the function $f(x)$ shall have a minimum at $x = 2$.

$\therefore$ Required minimum $= 2(2)^3 - 24(2) + 107 = 75.$

Example 25.15 Find the maximum and minimum value of the function

$$f(x) = \sin x\,(1 + \cos x) \text{ in } (0, \pi)$$

Solution : We have, $f(x) = \sin x \;= (1 + \cos x)$

$$f'(x) = \cos x - (1 + \cos x) + \sin x\,(-\sin x)$$

$$= \cos x + \cos^2 x - \sin^2 x$$

$$= \cos x + \cos^2 x - (1 - \cos^2 x)$$

$$= 2\cos^2 x + \cos x - 1$$

For stationary points, $f'(x) = 0$

$$\Rightarrow \quad 2\cos^2 x + \cos x - 1 = 0$$

$$\Rightarrow \quad \cos x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4} = -1, \frac{1}{2}.$$

$$\therefore \qquad x = \pi, \frac{\pi}{3}$$

Now, $\qquad f(0) = 0$

$$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3}\left(1 + \cos\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$$

and $\qquad f(\pi) = 0$

$\therefore f(x)$ has maximum value $x = \dfrac{\pi}{3}$ at $x = \dfrac{3\sqrt{3}}{4}$.

and minimum value 0 at $x = 0, x = \pi$.

EXERCISE 25.4

Find local maximum and local minimum for each of the following functions using second order derivatives.

1. $2x^3 + 3x^2 - 36x + 10$.
2. $-x^3 + 12x^2 - 5$
3. $(x - 1)\,(x + 2)^2$
4. $x^5 - 5x^4 + 5x^3 - 1$
5. $\sin x\,.\,(1 + \cos x)\,.\; 0 < x < \dfrac{\pi}{2}$
6. $\sin x + \cos x,\; 0 < x < \dfrac{\pi}{2}$
7. $\sin 2x \;- x,\; -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$.

MODULE - V
Calculus

Notes

25.8 APPLICATIONS OF MAXIMA AND MINIMA TO PRACTICAL PROBLEMS

The application of derivative is a powerful tool for solving problems that call for minimising or maximising a function. In order to solve such problems, we follow the steps in the following order :

(i) Frame the function in terms of variables discussed in the data.

(ii) With the help of the given conditions, express the function in terms of a single variable.

(iii) Lastly, apply conditions of maxima or minima as discussed earlier.

Example 25.16 Find two positive real numbers whose sum is 70 and their product is maximum.

Solution : Let one number be x. As their sum is 70, the other number is $70 - x$. As the two numbers are positive, we have, $x > 0$, $(70 - x) > 0$.

$$70 - x > 0 \Rightarrow x < 70$$

$$0 < x < 70$$

Let their product be $f(x)$

Then $f(x) = x(70 - x) = 70x - x^2$

We have to maximize the prouct $f(x)$.

We, therefore, find $f'(x)$ and put that equal to zero.

$$f'(x) = 70 - 2x$$

For maximum product, $f'(x) = 0$

or, $70 - 2x = 0$

or, $x = 35$.

Now, $f''(x) = -2x$ which is negative. Hence $f(x)$ is maximum at $x = 35$

The other number is $70 - x = 35$

$\therefore$ Hence the required numbers are 35, 35.

Example 25.17 Show that among rectangles of given area, the square has the least perimeter.

Solution : Let x, y be the length and breadth of the rectangle respectively.

Its area = xy

Since its area is given, represent it by A, so that we have

$A = xy.$

or, $y = \frac{A}{x}$ ...(1)

Now, perimeter say P of the rectangle $= 2(x + y)$

or $P = 2\left(x + \frac{A}{x}\right)$

$\therefore \quad \frac{dp}{dx} = 2\left(1 - \frac{A}{x^2}\right)$

For a minimum P, $\frac{dp}{dx} = 0$

i.e., $2\left(1 - \frac{A}{x^2}\right) = 0$

or $A = x^2$ or $\sqrt{A} = x$

Now, $\frac{d^2p}{dx^2} = \frac{4A}{x^3}$, which is positive.

Hence perimeter is minimum when $x = \sqrt{A}$

$\therefore \quad y = \frac{A}{x}$

$= \frac{x^2}{x} = x \qquad (\because A = x^2)$

Thus, the perimeter is minimum when rectangle is a square.

Example 25.18 An open box with a square base is to be made out of a given quantity of sheet of area a^2. Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.

Solution : Let x be the side of the square base of the box and y its height.

Total surface area of othe box = $x^2 + 4xy$.

$$x^2 + 4xy = a^2 \text{ or } y = \frac{a^2 - x^2}{4x}.$$

Volume of the box, V = base area $\times$ height

$$= x^2 y = x^2\left(\frac{a^2 - x^2}{4x}\right)$$

or, $$V = \frac{1}{4}(ax^2 - x^3) \qquad \text{...(i)}$$

$$\frac{dV}{dx} = \frac{1}{4}(a^2 - 3x^2)$$

For maxima/minima $\frac{dV}{dx} = 0$

$$\Rightarrow \frac{1}{4}(a^2 - 3x^2) = 0$$

$$\Rightarrow x^2 = \frac{a^2}{3}$$

$$\Rightarrow x = \frac{a}{\sqrt{3}} \qquad \text{...(ii)}$$

From (1) and (2), we get $V = \frac{1}{4}\left(\frac{a^3}{\sqrt{3}} - \frac{a^3}{3\sqrt{3}}\right) = \frac{a^3}{6\sqrt{3}}$ ...(iii)

Again $$\frac{d^2V}{dx^2} = \frac{d}{dx}\left[\frac{1}{4}\left(a^2 - 3x^2\right)\right] = -\frac{3}{2}x$$

x being the length of the side, is positive.

$\therefore \qquad \frac{d^2V}{dx^2} < 0.$

$\therefore$ The volume is maximum.

Hence maximum volume of the box $= \frac{a^3}{6\sqrt{3}}$.

Example 25.19 Show that of all rectangles inscribed in a given circle, the square has the maximum area.

Solution : Let ABCD be a rectangle inscribed in a circle of radius r. Then diameter AC= $2r$.

Let $\quad AB = x$ and $BC = y$

Then $\quad AB^2 + BC^2 = AC^2$ or $x^2 + y^2 = (2r)^2 = 4r^2$

Now area A of the rectangle = xy

$$A = x\sqrt{4r^2 - x^2}$$

$$\frac{dA}{dx} = \frac{x(-2x)}{2\sqrt{4r^2 - x^2}} + \sqrt{4r^2 - x^2}.1$$

$$= \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

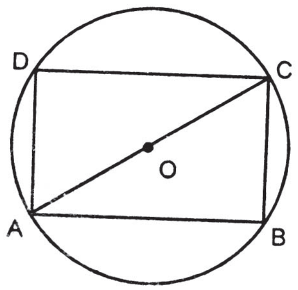


Fig. 25.8

For maxima/minima, $\frac{dA}{dx} = 0$

$$\frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0 \Rightarrow x = \sqrt{2}r$$

Now $$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}(-4x) - (4r^2 - 2x^2)\frac{(-2x)}{2\sqrt{4r^2 - x^2}}}{(4r^2 - x^2)}$$

$$= \frac{-4x(4r^2 - x^2) + x(4r^2 - 2x^2)}{\left(4r^2 - x^2\right)^{\frac{3}{2}}}$$

$$= \frac{-4\sqrt{2}\left(2r^2\right)+0}{\left(2r^2\right)^{\frac{3}{2}}} \qquad \text{... (Putting } x=\sqrt{2}r\text{)}$$

$$= \frac{-8\sqrt{2}r^3}{2\sqrt{2}r^3} = -4 < 0$$

Thus, A is maximum when $x=\sqrt{2}r$

Now, from (i), $y=\sqrt{4r^2-2r^2} = \sqrt{2}r$

$x = y$. Hence, rectangle ABCD is a square.

Example 25.20 Show that the height of a closed right circular cylinder of a given volume and least surface is equal to its diameter.

Solution : Let V be the volume, r the radius and h the height of the cylinder.

Then, $\quad V = \pi r^2 h$

or $\quad h = \dfrac{V}{\pi r^2}$

Now surface area $\quad S = 2\pi rh + 2\pi r^2$

$$= 2\pi r.\frac{V}{\pi r^2} + 2\pi r^2 = \frac{2V}{r} + 2\pi r^2$$

$$\frac{dS}{dr} = \frac{-2V}{r^2} + 4\pi r$$

For minimum surface area, $\dfrac{dS}{dr} = 0$

$\therefore \quad \dfrac{-2V}{r^2} + 4\pi r = 0$

or $\quad V = 2\pi r^3$

From (i) and (ii), we get $h = \dfrac{2\pi r^3}{\pi r^2} = 2r$ ... (ii)

Again, $\quad \dfrac{d^2S}{dr^2} = \dfrac{4V}{r^3} + 4\pi = 8\pi + 4\pi$ ... [Using (ii)]

$\therefore$ S is least when $h = 2r$

Thus, height of the cylidner = diameter of the cylinder.

Example 25.21 Show that a closed right circular cylinder of given surface has maximum volume if its height equals the diameter of its base.

Solution : Let S and V denote the surface area and the volume of the closed right circular cylinder of height h and base radius r.

Then $\quad S = 2\pi rh + 2\pi r^2 \quad$ ...(i)

(Here surface is a constant quantity, being given)

$$V = \pi r^2 h$$

$$V = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right)$$

$$= \frac{r}{2}(S - 2\pi r^2)$$

$$V = \frac{Sr}{2} - \pi r^3$$

$$\frac{dV}{dr} = \frac{S}{2} - \pi(3r^2)$$

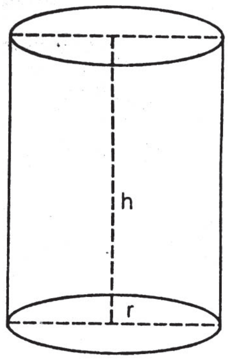


Fig. 25.9

For maximum or minimum, $\quad \frac{dV}{dr} = 0$

i.e., $\quad \frac{S}{2} - \pi(3r^2) = 0$

$$S = 6\pi r^2$$

From (i), we have $\quad 6\pi r^2 = 2\pi rh + 2\pi r^2.$

$\Rightarrow \quad 4\pi r^2 = 2\pi rh$

$\Rightarrow \quad 2r = h \quad$ ...(ii)

Also, $\quad \frac{d^2V}{dr^2} = \frac{d}{dr}\left[\frac{S}{2} - 3\pi r^2\right]$

$$= -6\pi r \qquad \left[\because \frac{d}{dr}\left(\frac{S}{2}\right) = 0\right]$$

= a negative quantity

Hence the volume of the right circular cylinder is maximum when its height is equal to twice its radius i.e. when $h = 2r$.

Example 25.22 A square metal sheet of side 48 cm. has four equal squares removed from the corners and the sides are then turned up so as to form an open box. Determine the size of the square cut so that volume of the box is maximum.

Solution : Let the side of each of the small squares cut be x cm, so that each side of the box to be made is $(48 - 2x)$ cm. and height x cm.

Now $x > 0$, $48 - 2x > 0$, $\quad x < 24$

$\therefore$ x lies between 0 and 24 or $\quad 0 < x < 24$

Now, Volume V of the box

$V = (48 - 2x)(48 - 2x)(x)$

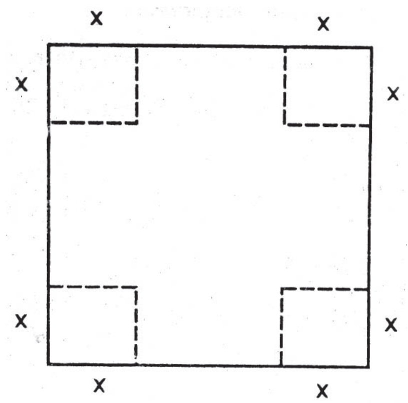


Fig. 25.10

i.e., $$V = (48 - 2x)^2 (x)$$

$\therefore$ $$\frac{dV}{dx} = (48-2x)^2 + 2(48-2x)(-2)(x)$$

$$= (48 - 2x)(48 - 6x)$$

Condition for maximum or minimum is $\frac{dV}{dx} = 0$

i.e., $$(48-2x)(48-6x) = 0$$

We have either $x = 24$ or $x = 8$

$\because$ $0 < x < 24$

$\therefore$ Rejecting $x = 24$, we have, $x = 8$ cm.

Now, $$\frac{d^2V}{dx^2} = 24x - 384$$

$$\left(\frac{d^2V}{dx^2}\right)_{x=8} = 192 - 384 = -192 < 0.$$

Hence for $x = 8$, the volume is maximum.

Hence the square of side 8 cm. should be cut from each corner.

Example 25.23 The profit function P(x) of a firm, selling x items per day is given by

$$p(x) = (150 - x)x - 1625$$

Find the number of items the firm should manufacture to get maximum profit. Find the maximum profit.

Solution : It is given that 'x' is the number of items produced and sold out by the firm every day.

In order to maximize profit,

$$p'(x) = 0 \text{ i.e., } \frac{dP}{dx} = 0$$

or $$\frac{d}{dx}\left[(150 - x)x - 1625\right] = 0$$

or $$150 - 2x = 0$$

or $$x = 75$$

Now, $$\frac{d}{dx}\left[p'(x)\right] = p''(x) = -2 \text{ = a negative quantity.}$$

Hence P (x) is maximum for $x = 75$.

Thus, the firm should manufacture only 75 items a day to make maximum profit.

Now, Maximum Profit $= p(75) = (150 - 75)\ 75 - 1625$

$= \text{Rs. } [(75)\ (75) - 1625]$

$= \text{Rs. } [5625 - 1625]$

$= \text{Rs. } 4000.$

MODULE - V
Calculus

Notes

Example 25.24 Find the volume of the largest cylinder that can be inscribed in a sphere of radius '*r*' cm.

Solution : Let h be the height and R the radius of the base of the inscribed cylinder. Let V be the volume of the cylinder.

Then, $$V = \pi R^2 h \qquad ...(i)$$

From ΔOCB, we have $$r^2 = \left(\frac{h}{2}\right)^2 + R^2 \qquad ... (\because OB^2 = OC^2 + BC^2)$$

$$R^2 = r^2 - \frac{h^2}{4}$$

Now, $$V = \pi\left(r^2 - \frac{h^2}{4}\right)h =$$

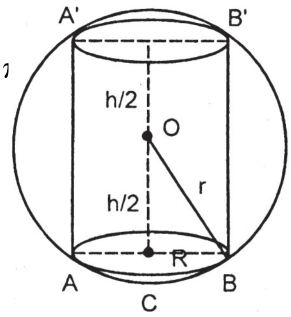


Fig. 25.11

$$\therefore \quad \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4}$$

For maxima/minima $$\frac{dV}{dh} = 0$$

$$\therefore \quad \pi r^2 - \frac{3\pi h^2}{4} = 0$$

$$\Rightarrow \quad h^2 = \frac{4r^2}{3} \quad \Rightarrow h = \frac{2r}{\sqrt{3}}$$

$$\frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$$

$$\therefore \quad \frac{d^2V}{dh^2}\left(\text{at } h = \frac{2r}{\sqrt{3}}\right) = -\frac{3\pi \times 2r}{2 \times \sqrt{3}}$$

$$= -\sqrt{3}\pi r < 0$$

$\therefore$ V is maximum at $h = \frac{2r}{\sqrt{3}}$

$\therefore$ Putting $h = \frac{2r}{\sqrt{3}}$ in (ii) we get

$$R^2 = r^2 - \frac{4r^2}{4\times 3} = \frac{2r^2}{3}$$

$$\therefore \quad R = \sqrt{\frac{2}{3}}\, r$$

Maximum volume of the cylinder = $\pi R^2 h$

$$= \pi.\left(\frac{2}{3}r^2\right)\frac{2r}{\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}\text{cm}^3.$$

EXERCISE 25.5

1. Find two numbers whose sum is 15 and the square of one multiplied by the cube of the other is maximum.
2. Divide 15 into two parts such that the sum of their squares is minimum.
3. Show that among the rectangles of given perimeter, the square has the greatest area.
4. Prove that the perimeter of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles.
5. A window is in the form of a rectangle surmounted by a semi-circle. If the perimeter be 30 m, find the dimensions so that the greatest possible amount of light may be admitted.
6. Find the radius of a closed right circular cylinder of volume 100 c.c. which has the minimum total surface area.
7. A right circular cylinder is to be made so that the sum of its radius and its height is 6 m. Find the maximum volume of the cylinder.
8. Show that the height of a right circular cylinder of greatest volume that can be inscribed in a right circular cone is one-third that of the cone.
9. A conical tent of the given capacity (volume) has to be constructed.

Find the ratio of the height to the radius of the base so as to minimise the canvas requried for the tent.

10. A manufacturer needs a container that is right circular cylinder with a volume 16π cubic meters. Determine the dimensions of the container that uses the least amount of surface (sheet) material.

11. A movie theatre's management is considering reducing the price of tickets from Rs.55 in order to get more customers. After checking out various facts they decide that the average number of customers per day 'q' is given by the function where x is the amount of ticket price reduced. Find the ticket price othat result in maximum revenue.

$$q = 500 + 100x$$

where x is the amount of ticket price reduced. Find the ticket price that result is maximum revenue.

KEY WORDS

- **Increasing function :** A function $f(x)$ is said to be increasing in the closed interval $[a, b]$ if $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$.
- **Decreasing function :** A function $f(x)$ is said to be decreasing in the closed interval $[a, b]$ if $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$.
- $f(x)$ is increasing in an open interval $]a, b[$

  if $f'(x) > 0$ for all $x \in [a, b]$
- $f(x)$ is decreasing in an open interval $]a, b[$

  if $f'(x) < 0$ for all $x \in [a, b]$
- **Monotonic function :**

  (i) A function is said to be monotonic (increasing) if it increases in the given interval.

(ii) A function $f(x)$ is said to be monotonic (decreasing) if it decreases in the given interval.

A function $f(x)$ which increases and decreases in a given interval, is not monotonic.

- In an interval around the point $x = a$ of the function

(i) if $f'(x) > 0$ on the left of the point 'a' and $f'(x) < 0$ on the right of the point $x = a$, then $f(x)$ has a local maximum.

(ii) if $f'(x) < 0$ on the left of the point 'a' and $f'(x) > 0$ on the right of the point $x = a$, then $f(x)$ has a local minimum.

- If $f(x)$ has a local maximum or local minimum at $x = a$ and $f(x)$ is derivable at $x = a$, then $f'(x) = 0$

(i) If $f'(x)$ changes sign from positive to negative as x passes through 'a', then $f(x)$ has a local maximum at $x = a$.

(ii) If $f'(x)$ changes sign from negative to positive as x passes othrough 'a', then $f(x)$ has a local minimum at $x = a$.

- **Second order derivative Test :**

(i) If $f'(a) = 0$, and $f'(a) < 0$; then $f(x)$ has a local maximum at $x = a$.

(ii) If $f'(a) = 0$, and $f'(a) > 0$; then $f(x)$ has a local minimum at $x = a$.

(iii) In case $f'(a) = 0$, and $f'(a) = 0$; then to determine maximum or minimum at $x = a$, we use the method of change of sign of $f'(a)$ as x passes through 'a' to.

SUPPORTIVE WEB SITES

http : //www.wikipedia.org

http:// math world . wolfram.com

PRACTICE EXERCISE

1. Show that $f(x) = x^2$ is neither increasing nor decreasing for all $x \in R$.

   Find the intervals for which the following functions are increasing or decreasing :

2. $2x^3 - 3x^2 - 12x + 6$

3. $\frac{x}{4} + \frac{4}{x},\ x \neq 0$

4. $x^4 - 2x^2$

5. $\sin x - \cos x,\quad 0 \leq x \leq 2\pi$

Find the local maxima or minima of the following functions :

6. (a) $x^3 - 6x^2 + 9x + 7$ (b) $2x^3 - 24x + 107$

   (c) $x^3 + 4x^2 - 3x + 2$ (d) $x^4 - 62x^2 + 120x + 9$

7. (a) $\frac{1}{x^2+2}$ (b) $\frac{x}{(x-1)(x-4)},\ 1 < x < 4$ π

   (c) $x\sqrt{1-x},\ x<1$

8. (a) $\sin x + \frac{1}{2}\cos 2x,\ 0 \leq x \leq \frac{\pi}{2}$ (b) $\sin 2x,\quad 0 \leq x \leq 2\pi$

   (c) $-x + 2\sin x,\quad 0 \leq x \leq 2\pi$

9. For what value of x lying in the closed interval [0, 5] the slope of the tangent to

$$x^3 - 6x^2 + 9x + 4$$

   is maximum. Also, find the point.

10. Find the vlaue of the greatest slope of a tangent to

$$-x^3 + 3x^2 + 2x - 27$$

   at a point of othe curve. Find also the point.

11. A container is to be made in the shape of a right circular cylinder with total surface area of 24π sq. m. Determine the dimensions of the container if the volume is to be as large as possible.

12. A hotel complex consisting of 400 two bedroom apartments has 300 of them rented and the rent is Rs. 360 per day. Management's research indicates that if the rent is reduced by x rupees then the number of apartments rented q will be $q = \frac{5}{4}x + 300,\ 0 \le x \le 80$.

    Determine the rent that results in maximum revenue. Also find the maximum revenue.

ANSWERS

EXERCISE 25.2

1. (a) Increasing for $x > \frac{7}{2}$, Decreasing for $x < \frac{7}{2}$

   (b) Increasing for $x > \frac{5}{2}$, Decreasing for $x < \frac{5}{2}$

2. (a) Increasing for $x > 6$ or $x < -2$, Decreasing for $-2 < x < 6$

   (b) Increasing for $x > 4$ or $x < 2$, Decreasing for x in the interval $]2, 4[$

3. (a) Increasing for $x < -2$; Decreasing for $x > -2$

   (b) Increasing in the interval $-1 < x < -2$, Decreasing for $x > -1$ or $x < -2$

4. (a) Increasing always.

   (b) Increasing for $x > 2$, Decreasing in the interval $0 < x < 2$

   (c) Increasing for $x > 2$ or $x < -2$ Decreasing in the interval $-2 < x < 2$

5. (a) Increasing in the interval : $\frac{3\pi}{8} \leq x \leq \frac{7\pi}{8}$

Decreasing in the interval : $0 \leq x \leq \frac{3\pi}{8}$

Points at which the tangents are parallel to x-axis are $x = \frac{3\pi}{8};\ x = \frac{7\pi}{8}$

EXERCISE 25.3

1. Local minimum is –4 at $x = 4$
2. Local minimum is 15 at $x = 3$, Local maximum is 19 at $x = 1$
3. Local minimum is –128 at $x = 6$, Local maximum is –3 at $x = 1$
4. Local minimum is –1647 at $x = -6$,

   Local maximum is –316 at $x = 5$.

   Local maximum is 68 at $x = 1$
5. Local minimum at $x = 0$ is –4, Local maximum at $x = -2$ is 0.
6. Local minimum at $x = -1$, value –1;

   Local maximum at $x = 3$, value $\frac{1}{7}$

EXERCISE 25.4

1. Local minimum is –34 at $x = 2$ Local maximum is 91 at $x = -3$
2. Local minimum is –5 at $x = 0$ Local maximum is 251 at $x = 8$
3. Local minimum –4 at $x = 0$ Local maximum 0 at $x = -2$
4. Local minimum = –28; $x = 3$ Local maximum 0; $x = 1$

   Neither maximum nor minimum at $x = 0$
5. Local maximum $x = \frac{\pi}{3};\ x = \frac{3\sqrt{3}}{4}$.

6. Local maximum $=\sqrt{2}$, $x=\frac{\pi}{4}$

7. Local minimum $= -\frac{\sqrt{3}}{2}+\frac{\pi}{6}$, $x=-\frac{\pi}{6}$

   Local maximum $= \frac{\sqrt{3}}{2}-\frac{\pi}{6}$, $x=\frac{\pi}{6}$

EXERCISE 25.5

1. Numbers are 6, 9. 2. Parts are 7.5, 7.5

5. Dimensions are : $\frac{30}{\pi+4}, \frac{30}{\pi+4}$ meters each.

6. Radus = $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm; height = $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm

7. Maximum Volume 32π cubic meters 9. $h=\sqrt{2}r$

10. $r = 2$ meters, $h = 4$ meters

11. Rs. 30.00

PRACTICE EXERCISE

2. Increasing for $x > 2$ or $x < -1$

   Decreasing in the interval $-1 < x < 2$

3. Increasing for $x > 4$ or $x < -4$

   Decreasing in the interval $]-4, 4[$

4. Increasing for $x > 1$ or $-1 < x < 0$

   Decreasing for $x > -1$ or $0 < x < 1$

5. Increasing for $0 \le x \le \frac{3\pi}{4}$ or $\frac{7\pi}{4} \le x \le 2\pi$

   Decreasing for $\frac{3\pi}{4} \le x \le \frac{7\pi}{4}$

6. (a) Local maximum is 11 at $x = 1$ Local minimum is 7 at $x = 3$

   (b) Local maximum is 139 at $x = -2$ Local minimum is 75 at $x = 2$

   (c) Local maximum is 20 at $x = -3$ Local minimum is $\frac{40}{27}$ at $x = \frac{1}{3}$

   (d) Local maximum is 68 at $x = 1$

   Local minimum is -316 at $x = 5$ and -1647 at $x = -6$

7. (a) Local minimum is $\frac{1}{2}$ at $x = 0$

   (b) Local maximum is -1 at $x = 2$

   (c) Local maximum is $\frac{2}{3\sqrt{3}}$ at $x = \frac{2}{3}$

8. (a) Local maximum is $\frac{3}{4}$ at $x = \frac{\pi}{6}$

   Local minimum is $\frac{1}{2}$ at $x = \frac{\pi}{2}$

   (b) Local maximum is 1 at $x = \frac{\pi}{4}, \frac{5\pi}{4}$

   Local minimum is -1 at $x = \frac{3\pi}{4}$

   (c) Local maximum is $-\frac{\pi}{4} + \sqrt{3}$ at $x = \frac{\pi}{3}$

   Local maximum is $-\frac{5\pi}{3} - \sqrt{3}$ at $x = \frac{5\pi}{3}$

9. Greatest slope is 24 at $x = 5$

   Coordinates of the point: (5, 24).

10. Greatest slope of a tangent is 5 at $x = 1$, The point is $(1, -23)$.

11. Radius of base = 2 m, Height of cylinder = 4 m.

12. Rent reduced to Rs. 300, The maximum revenue = Rs. 1,12,500

Chapter

26

INTEGRATION

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- explain integration as inverse process (anti-derivative) of differentiation;
- find the integral of simple functions like x^n, $\sin x$, $\cos x$, $\sec^2 x$, $\mathrm{cosec}^2 x$, $\sec x, \tan x, \mathrm{cosec}\, x \cot x, \frac{1}{x}\; e^x$ etc.;
- state the following results :

  (i) $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

  (ii) $\int [\pm k\;\; f(x)]\, dx = \pm k \int f(x)\, dx$
- find the integrals of algebraic, trigonometric, inverse trigonometric and exponential functions;
- find the integrals of functions by substitution method.
- evaluate integrals of the type

  $\int \frac{dx}{x^2 \pm a^2},\; \int \frac{dx}{a^2 - x^2},\; \int \frac{dx}{\sqrt{x^2 \pm a^2}},\; \int \frac{dx}{\sqrt{a^2 - x^2}}$

$$\int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \frac{(px+q)\,dx}{ax^2 + bx + c}$$

$$\int \frac{(px+q)\,dx}{\sqrt{ax^2 + bx + c}}$$

- derive and use the result

$$\int \frac{f'(x)}{f(x)}\, dx = \log\ |f(x)| + c$$

- state and use the method of integration by parts;

  evaluate integrals of the type:

$$\int \sqrt{x^2 \pm a^2}\,dx,\ \int \sqrt{a^2 - x^2}\,dx, \int e^{ax}\ \sin bx\ dx, \int e^{ax} \cos bx\ dx,$$

$$\int (px+q)\ \sqrt{ax^2 + bx + c}\ dx,\ \int \sin^{-1} x\ dx, \int \cos^{-1}\ x\ dx,$$

$$\int \sin^n x \cos^m\ x\ dx,\ \int \frac{dx}{a + b\sin\ x}, \int \frac{dx}{a + b\cos\ x}$$

- derive and use the result

  $\int e^x\ [f(x) + f'(x)]\ dx = e^x . f(x) + c$: =°i†Çò

- integrate rational expressions using partial fractions

PREREQUISITES

- Differentiation of various functions
- Basic knowledge of plane geometry
- Factorization of algebraic expression
- Knowledge of inverse trigonometric functions

INTRODUCTION

We have learnt the concept of derivative of a function in the previous lesson. You have also learnt the application of derivative in various situations.

Concept of Integration as the inverse process of differentiation. In this lesson we discuss standard forms and properties of integrals.

26.1 INTRODUCTION INTEGRATION

Integration literally means summation. Consider, the problem of finding area of region ABCB as shown in Fig. 26.1.

• .

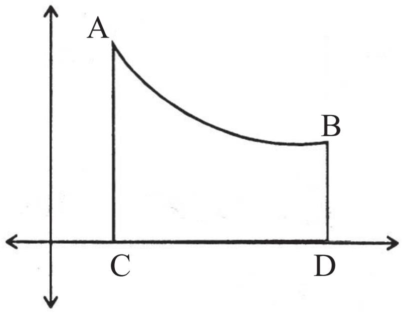


Fig. 26.1

We will try to find this area by some practical method. But that may not help every time. To solve such a problem, we take the help of integration (summation) of area. For that, we divide the figure into small rectangles (See Fig.26.2).

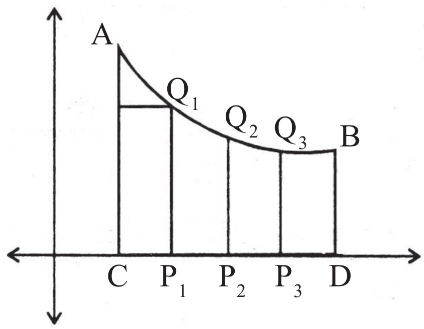


Fig. 26.2

Unless these rectangles are having their width smaller than the smallest possible, we cannot find the area.

In this lesson, we shall learn about methods of integrating polynomial, trigonometric, exponential and logarithmic and rational functions using different techniques of integration.

26.2 INTEGRATION AS INVERSE OF DIFFERENTIATION

Definition

*Let E be a subset of **R** such that E contains a right or a left neighbourhood of each of its points and let $f : E \to \mathbf{R}$ be a function. If* *there is a function F on E such that* $F'(x) = f(x)$ *for all* $x \in E$, *then we call F an* ***antiderivative of f*** *or a* ***primitive of f***.

For example, we know that

$$\frac{d}{dx}(\sin x) = \cos x,\ x \in \mathbf{R}.$$

Hence, if f is the function given by $f(x) = \cos x$, $x \in \mathbf{R}$, then the function F given by $F(x) = \sin x$, $x \in \mathbf{R}$ is an antiderivative or a primitive of f.

If F is an antiderivative of f on E, then for any real number k, we have

$$(F + k)'\ (x) = f(x) \text{ for all } x \in E.$$

Hence $F + k$ is also an antiderivative of f.

Thus, in the above example, if c is any real constant then the function G given by $G(x) = \sin x + c$, $x \in \mathbf{R}$ is also an antiderivative of $\cos x$.

Consider the following examples :

(i) $\frac{d}{dx}(x^2) = 2x$ (ii) $\frac{d}{dx}(\sin x) = \cos x$ (iii) $\frac{d}{dx}(e^x) = e^x$.

Let us consider the above examples in a different perspective

(i) $2x$ is a function obtained by differentiation of x^2.

$\Rightarrow$ x^2 is called the antiderivative of $2x$.

(ii) cos x is a function obtained by differentiation of sin x

$\Rightarrow$ $\sin x$ is called the antiderivative of $\cos x$

(iii) Similarly, e^x is called the antiderivative of e^x

Generally we express the notion of antiderivative in terms of an operation. This operation is called the operation of integration. We write

(1) Integration of $2x$ is x^2 (2) Integration of $\cos x$ is $\sin x$

(3) Integration of e^x is e^x.

The operation of integration is denoted by the symbol $\int$

Thus

(1) $\int 2x.dx = x^2$ (2) $\int \cos x.dx = \sin x$ (3) $\int e^x.dx = e^x$

Remember that dx is symbol which together with symbol $\int$ denotes the operation of integration.

The function to be integrated is enclosed between $\int$ and dx.

Definition of (Indefinite integral)

Let $f: I \to \mathbf{R}$. Suppose that f has an antiderivative F on I. Then we say that f has an integral on I and for any real constant c, we call F + c an indefinite integral of f over I, denote it by $\int f(x)\,dx$ and read it as 'integral f(x) dx'. We also denote $\int f(x)\,dx$ as $\int f$. Thus we have

$$\int f = \int f(x)dx = \mathrm{F}(x) + c.$$

Here c is called a 'constant of integration'.

In the indefinite integral $\int f(x)dx$, f is called the 'integrand' and x is called the 'variable of integration'.

Note: If $\frac{d}{dx}[f(x)] = f'(x)$, then $f(x)$ is said to be an integral of $f'(x)$ and is written as $\int f'(x)\,dx = f(x)$.

The function $f'(x)$ which is integrated is called the integrand.

Constant of integration

If $y = x^2$, then $\frac{dy}{dx} = 2x$

$\therefore \quad \int 2x\,dx = x^2$

Now consider $\frac{d}{dx}(x^2+2)$ or $\frac{d}{dx}(x^2+c)$ where c is any real constant. Thus, we see that integral of $2x$ is not unique. The different values of $\int 2x\,dx$ differ by some constant. Therefore, $\int 2x\,dx = x^2 + c$, where c is called the constant of integration.

Thus $\int e^x\,dx = e^x + c$, $\int \cos x\,dx = \sin x + c$

In general $\int f'(x)\,dx = f(x) + c$ The constant c can take any value.

We observe that the derivative of an integral is equal to the integrand.

Note: $\int f(x)\,dx$, $\int f(y)\,dy$, $\int f(z)\,dz$ but not like $\int f(z)\,dx$

Example 26.1: Find the integral of the following :

(i) x^3 (ii) x^{30} (iii) x^n

Solution:

(i) $\int x^3\,dx = \dfrac{x^4}{4} + C$, since $\dfrac{d}{dx}\left(\dfrac{x^4}{4}\right) = \dfrac{4x^3}{4} = x^3$

(ii) $\int x^{30}\,dx = \dfrac{x^{31}}{31} + C$, since $\dfrac{d}{dx}\left(\dfrac{x^{31}}{31}\right) = \dfrac{31x^{30}}{31} = x^{30}$

(iii) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$,

since $\dfrac{d}{dx}\dfrac{x^{n+1}}{n+1} = \dfrac{1}{n+1}\dfrac{d}{dx}x^{n+1} = \dfrac{1}{n+1}.(n+1)x^n = x^n$

Example 26.2:

(i) If $\dfrac{dy}{dx} = \cos x$ find y. (ii) If $\dfrac{dy}{dx} = \sin x$, find y.

Solution :

(i) $\int \dfrac{dy}{dx}dx = \int \cos x\,dx \Rightarrow y = \sin x + C$

(ii) $\int \dfrac{dy}{dx}dx = \int \sin x\,dx \Rightarrow y = -\cos x + C$

We have already seen that if $f(x)$ is any integral of $f'(x)$, then functions of the form $f(x) + C$ provide integral of $f'(x)$. We repeat that C can take any value including 0 and thus

$$\int f'(x)\,dx = f(x) + C$$

which is an indefinite integral and it becomes a definite integral with a defined value of C.

Example 26.3: Write any 4 different values of $\int 4x^3\,dx$.

Solution: $\int 4x^3\,dx = x^4 + C$, Where C is a constant.

The four different values of $\int 4x^3\,dx$ may be x^4+1, x^4+2, x^4+3 and x^4+4 etc.

26.3 INTEGRATION OF SIMPLE FUNCTIONS

- Integrals of some simple functions given below. The validity of the integrals is checked by showing that the derivative of the integral is equal to the integrand.

Integral	Verification
1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	$\because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = x^n$

where n is a constant and $n \neq -1$.

2. $\int \sin x\,dx = -\cos x + C$	$\because \frac{d}{dx}(-\cos x + C) = \sin x$
3. $\int \cos x\,dx = \sin x + C$	$\because \int \frac{d}{dx}(\sin x + C) = \cos x$
4. $\int \sec^2 x\,dx = \tan x + C$	$\because \frac{d}{dx}(\tan x + C) = \sec^2 x$
5. $\int \text{cosec}^2 x\,dx = -\cot x + C$	$\because \frac{d}{dx}(-\cot x + C) = \text{cosec}^2 x$
6. $\int \sec x \tan x\,dx = \sec x + C$	$\because \frac{d}{dx}(\sec x + C) = \sec x \tan x$

7. $\int \text{cosec}\, x \cot x\,dx = -\text{cosec}\, x + C$

$\because \frac{d}{dx}(-\text{cosec}\, x + C) = \text{cosec}\, x \cot x$

8. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$	$\because \frac{d}{dx}(\sin^{-1} x + C) = \frac{1}{\sqrt{1-x^2}}$
9. $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$	$\because \frac{d}{dx}(\tan^{-1} x + C) = \frac{1}{1-x^2}$

10. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$ $\qquad \because \frac{d}{dx}(\sec^{-1} x + C) = \frac{1}{x\sqrt{x^2-1}}$

11. $\int e^x dx = e^x + C$ $\qquad \because \frac{d}{dx}(e^x + C) = e^x$

12. $\int a^x dx = \frac{a^x}{\log a} + C$ $\qquad \because \frac{d}{dx}\left(\frac{a^x}{\log a} + C\right) = a^x = \frac{1}{x}$ if $x > 0$

13. $\int \frac{1}{x} dx = \log|x| + C$ $\qquad \because \frac{d}{dx}(\log|x| + C)$

EXERCISE 26.1

1. Write indefinite integral of the following.

   a) x^5 b) $\cos x$ c) 0

2. Evaluate

   a) $\int x^3 dx$ b) $\int x^{-7} dx$ c) $\int \sqrt[3]{x^2} dx$

   d) $\int \frac{1}{\sqrt{x}} dx$ e) $\int \sqrt[3]{x^4} dx$ f) $\int \sqrt[9]{x^{-8}} dx$

3. Evaluate

   a) $\int \frac{\cos\theta}{\sin^2\theta} dx$ b) $\int \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} d\theta$

26.4 PROPERTIES OF INTEGRALS

If a function can be expressed as a sum of two or more functions then we can write the integral of such a function as the sum of the integral of the component functions.

e.g. If $f(x) = x^7 + x^3$

$$\int f(x) = \int \left[x^7 + x^3\right] dx$$

$$= \int x^7 dx + \int x^3 dx$$

$$= \frac{x^8}{8} + \frac{x^4}{4} + c$$

So, in general the integral of the sum of two functions is equal to the sum of their integrals.

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

Similarly, If the given function

$$f(x) = x^7 - x^3$$

We can write

$$\int f(x)\, dx = \int [x^7 - x^3] dx$$

$$= \int x^7 dx - \int x^3\, dx$$

$$= \frac{x^8}{8} - \frac{x^4}{4} + C$$

The integral of the difference of two functions is equal to the difference of their integrals

$$\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$$

If we have a function $f(x)$ as a product of a constant k and another function $g(x)$.

$$f(x) = k\ g(x)$$

then we can integrate $f(x)$ as

$$\int f(x)\, dx = \int k\ g(x)\, dx$$

$$= k \int g(x)\, dx$$

Example 26.3: Evaluate $\int \left(e^x - \frac{1}{x} + \frac{2}{\sqrt{x^2 - 1}} \right) dx$

$$= \int e^x dx - \int \frac{1}{x} dx + 2\int \frac{1}{\sqrt{x^2 - 1}}\, dx$$

$$= e^x - \log x + 2\cosh^{-1} x + c$$

Example 26.4: Evaluate $\int \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1+x^2}}\, dx$

$= \sin^{-1}x + \sinh^{-1}x + c$

Example 26.5: Evaluate $\int e^{\log(1+\tan^2 x)}\, dx$

$= \int (1+\tan^2 x)\, dx$

$= \int \sec^2 x\, dx$

$= \tan x + c$

Example 26.6: Evaluate $\int 4^x\, dx = \frac{4^x}{\log 4} + C$

Example 26.7: Evaluate $\int (\sin x + \cos x)\, dx$

$= -\cos x + \sin x + c$

Example 26.8:Evaluate $\int \frac{x^2+1}{x^3} dx$

$= \int \frac{1}{x} + \frac{1}{x^3} dx$

$= \int \frac{1}{x} + \int \frac{1}{x^3} dx$

$= \log x + \frac{x^{-3+1}}{-3+1} + C$

$= \log x - \frac{1}{2x^2} + C$

Example 26.9: Evaluate : $\int \sqrt{1-\sin 2\theta}\, d\theta$

$= \int \sqrt{\cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta}\, d\theta$

$= \int \sqrt{(\cos\theta - \sin\theta)^2}\, d\theta$

$\int \sqrt{1-\sin 2\theta}\, d\theta = \pm \int (\cos\theta - \sin\theta)\, d\theta = \pm (\sin\theta + \cos\theta) + c$

(or)

If $\int \sqrt{1-\sin 2\theta}\; d\theta = \int \cos\theta - \sin\theta d\theta$

$= \sin\theta + \cos\theta + C$

If $\int \sqrt{1-\sin 2\theta}\; d\theta = -\int (\cos\theta - \sin\theta) d\theta$

$= -\int \cos\theta\; d\theta + \int \sin\theta d\theta$

$= -\sin\theta - \cos\theta + C$

Example 26.10: Evaluate $\int \frac{1}{\cosh x + \sinh x}\, dx$

$$= \int \frac{\cosh^2 x - \sinh^2 x}{\cosh x + \sinh x}\, dx$$

$$= \int \frac{(\cosh x - \sinh x)\,\cancel{(\cosh x + \sinh x)}}{\cancel{(\cosh x + \sinh x)}}\, dx$$

$$= \int (\cosh x - \sinh x)\; dx$$

$$= \int \cosh x\, dx - \sinh x\; dx$$

$= \sinh x - \cosh x + C.$

Example 26.11: Evaluate $\int \frac{\sin^2 x}{1+\cos 2x}\, dx$

$$= \int \frac{1-\cos^2 x}{2\cos^2 x}\, dx$$

$$= \frac{1}{2}\left[\int \frac{1}{\cos^2 x}\, dx - \int \frac{\cos^2 x}{\cos^2 x} dx\right]$$

$$= \frac{1}{2}\left[\int \sec^2 x\; dx - \int 1\; dx\right]$$

$$= \frac{1}{2}\left[\tan x - x\right] + C$$

Example 26.12: Evaluate $\int \sec^2 x \operatorname{cosec}^2 x \, dx$

$$= \int \frac{1}{\cos^2 x \sin^2 x} dx$$

$$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x} dx$$

$$= \int \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} dx$$

$$= \int \operatorname{cosec}^2 x \, dx + \int \sec^2 x \, dx$$

$$= -\cot x + \tan x + C$$

Example 26.13: Evaluate $\int (x + \cos x) dx$

$$= \int x \, dx + \int \cos x \, dx$$

$$= \frac{x^2}{2} + \sin x + c$$

Example 26.14: Evaluate $\int \frac{1}{1 + \cos x} dx$

$$= \int \frac{1 - \cos x}{1 - \cos^2 x} dx$$

$$= \int \frac{1 - \cos x}{\sin^2 x} dx$$

$$= \int \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} dx$$

$$= \int \operatorname{cosec}^2 x \, dx - \int \operatorname{cosec} x \cot x \, dx$$

$$= -\cot x + \operatorname{cosec} x + C$$

Example 26.15: Evaluate $\int \left(\sec x \tan x + \frac{3}{x} - 4 \right) dx$

$$= \int \sec x \tan x \, dx + 3\int \frac{1}{x} dx - 4\int 1 \, dx$$

$$= \sec x + 3\log x - 4x + c$$

Example 26.16: Evaluate $\int e^{x+7} dx$

$$\int e^{x+7} dx = \int e^{x+7} . e^7 dx$$

$$= e^7 \int e^x \, dx$$

$$= e^7 \; . \; e^x + C$$

$$= e^{x+7} + C$$

EXERCISE 26.2

1. Evaluate

(i) $\int \left(x + \frac{1}{2} \right) dx$ (ii) $\int \frac{x^2}{1+x^2} dx$

(iii) $\int \left(\sqrt{x} + \frac{2}{\sqrt{x}} \right) dx$ (iv) $\int \left(\frac{3}{\sqrt{x}} - \frac{2}{x} + \frac{1}{3x^2} \right) dx$

2. Evaluate

(i) $\int \frac{1}{1+\cos 2x} dx$ (ii) $\int \tan^2 x \, dx$ (iii) $\int \frac{\sin x}{\cos^2 x} dx$

(iv) $\int \sqrt{1+\cos 2x} \, dx$

26.5 TECHNIQUES OF INTEGRATION

26.5.1 Integration By Substitution

This method consists of expressing $\int f(x) \, dx$ in terms of another variable so that the resultant function can be integrated using one of the standard results discussed in the previous lesson.

First, we will consider the functions of the type $f(ax + b)$, $a \neq 0$ where $f(x)$ is a standard function.

Example 26.17 : Evaluate

(i) $\int \sin(ax + b) \, dx$ (ii) $\int \cos\left(7x + \frac{\pi}{4}\right) dx$

(iii) $\int \sin\left(\frac{\pi}{4} - \frac{x}{2}\right) dx$

Solution: (i) $\int \sin(ax + b)\, dx$

Put $ax + b = t.$

Then $a = \frac{dt}{dx}$ or $dx = \frac{dt}{a}$

$\therefore \int \sin(ax + b)\, dx = \int \sin t \,.\, \frac{dt}{a}$

(Here the integration factor will be replaced by dt.)

$= \frac{1}{a}\int \sin t\, dt$

$= \frac{1}{a}(\cot t) + C$

$= \frac{-\cos(ax + b)}{a} + C$

(ii) $\int \cos\left(7x + \frac{\pi}{4}\right) dx$

Put $7x + \frac{\pi}{4} = t \Rightarrow 7dx = dt$

$\therefore \int \cos\left(7x + \frac{\pi}{4}\right) dx = \int \cos t\, \frac{dt}{7}$

$= \frac{1}{7}\int \cos t\, dt$

$= \frac{1}{7}\sin t + C$

$= \frac{1}{7}\sin\left(7x + \frac{\pi}{4}\right) + C$

(iii) $\int \sin\left(\frac{\pi}{4} - \frac{x}{2}\right) dx$

Put $\frac{\pi}{4} - \frac{x}{2} = t$

Then $-\frac{1}{2} = \frac{dt}{dx}$ or $dx = -\,2dt$

$$\int \sin\left(\frac{\pi}{4} - \frac{\pi}{2}\right) dx = -2\int \sin t\, dt$$

$$= -2(-\cos t) + C$$

$$= 2\cos t + C$$

$$= 2\cos\left(\frac{\pi}{4} - \frac{x}{2}\right) + C$$

Similarly, the integrals of the following functions will be

$$\int \sin 2x\; dx = -\frac{1}{2}\cos 2x + C$$

$$\int \sin\left(3x + \frac{\pi}{3}\right) dx = -\frac{1}{3}\cos\left(3x + \frac{\pi}{3}\right) + C$$

$$\int \sin\left(\frac{\pi}{4} - \frac{x}{4}\right) dx = 4\cos\left(\frac{\pi}{4} - \frac{x}{4}\right) + C$$

$$\int \cos(ax + b)\; dx = \frac{1}{a}\sin(ax + b) + C$$

$$\int \cos 2x\; dx = \frac{1}{2}\sin 2x + C$$

Example 26.18 : Evaluate

(i) $\int (ax + b)^n\, dx$, where $n \neq -1$ (ii) $\int \frac{1}{(ax + b)} dx$

Solution: (i) $\int (ax + b)^n\, dx$, where $n \neq -1$

Put $ax + b = t \Rightarrow a = \frac{dt}{dx}$ or $dx = \frac{dt}{a}$

$$\therefore \int (ax + b)^n\, dx = \frac{1}{a}\int t^n\, dt$$

$$= \frac{1}{a} \cdot \frac{t^{n+1}}{(n+1)} + C$$

$$= \frac{1}{a} \cdot \frac{(ax + b)^{n+1}}{n + 1} + C \qquad \text{where } n \neq -1$$

(ii) $\int \frac{1}{(ax+b)} dx$

Put $ax + b = t \quad \Rightarrow \quad dx = \frac{1}{a} dt$

$$\therefore \quad \int \frac{1}{(ax+b)} dx = \int \frac{1}{a} \cdot \frac{dt}{t}$$

$$= \frac{1}{a} \log |t| + C$$

$$= \frac{1}{a} \log |ax + b| + C$$

Example 26.19: Evaluate

(i) $\int e^{5x+7} dx$ (ii) $\int e^{-3x-3} dx$

Solution: (i) $\int e^{5x+7} dx$

Put $5x + 7 = t \Rightarrow dx = \frac{dt}{5}$

$$\therefore \quad \int e^{5x+7} dx + \frac{1}{5} \int e^{t} dt$$

$$= \frac{1}{5} e^{t} + C$$

$$= \frac{1}{5} e^{5x+7} + C$$

(ii) $\int e^{-3x-3} dx$

Put $-3x - 3 = t \Rightarrow dx = \frac{1}{-3} dt$

$$\therefore \quad \int e^{-3x-3} dx = -\frac{1}{3} \int e^{t} dt$$

$$= -\frac{1}{3} e^{t} + C$$

$$= -\frac{1}{3} e^{-3x-3} + C$$

Likewise $\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + C$

Similarly, using the substitution $ax + b = t$, the integrals of the following functions will be :

$$\int (ax+b)^n\, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{(ax+b)} dx = \frac{1}{a}\log|ax+b| + C$$

$$\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$$

$$\int \operatorname{cosec}^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b) + C$$

$$\int \sec(ax+b)\tan(ax+b)\, dx = \frac{1}{a}\sec(ax+b) + C$$

$$\int \operatorname{cosec}(ax+b)\cot(ax+b)\, dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$$

Example 26.20: Evaluate

(i) $\int \sin^2 x\, dx$ (ii) $\int \sin^3 x\, dx$ (iii) $\int \cos^3 x\, dx$

(iv) $\int \sin 3x \sin 2x\, dx$

Solution: We use trigonometrical identities and express the functions in terms of sines and cosines of multiples of x

(i) $\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx$ $\left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$

$$= \frac{1}{2}\int (1-\cos 2x)\, dx$$

$$= \frac{1}{2}\int 1\, dx - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

(ii) $\int \sin^3 x\, dx = \int \frac{3\sin x - \sin 3x}{4}\, dx$

$\left[\because \quad \sin 3x = 3\sin x - 4\sin^3 x\right]$

$$= \frac{1}{4}\int (3\sin x - \sin 3x)\, dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + C$$

(iii) $\int \cos^3 x\; dx = \int \frac{\cos 3x + 3\cos x}{4} dx \quad \left[\because \quad \cos 3x = 4\cos^3 x - 3\cos x\right]$

$$= \frac{1}{4}\int (\cos 3x + 3\cos x)\, dx$$

$$= \frac{1}{4}\frac{\sin 3x}{3} + \frac{3}{4}\sin x\; dx + C$$

(iv) $\int \sin 3x\; \sin 2x\; dx = \frac{1}{2}\int 2\sin 3x\; \sin 2x\; dx$

$[\because\ 2\sin A\ \sin B = \cos(A - B) - \cos(A - B)]$

$$= \frac{1}{2}\int (\cos x - \cos 5x)\, dx$$

$$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right] + C$$

Example 26.21: Evaluate (i) $\int e^{3-8x} dx$ (ii) $\int (4x-5)^3\, dx$

(i) $\int e^{3-8x} dx$

$$= \frac{e^{3-8x}}{-8} + C$$

(ii) $\int (4x-5)^3 dx$

$$= \frac{(4x-5)^4}{4} + C$$

Example 26.22: Evaluate (i) $\int \cos(x+5)dx$ (ii) $\int \sec(3x+5)\ \tan(3x+5)dx$

(i) $\int \cos(x+5)dx$

$= \sin(x + 5) + C$

(ii) $\int \sec(3x+5)\ \tan(3x+5)dx$

$$= \frac{\sec(3x+5)}{3} + C$$

EXERCISE 26.3

1. Evaluate

(i) $\int \sin(4 - 5x)\, dx$ (ii) $\int \sec^2 (2 + 3x)\, dx$

(iii) $\int \sec\left(x + \frac{\pi}{4}\right) dx$ (iv) $\int \cos\,(4x+5)\, dx$

(v) $\int \text{cosec}\,(2+5x) \cot(2+5x)\; dx$

2. Evaluate

(i) $\int \frac{1}{(3-4x)^4}\, dx$ (ii) $\int (x+1)^4\, dx$

(iii) $\int (4-7x)^{10}\, dx$ (iv) $\int \frac{1}{3x-5}\, dx$

(v) $\int \frac{1}{\sqrt{5-9x}}\, dx$ (vi) $\int (2x+1)^2\, dx$ (vii) $\int \frac{1}{x+1}\, dx$

3. Evaluate

(i) $\int e^{2x+1}\, dx$ (ii) $\int \frac{1}{e^{7+4x}}\, dx$

4. Evaluate $\int \cos^2 x . dx$

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Calculus

Notes

26.5.2 Integration of the function of the type

$$\frac{f'(x)}{f(x)}$$

To evaluate $\int \frac{f'(x)}{f(x)}\,dx$ we put $f(x) = t$

$$f'(x)\,dx = dt$$

$$\int \frac{1}{t}\,dt = \log|t| + C$$

$$= \log|f(x)| + C$$

Similarly, $\int [f(x)]^n\ f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C$$

$$\int f'(ax+b)\,dx = \frac{f(ax+b)}{a} + C$$

Example 26.23: Evaluate $\int \frac{2x}{x^2+1}\,dx$ $\quad \left[\begin{array}{c} \because f(x) = x^2+1 \\ f'(x) = 2x \end{array}\right]$

$$= \log|1 + x^2| + C$$

Example 26.24: Evaluate $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$ $\quad \left[\begin{array}{c} \because f(x) = e^x - e^{-x} \\ f'(x) = e^x + e^{-x} \end{array}\right]$

$$= \log|e^x - e^{-x}| + C$$

Example 26.25: Evaluate $\int \frac{2x+1}{x^2+x+1}\,dx$

$$= \log|x^2 + x + 1| + C$$

EXERCISE 26.4

1. Evaluate

(i) $\int \frac{x}{3x^2-2}\,dx$ (ii) $\int \frac{2x+9}{x^2+9x+30}\,dx$

(iii) $\int \frac{x^2+1}{x^3+3x+3}\,dx$ (iv) $\int \frac{1}{x(8+\log x)}\,dx$

2. Evaluate

(i) $\int \frac{e^x}{2+be^x}\,dx$ (ii) $\int \frac{1}{e^x-e^{-x}}\,dx$

26.5.3 Integration by Substitution

Example 26.26 :

(i) $\int \tan x\,dx$ (ii) $\int \sec x\,dx$ (iii) $\int \frac{1-\tan x}{1+\tan x}\,dx$

(iv) $\int \frac{(1-\sin x)}{(1+\cos x)}\,dx$ (v) $\int \text{cosec}^5 x \cot x\,dx$

(vi) $\int \frac{\sin x}{\sin(x-a)}\,dx$

Solution: (i) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$= -\int \frac{-\sin x}{\cos x}\,dx$

$= -\log|\cos x| + C$ ($\because$ $\sin x$ is derivative of $\cos x$)

$= \log\left|\frac{1}{\cos x}\right| + C$ or $= \log|\sec x| + C$

$\therefore \int \tan x\,dx = \log|\sec x| + C$

Alternatively,

$$\int \tan x\, dx = \int \frac{\sin x\, dx}{\cos x}$$

Put $\cos x = t$

Then $-\sin x\, dx = dt$

$$\therefore \int \tan x\, dx = -\int \frac{dt}{t}$$

$$= -\log |t| + C$$

$$= -\log |\cos x| + C$$

$$= \log \left|\frac{1}{\cos x}\right| + C$$

$$= \log |\sec x| + C$$

(ii) $\int \sec x . dx$

sec x can not be integrated as such because sec x by itself is not derivative of any function. But this is not the case with sec2 x and sec x tan x. Now $\int \sec x . dx$ can be written as

$$\int \sec x \frac{\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x} dx$$

Put $\sec x + \tan x = t$

Then $(\sec x \tan x + \sec^2 x)\, dx = dt$

$$\therefore \int \sec x\, dx = \int \frac{dt}{t}$$

$$= \log |t| + C$$

$$= \log |\sec x + \tan x| + C$$

(iii) $\int \frac{1-\tan x}{1+\tan x}\, dx = \int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx$

Put $\cos x + \sin x = t$

So that $(-\sin x + \cos x)\, dx = dt$

$$\therefore \quad \int \frac{1-\tan x}{1+\tan x} dx = \int \frac{1}{t} dt$$

$$= \log |t| + C$$

$$= \log |\cos x + \sin x| + C$$

(iv) $\int \frac{1+\sin x}{1+\cos x}\, dx = \int \frac{1}{1+\cos x}\, dx + \int \frac{\sin x}{1+\cos x}\, dx$

$$= \int \frac{1}{2\cos^2\left(\frac{x}{2}\right)} dx + \int \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)} dx$$

$$= \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right) dx + \int \tan\frac{x}{2}\, dx$$

Put $\frac{x}{2} = t \Rightarrow \frac{1}{2} dx = dt$

$$\int \frac{1+\sin x}{1+\cos x} dx = \int \sec^2 t\, dt + 2\int \tan t\, dt$$

$$= \tan t - 2\log|\cos t| + C$$

$$= \tan\frac{x}{2} - 2\log\left|\cos\left(\frac{x}{2}\right)\right| + C$$

(v) $\int \operatorname{cosec}^5 x \cot x\, dx = -\int -\operatorname{cosec}^4 x \operatorname{cosec} x \cot x\, dx$

Put $\operatorname{cosec} x = t$

Then $-\operatorname{cosec} x \cot x\, dx = dt$

$$\therefore \quad \int \operatorname{cosec}^5 x \cot x\, dx = -\int t^4 dt$$

$$= -\frac{t^5}{5} + C$$

$$= \frac{-(\operatorname{cosec} x)^5}{5} + C$$

(vi) $\int \frac{\sin x}{\sin(x-a)}\, dx$

Put $x - a = t$

Then $dx = dt$ and $x = t + a$

$$\therefore \int \frac{\sin x}{\sin(x-a)}\, dx = \int \frac{\sin(t+a)}{\sin t}\, dt$$

$$= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t}\, dt$$

$(\because \sin(A + B) = \sin A \cos B + \cos A \sin B)$

$$= \cos a \int dt + \sin a \int \cot t\, dt$$

(cos a and sin a are constants.)

$$= \cos a \,.\, t + \sin a \log|\sin t| + C$$

$$= (x - a)\cos a + \sin a \log|\sin(x - a)| + C$$

26.5.4 Evaluation of integrals of algebraic functions of special forms

In the following integrals, a is a positive real number.

1. Let us show that $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\,\mathrm{Tan}^{-1}\frac{x}{a} + c$ on **R**.

$$\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a^2}\int \frac{1}{1+(\frac{x}{a})^2}\, dx + c$$

$$= \frac{1}{a^2}\,.\, a\,\mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + c$$

(by Corollary 6.2.6)

$$= \frac{1}{a}\,\mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + c.$$

We can also evaluate the same integral by putting $x = a\tan\theta$.

2. Let us show that $\int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$ on any interval containing neither $-a$ nor a.

$$\frac{1}{x^2 - a^2} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right].$$

Hence $$\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a}\left[\int \frac{1}{x-a} dx - \int \frac{1}{x+a} dx\right] + c$$

$$= \frac{1}{2a}\left[\log|x-a| - \log|x+a|\right] + c$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c.$$

3. Let us show that $\int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \sinh^{-1}\left(\frac{x}{a}\right) + c$ on **R**.

$$\int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \frac{1}{a}\int \frac{1}{\sqrt{1 + \left(\frac{x}{a}\right)^2}}\, dx$$

$$= \frac{1}{a} \cdot a\sinh^{-1}\left(\frac{x}{a}\right) + c$$

$$= \sinh^{-1}\left(\frac{x}{a}\right) + c.$$

Also $$\int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \log\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) + c.$$

(since $a > 0$ and $x + \sqrt{x^2 + a^2}$ is positive for all x in **R**, we need not write modulus for the expression $\frac{x + \sqrt{x^2 + a^2}}{a}$).

We can also evaluate the same integral by using the method of substitution.

For example, to evaluate $\int \frac{dx}{\sqrt{a^2 + x^2}}$ on **R**, we substitute $x = \varphi(\theta) = a \sinh \theta$, $\theta \in$ **R**.

Observe that in this example I = **R** and J = **R**, $dx = a \cosh \theta \, d\theta$ and

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a \cosh \theta}{a \cosh \theta} d\theta = \int d\theta = \theta + c$$

$$= \sinh^{-1}\left(\frac{x}{a}\right) + c = \log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) + c.$$

(OR)

We substitute $x = \varphi(\theta) = a \tan \theta$ for $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In this case, $J = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\theta : J \to \mathbf{R}$ is a bijection. φ and φ^{-1} are differentiable on their respective domains.

$$\sqrt{x^2+a^2} = \sqrt{a^2 \tan^2\theta + a^2} = a \sec \theta;$$

$$dx = a \sec^2 \theta \; d\theta.$$

Therefore $$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a \sec^2\theta}{a \sec \theta} d\theta = \int \sec \theta \, d\theta$$

$$= \log|\sec \theta + \tan\theta| + c$$

$$= \log\left|\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right| + c$$

$$= \log\left|\frac{\sqrt{a^2+x^2}+x}{a}\right| + c.$$

4. Let us show that $\int \frac{dx}{\sqrt{a^2-x^2}} = \mathrm{Sin}^{-1}\frac{x}{a} + c$ for $x \in (-a, a)$.

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{a}\int \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} + c = \mathrm{Sin}^{-1}\left(\frac{x}{a}\right) + c.$$

Here we note that $\int \frac{dx}{\sqrt{a^2-x^2}}$ can also be evaluated by substituting $x = a \sin \theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

5. Let us evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$ on I, where I = (a, ∞) or $(-\infty, -a)$.

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \frac{1}{a} \int \frac{dx}{\sqrt{\left(\frac{x}{a}\right)^2 - 1}} + c$$

$$= \begin{cases} \cosh^{-1}\left(\frac{x}{a}\right) + c \text{ on } (a, \infty) \\ -\cosh^{-1}\left(-\frac{x}{a}\right) + c \text{ on } (-\infty, -a) \end{cases}$$

$$= \begin{cases} \log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) + c \text{ on } (a, \infty) \\ -\log\left(\frac{-x + \sqrt{x^2 - a^2}}{a}\right) + c \text{ on } (-\infty, -a) \end{cases} \text{ . (from 26.1.9 (21))}$$

Alternative method: The function $\frac{1}{\sqrt{x^2 - a^2}}$ is defined on

$(-\infty, -a) \cup (a, \infty)$, $a > 0$. We can evaluate the integral on an interval I only when $I \subset (-\infty, -a) \cup (a, \infty)$.

Let $I \subset (a, \infty)$, put $x = \varphi(\theta) = a \cosh \theta$, $\theta \in (0, \infty)$.

Then $\varphi : (0, \infty) \to (a, \infty)$ is a bijective function, φ and φ^{-1} are differentiable,

$dx = a \sinh \theta \, d\theta$ and

$$\sqrt{x^2 - a^2} = \sqrt{a^2 \cosh^2 \theta - a^2} = a\sqrt{\cosh^2 \theta - 1} = a \sinh \theta.$$

Hence $\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sinh \theta}{a \sinh \theta} d\theta = \int d\theta = \theta + c = \cosh^{-1}\left(\frac{x}{a}\right) + c$ on (a, ∞).

Now let $I \subset (-\infty, -a)$.

On substituting $x = -y$, $y \in (a, \infty)$, we observe that

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = -\int \frac{dy}{\sqrt{y^2 - a^2}} = -\cosh^{-1}\left(-\frac{x}{a}\right) + c \text{ on } (-\infty, -a).$$

We know from hyperbolic functions (Intermediate Mathematics - I(A) Text Book) that

$$\cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right) \quad \text{if } x > 1.$$

Hence for $x > a$ we have

$$\cosh^{-1}\left(\frac{x}{a}\right) = \log\left(\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right) = \log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right).$$

If $x < -a$ then $-\frac{x}{a} > 1$.

Hence

$$\cosh^{-1}\left(-\frac{x}{a}\right) = \log\left(-\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right)$$

$$= \log\left(\frac{-x+\sqrt{x^2-a^2}}{a}\right) = -\log\left(\frac{-x-\sqrt{x^2-a^2}}{a}\right).$$

Thus it follows that

$$\int\frac{dx}{\sqrt{x^2-a^2}} = \begin{cases} \log\left(\dfrac{x+\sqrt{x^2-a^2}}{a}\right)+c \text{ if } I\subset(a,\infty) \\ \log\left(\dfrac{-x-\sqrt{x^2-a^2}}{a}\right)+c \text{ if } I\subset(-\infty,-a) \end{cases}.$$

Hence , $\int\frac{dx}{\sqrt{x^2-a^2}} = \ln\frac{|x+\sqrt{x^2-a^2}|}{a}+c$ on $I \subset \mathbf{R} \setminus [-a, a]$.

6. Let us show that $\int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\,\text{Sin}^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$ on $(-a, a)$.

Put $x = a \sin\theta$ for $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ then $dx = a\cos\theta\, d\theta$.

Hence $\int\sqrt{a^2-x^2}\,dx = \int\sqrt{a^2-a^2\sin^2\theta}\,.\,a\cos\theta\,d\theta$

$$= a^2\int\cos^2\theta\,d\theta = a^2\int\frac{1+\cos 2\theta}{2}d\theta$$

$$= \frac{a^2}{2}\left[\int d\theta+\int\cos 2\theta\,d\theta\right]+c$$

$$= \frac{a^2}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]+c$$

$$= \frac{a^2}{2}\left[\theta + \sin\theta\cos\theta\right] + c$$

$$= \frac{a^2}{2}\left[\text{Sin}^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right] + c$$

$$= \frac{a^2}{2}\,\text{Sin}^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + c.$$

Note : This integral $\int\sqrt{a^2 - x^2}\,dx$ can also be evaluated by using the formula for integration by parts (see 6.2.26(1)).

7. Let us show that

$$\int\sqrt{x^2 - a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c \quad \text{on } [a, \infty).$$

put $x = a\cosh\theta$ for $\theta \in [0, \infty)$. Then $dx = a\sinh\theta\,d\theta$

and $\sqrt{x^2 - a^2} = \sqrt{a^2\cosh^2\theta - a^2} = a\sinh\theta$.

$$\int\sqrt{x^2 - a^2}\,dx = \int a\sinh\theta \,.\, a\sinh\theta\,d\theta = a^2\int\sinh^2\theta\,.\,d\theta$$

$$= a^2\int\left(\frac{\cosh 2\theta - 1}{2}\right)d\theta = \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} - \theta\right] + c$$

$$= \frac{a^2}{2}[\sinh\theta\,\cosh\theta - \theta] + c$$

$$= \frac{a^2}{2}\left[\sqrt{\cosh^2\theta - 1}\,.\,\cosh\theta - \theta\right] + c$$

$$= \frac{a^2}{2}\left[\sqrt{\frac{x^2}{a^2} - 1}\,.\,\frac{x}{a} - \cosh^{-1}\frac{x}{a}\right] + c$$

$$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c. \quad \text{(Also see 6.2.26(2)).}$$

Similarly, it can be shown that

$$\int\sqrt{x^2 - a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\cosh^{-1}\left(-\frac{x}{a}\right) + c \quad \text{on } (-\infty, -a) \text{ by}$$

substituting $x = -a\cosh\theta,\ \theta \in [0, \infty)$.

8. Let us show that

$$\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+c \text{ on } \mathbf{R}.$$

The given integral can be evaluated by substituting $x = a \sinh\theta$, $\theta \in \mathbf{R}$ or by substituting

$$x = a\tan\theta,\quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).\quad \text{(Also, see 6.2.26(3))}.$$

Example 26.27 : Evaluate $\int \frac{dx}{\sqrt{4-9x^2}}$ *on* $I = \left(-\frac{2}{3}, \frac{2}{3}\right)$.

Solution : $\int \frac{dx}{\sqrt{4-9x^2}} = \int \frac{dx}{\sqrt{2^2-(3x)^2}}$.

Put $x = \varphi(\theta) = \frac{2}{3}\sin\theta$ for $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $dx = \frac{2}{3}\cos\theta\, d\theta$.

Hence
$$\int \frac{dx}{\sqrt{4-9x^2}} = \int \frac{\frac{2}{3}\cos\theta\, d\theta}{\sqrt{4-9.\frac{4}{9}\sin^2\theta}} = \int \frac{\frac{2}{3}\cos\theta}{2\cos\theta}\, d\theta$$

$$= \frac{1}{3}\int d\theta = \frac{1}{3}\theta + c = \frac{1}{3}\,\mathrm{Sin}^{-1}\left(\frac{3x}{2}\right) + c.$$

Example 26.28: Evaluate $\int \frac{1}{a^2-x^2}\, dx$ *for* $x \in I = (-a, a)$.

Solution : We have $\frac{1}{a^2-x^2} = \frac{1}{(a-x)(a+x)} = \frac{1}{2a}\left(\frac{1}{a-x}+\frac{1}{a+x}\right)$.

Hence
$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\left[\int \frac{1}{a-x}\,dx + \int \frac{1}{a+x}\,dx\right] + c$$

$$= \frac{1}{2a}\left[-\log|a-x| + \log|a+x|\,\right] + c$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c.$$

Example 26.29: Evaluate $\int \frac{1}{1+4x^2}dx$ on R

Solution : $\int \frac{1}{1+4x^2}dx = \int \frac{dx}{4\left[\left(\frac{1}{2}\right)^2 + x^2\right]} = \frac{1}{4}\int \frac{dx}{\left(\frac{1}{2}\right)^2 + x^2}$

$$= \frac{1}{4}\cdot\left[2\ \mathrm{Tan}^{-1}\ 2x\right] + c \quad \text{(by 6.2.18(1))}$$

$$= \frac{1}{2}\ \mathrm{Tan}^{-1}\ (2x) + c.$$

Example 26.30: *Find* $\int \frac{1}{\sqrt{4-x^2}}\ dx$ *on* $(-2, 2)$.

Solution : $\int \frac{1}{\sqrt{4-x^2}}\ dx = \int \frac{1}{\sqrt{2^2-x^2}}\ dx = \mathrm{Sin}^{-1}\left(\frac{x}{2}\right) + c.$

Example 26.31: *Evaluate* $\int \sqrt{4x^2+9}\ dx$ *on* **R**.

Solution : $\int \sqrt{4x^2+9}\ dx = 2\int \sqrt{x^2 + \left(\frac{3}{2}\right)^2}\ dx$

$$= 2\left[\frac{x\sqrt{(\frac{3}{2})^2 + x^2}}{2} + \frac{(\frac{3}{2})^2}{2}\sinh^{-1}\left(\frac{x}{(\frac{3}{2})}\right)\right] + c \quad \text{(by 6.2.18(8))}$$

$$= \frac{1}{2}x\sqrt{4x^2+9} + \frac{9}{4}\sinh^{-1}\left(\frac{2x}{3}\right) + c.$$

Example 26.32: *Evaluate* $\int \sqrt{9x^2-25}\ dx$ *on* $\left[\frac{5}{3}, \infty\right)$.

Solution: $\int \sqrt{9x^2-25}\ dx = 3\int \sqrt{x^2 - \left(\frac{5}{3}\right)^2}\ dx$

$$= 3\left[\frac{x\sqrt{x^2 - (\frac{5}{3})^2}}{2} - \frac{(\frac{5}{3})^2}{2}\cosh^{-1}\left(\frac{x}{(\frac{5}{3})}\right)\right] + c$$

(by 6.2.18(7))

$$= \frac{1}{2}x\sqrt{9x^2-25} - \frac{25}{6}\cosh^{-1}\left(\frac{3x}{5}\right) + c.$$

Example 26.33: *Evaluate* $\int\sqrt{16-25x^2}\,dx$ *on* $\left(-\frac{4}{5},\frac{4}{5}\right)$.

Solution : $\int\sqrt{16-25x^2}\,dx = 5\int\sqrt{\left(\frac{4}{5}\right)^2-x^2}\,dx$

$$= 5\left[\frac{x}{2}\sqrt{\left(\frac{4}{5}\right)^2-x^2}+\frac{\left(\frac{4}{5}\right)^2}{2}\operatorname{Sin}^{-1}\frac{x}{\left(\frac{4}{5}\right)}\right]+c$$

$$= \frac{x}{2}\sqrt{16-25x^2}+\frac{16}{10}\operatorname{Sin}^{-1}\left(\frac{5x}{4}\right)+c$$

$$= \frac{x}{2}\sqrt{16-25x^2}+\frac{8}{5}\operatorname{Sin}^{-1}\left(\frac{5x}{4}\right)+c.$$

26.5.5. Evaluation of integrals of the form

$\int\frac{1}{ax^2+bx+c}dx$ where a, b, c are real numbers $a \neq 0$

Working Rule:

Reduce $ax^2 + bx + c$ to the form $a[(x + \alpha)^2 + \beta^2]$ and then integrate using the substitution $t = x + \alpha$.

Example 26.34: Evaluate $\int\frac{1}{x^2+x+1}dx$

$$x^2 + x + 1 = x^2+x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1$$

$$= \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$= \left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$$

Hence $\int\frac{1}{x^2+x+1}dx = \int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx \qquad t = x+\frac{1}{2}$

$$= \int\frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt \qquad dt = dx.$$

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \tan^{-1}\left(\frac{t}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Example 26.35: Evaluate $\int \frac{1}{3x^2+x+1} dx$

$$3x^2+x+1 = 3\left[x^2+\frac{x}{3}+\frac{1}{3}\right]$$

$$= 3\left[x^2+\frac{x}{3}+\left(\frac{1}{6}\right)^2-\left(\frac{1}{6}\right)^2+\frac{1}{3}\right]$$

$$= 3\left[\left(x+\frac{1}{6}\right)^2+\left(\frac{\sqrt{11}}{6}\right)^2\right]$$

Hence $\int \frac{1}{3x^2+x+1} dx = \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{6}\right)^2+\left(\frac{\sqrt{11}}{6}\right)^2} dx$

$t = x+\frac{1}{6}$

$dt = dx$

$$= \frac{1}{3}\int \frac{1}{t^2+\left(\frac{\sqrt{11}}{6}\right)^2} dt$$

$$= \frac{1}{3} \cdot \frac{1}{\sqrt{11}/6} \tan^{-1}\left(\frac{t}{\sqrt{11}/6}\right)+C$$

$$= \frac{2}{\sqrt{11}} \tan^{-1}\left(\frac{6x+1}{\sqrt{11}}\right)+C$$

26.5.6 Evaluation of integrals of the form

(i) $\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$ (ii) $\int \sqrt{ax^2+bx+c}\,dx$

where a, b, c are real numbers and $a \neq 0$

Working Rule:

Case I : If $a > 0$ and $b^2 - 4ac < 0$, then reduce $ax^2 + bx + c$ to the form $a[(x+\alpha)^2 + \beta^2]$ and then integrate.

Case II: Reduce $ax^2 + bx + c > 0$ then write $ax^2 + bx + c$ as $(-a)$ $[\beta^2 - (x+\alpha)^2]$and then integrate.

Example 26.36 : Evaluate $\int \frac{1}{\sqrt{2x-3x^2+1}}\,dx$

$$2x-3x^2+1 = (-3)\left[x^2-\frac{2}{3}x-\frac{1}{3}\right]$$

$$= (-3)\left[x^2-\frac{2}{3}x+\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2-\frac{1}{3}\right]$$

$$= (-3)\left[\left(x-\frac{1}{3}\right)^2-\left(\frac{2}{3}\right)^2\right]$$

$$= 3\left[\left(\frac{2}{3}\right)^2-\left(x-\frac{1}{3}\right)^2\right]$$

Hence $\int \frac{1}{\sqrt{2x-3x^2+1}}\,dx$

$$= \int \frac{1}{\sqrt{3}} \frac{1}{\sqrt{\left[\left(\frac{2}{3}\right)^2-\left(x-\frac{1}{3}\right)^2\right]}}\,dx$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{x-\frac{1}{3}}{\frac{2}{3}}\right)+C$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3x-1}{2}\right)+C$$

Example 26.37: Evaluate $\int\sqrt{1+3x-x^2}\,dx$

$$1+3x-x^2 = (-1)\left[x^2-3x-1\right]$$

$$= (-1)\left[x^2-3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2-1\right]$$

$$= (-1)\left[\left(x-\frac{3}{2}\right)^2-\left(\frac{\sqrt{13}}{2}\right)^2\right]$$

$$= \left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2$$

Hence $\int\sqrt{1+3x-x^2}\,dx = \int\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\,dx$

$$= \frac{\left(x-\frac{3}{2}\right)}{2}\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}+\frac{(\sqrt{13}/2)^2}{2}\sin^{-1}\left(\frac{x-3/2}{\sqrt{13}/2}\right)+C$$

$$= \frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right)+C$$

26.5.7 Evaluation of integrals of the forms

(i) $\int \frac{px+q}{ax^2+bx+c}dx$ (ii) $\int (px+q)\sqrt{ax^2+bx+c}\ dx$

(iii) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\ dx$ where a, b, c, p, q are real numbers, $a \neq 0$ and $p \neq 0$

Working Rule:

Write $px+q = A\frac{d}{dx}(ax^2+bx+c)+B$ and then integrate

Example 26.38: Evaluate $\int \frac{x+1}{x^2+3x+12}dx$

We write

$$x+1 = A.\frac{d}{dx}(x^2+3x+12)+B$$

$x + 1 = A(2x + 3) + B.$

On comparing the co-efficients in like powers of x on both sides on the above.

equation we get

$$A = \frac{1}{2},\ B = \frac{-1}{2}$$

Hence

$$x+1 = \frac{1}{2}(2x+3) - \frac{1}{2}$$

Now $\int \frac{x+1}{x^2+3x+12}dx = \frac{1}{2}\int \frac{2x+3}{(x^2+3x+12)}dx - \frac{1}{2}\int \frac{1}{(x^2+3x+12)}dx$

$$= \frac{1}{2}\log|x^2+3x+12| - \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2+\left(\frac{\sqrt{39}}{2}\right)^2}dx + C$$

$$= \frac{1}{2}\log|x^2+3x+12| - \frac{1}{2}\frac{2}{\sqrt{39}}\tan^{-1}\left(\frac{x+3/2}{\sqrt{39}/2}\right)+C$$

$$= \frac{1}{2}\log|x^2+3x+12| - \frac{1}{\sqrt{39}}\tan^{-1}\left(\frac{2x+3}{\sqrt{39}}\right)+C$$

Example 26.39 : *Evaluate* $\int(3x-2)\sqrt{2x^2-x+1}\,dx.$

Solution : We write $(3x-2) = A\frac{d}{dx}(2x^2-x+1)+B$

$$= A(4x-1)+B.$$

On comparing the coefficients of like powers of x on both sides of the above equation, we get

$A=\frac{3}{4}$ and $B=-\frac{5}{4}$. Hence $3x-2=\frac{3}{4}(4x-1)-\frac{5}{4}$.

Therefore $\int(3x-2)\sqrt{2x^2-x+1}\,dx = \int\left[\frac{3}{4}(4x-1)-\frac{5}{4}\right]\sqrt{2x^2-x+1}\,dx$

$$= \frac{3}{4}\int(4x-1)\sqrt{2x^2-x+1}\,dx - \frac{5}{4}\int\sqrt{2x^2-x+1}\,dx + c$$

$$= \frac{3}{4}\cdot\frac{2}{3}(2x^2-x+1)^{\frac{3}{2}} - \frac{5\sqrt{2}}{4}\int\sqrt{\left(x-\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}\,dx + c$$

$$= \frac{1}{2}(2x^2-x+1)^{\frac{3}{2}} - \frac{5\sqrt{2}}{4}$$

$$\left[\frac{1}{2}\left(x-\frac{1}{4}\right)\sqrt{\left(x-\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2} + \frac{7}{32}\sinh^{-1}\left(\frac{(x-\frac{1}{4})}{\frac{\sqrt{7}}{4}}\right)\right]+c$$

$$= \frac{1}{2}(2x^2-x+1)^{\frac{3}{2}} - \frac{5}{4\sqrt{2}}\left(x-\frac{1}{4}\right)\sqrt{\left(x-\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^4}$$

$$-\frac{35}{64\sqrt{2}}\sinh^{-1}\left(\frac{4x-1}{\sqrt{7}}\right)+c.$$

Example 26.40: *Evaluate* $\int \frac{2x+5}{\sqrt{x^2-2x+10}}\,dx.$

Solution : We write

$$2x+5 = A\frac{d}{dx}(x^2-2x+10)+B = A(2x-2)+B.$$

On comparing the coefficients of the like powers of x on both sides of the above equation, we get A = 1 and B = 7. Thus $2x+5 = (2x-2)+7$.

$$\text{Hence } \int \frac{2x+5}{\sqrt{x^2-2x+10}}\,dx = \int \frac{2x-2}{\sqrt{x^2-2x+10}}\,dx + 7\int \frac{dx}{\sqrt{x^2-2x+10}} + c$$

$$= 2\sqrt{x^2-2x+10} + 7\int \frac{dx}{\sqrt{(x-1)^2+3^2}} + c$$

$$= 2\sqrt{x^2-2x+10} + 7\sinh^{-1}\left(\frac{x-1}{3}\right) + c.$$

26.5.8 To evaluate integrals of the type

$\int \frac{dx}{(ax+b)\sqrt{px+q}}$ **where a, b, p and q are real numbers, $a \neq 0$ and $p \neq 0$**

Working rule : Put $t = \sqrt{px+q}$ and then integrate.

Example 26.41: *Evaluate* $\int \frac{dx}{(x+5)\sqrt{x+4}}$.

Solution : Put $t = \sqrt{x+4}$. Then $dt = \frac{1}{2\sqrt{x+4}}\,dx$.

We have $t^2 = x+4$. Hence $x+5 = t^2+1$.

$$\text{Therefore } \int \frac{dx}{(x+5)\sqrt{x+4}} = \int \frac{2}{t^2+1}\,dt = 2\,\text{Tan}^{-1}\,t + c$$

$$= 2\,\text{Tan}^{-1}(\sqrt{x+4}) + c.$$

26.5.9 To evaluate integrals of the type

(i) $\int \frac{1}{a+b\cos x}dx$ **(ii)** $\int \frac{1}{a+b\sin x}dx$

where a and b real numbers, $b \neq 0$

Working rule : We write $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

and $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

Put $t = \tan\frac{x}{2}$. Then $dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$.

Hence, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$.

We have $a+b\cos x = a+b\left(\frac{1-t^2}{1+t^2}\right) = \frac{a(1+t^2)+b(1-t^2)}{1+t^2}$.

Therefore $\int \frac{dx}{a+b\cos x} = \int \frac{\frac{1}{2}\sec^2\frac{x}{2}}{(a+b\cos x)\frac{1}{2}\sec^2\frac{x}{2}}dx$

$$= \int \frac{2}{a(1+t^2)+b(1-t^2)}\,dt$$

$$= 2\int \frac{dt}{(a+b)+(a-b)t^2},$$

and we can now integrate it by known methods.

The integral in (ii) can be evaluated in a similar way by using the expression $\frac{2t}{1+t^2}$ for $\sin x$.

Example 26.42 : *Evaluate* $\int \frac{dx}{5+4\cos x}$.

Solution: Put $t = \tan\frac{x}{2}$. Then $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$.

Now $\displaystyle\int \frac{dx}{5+4\cos x} = \int \frac{\frac{1}{2}\sec^2\frac{x}{2}}{(5+4\cos x)\frac{1}{2}\sec^2\frac{x}{2}}\,dx$

$$= \int \frac{2dt}{(5+4(\frac{1-t^2}{1+t^2}))\ (1+t^2)}$$

$$= 2\int \frac{dt}{5(1+t^2)+4(1-t^2)} \quad = 2\int \frac{dt}{9+t^2}$$

$$= \frac{2}{3}\ \mathrm{Tan}^{-1}\left(\frac{t}{3}\right)+c \quad = \frac{2}{3}\ \mathrm{Tan}^{-1}\left(\frac{\tan(\frac{x}{2})}{3}\right)+c.$$

Example 26.43: *Find* $\displaystyle\int \frac{dx}{3\cos x+4\sin x+6}$.

Solution : Put $t=\tan\frac{x}{2}$. Then $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt.$

$$\int \frac{dx}{3\cos x+4\sin x+6} = \int \frac{\frac{1}{2}\sec^2\frac{x}{2}}{(3\cos x+4\sin x+6)\frac{1}{2}\sec^2\frac{x}{2}}\,dx$$

$$= \int \frac{2dt}{(3\,(\frac{1-t^2}{1+t^2})+4\frac{2t}{1+t^2}+6)\ (1+t^2)}$$

$$= \int \frac{2dt}{3\,(1-t^2)+8t+6(1+t^2)} \quad = \int \frac{2dt}{3t^2+8t+9}$$

$$= \frac{2}{3}\int \frac{dt}{(t+\frac{4}{3})^2+\frac{11}{9}}+c$$

$$= \frac{2}{3}\cdot\frac{3}{\sqrt{11}}\ \mathrm{Tan}^{-1}\left(\frac{(t+\frac{4}{3})}{(\frac{\sqrt{11}}{3})}\right)+c$$

$$= \frac{2}{\sqrt{11}}\ \mathrm{Tan}^{-1}\left(\frac{3\tan\frac{x}{2}+4}{\sqrt{11}}\right)+c.$$

26.5.10 Evaluation of integrals of the type

$$\int \frac{a\ \cos x + b\sin x + c}{d\ \cos x + e\sin x + f}\,dx \qquad \text{...(A)}$$

where a, b, c, d, e, f are real numbers, $d \neq 0,\ e \neq 0$

Working rule : We find real numbers λ, μ and γ such that

$$(a\cos x + b\sin x + c) = \lambda[d\cos x + e\sin x + f]'$$

$$+ \mu\,[d\cos x + e\sin x + f] + \gamma$$

and then by substituting this expression in the integrand, we evaluate the given integral.

Example 26.44 : *Find* $\int \frac{dx}{d + e\tan x}$.

Solution : We have $\frac{1}{d + e\tan x} = \frac{\cos x}{d\cos x + e\sin x}$.

Let us find λ, μ and γ such that

$$\cos x \equiv \lambda(d\cos x + e\sin x)' + \mu(d\cos x + e\sin x) + \gamma$$

$$\equiv \lambda(-d\sin x + e\cos x) + \mu(d\cos x + e\sin x) + \gamma.$$

On comparing the coefficients of like terms on both sides of the above equation, we have

$$\lambda e + \mu d = 1,\ -\lambda d + \mu e = 0,\ \gamma = 0\,.$$

On solving these equations, we obtain $\lambda = \frac{e}{d^2+e^2},\ \mu = \frac{d}{d^2+e^2},\ \gamma = 0.$

Therefore

$$\int \frac{dx}{d + e\tan x} = \lambda \int \frac{(d\ \cos x + e\sin x)'}{(d\ \cos x + e\sin x)}\,dx + \mu \int \frac{d\ \cos x + e\sin x}{d\ \cos x + e\sin x}\,dx + c_1$$

$$= \lambda\ \log|d\ \cos x + e\ \sin x| + \mu x + c_1.$$

$$= \frac{1}{d^2+e^2}[dx + e\log|d\ \cos x + e\sin x|\,] + c_1.$$

$$= \frac{1}{d^2+e^2}[dx + e\log|d\ \cos x + e\sin x|\,] + c_1.$$

Example 26.45: Evaluate $\int \frac{\sin x}{d\cos x + e\sin x}\,dx$ *and* $\int \frac{\cos x}{d\cos x + e\sin x}\,dx.$

Solution : Let $A_1 = \int \frac{\sin x}{d\cos x + e\sin x}\,dx$ and $A_2 = \int \frac{\cos x}{d\cos x + e\sin x}\,dx$.

Now $$eA_1 + dA_2 = \int \frac{e\sin x + d\cos x}{d\cos x + e\sin x}\,dx = \int dx = x + c_1 \qquad ...(i)$$

and $$-dA_1 + eA_2 = \int \frac{(-d\ \sin x + e\ \cos x)}{d\cos x + e\ \sin x}\,dx$$

$$= \log|d\ \cos x + e\ \sin x| + c_2 \qquad ...(ii)$$

From (i) and (ii)

$$A_1 = \frac{1}{d^2+c^2}\,[ex - d\log|d\cos x + e\sin x|] + c_3 \quad \text{where } c_3 = \frac{ec_1 - dc_2}{d^2+e^2};$$

and

$$A_2 = \frac{1}{d^2+c^2}\,[dx + e\log|d\cos x + e\sin x|] + c_4$$

where $c_4 = \frac{dc_1 + ec_2}{d^2+e^2}.$

Example 26.46: *Evaluate* $\int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1}\,dx$.

Solution : Let us find real numbers λ, μ and γ such that

$$\cos x + 3\ \sin x + 7 = \lambda(\cos x + \sin x + 1)' + \mu(\cos x + \sin x + 1)\gamma$$

$= \lambda(-\sin x + \cos x) + \mu(\cos x + \sin x + 1)\ \gamma$

$= (\lambda + \mu) \cos x + (-\lambda + \mu) \sin x + (\gamma + \mu).$

On comparing the coefficients of like terms on both sides of the above equation, we have

$\lambda + \mu = 1; \quad -\lambda + \mu = 3; \quad \mu + \gamma = 7.$

On solving these equations, we have $\lambda = -1; \quad \mu = 2;$ and $\gamma = 5.$ Therefore

$$\int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1}\, dx$$

$$= -\int \frac{(\cos x + \sin x + 1)'}{\cos x + \sin x + 1}\, dx + 2\int \frac{\cos x + \sin x + 1}{\cos x + \sin x + 1}\, dx$$

$$+5\int \frac{1}{\cos x + \sin x + 1}\, dx + c$$

$$= -\log|\cos x + \sin x + 1| + 2x + 5\int \frac{1}{\cos x + \sin x + 1}\, dx + c. \qquad \text{... (A)}$$

We now evaluate $\int \frac{1}{\cos x + \sin x + 1}\, dx.$

$$\int \frac{1}{\cos x + \sin x + 1}\, dx = \int \frac{1}{2\cos^2 \frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2}}\, dx$$

$$= \frac{1}{2}\int \frac{\sec^2 \frac{x}{2}}{(1 + \tan\frac{x}{2})}\, dx$$

$$= \int \frac{dt}{1+t} \quad \left(\text{on substituting } t = \tan\frac{x}{2}\right)$$

$$= \log|1 + t| = \log\left|1 + \tan\frac{x}{2}\right|.$$

Hence from (A),

$$\int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1}\, dx = -\log|\cos x + \sin x + 1| + 2x + 5\log\left|1 + \tan\frac{x}{2}\right| + c.$$

EXERCISE 26.5

1. Evaluate

(i) $\int \frac{\cos x}{\sin^2 x + 4\sin x + 5} dx$

(ii) $\int \frac{1}{\sqrt{x^2 + 2x + 10}} dx$

(iii) $\int \frac{1}{\sqrt{1 + x - x^2}} dx$

(iv) $\int \sqrt{3 + 8x - 3x^2}\, dx$

(v) $\int \frac{1}{\sqrt{8 + 3x - x^2}} dx$

(vi) $\int \frac{1}{3x^2 + 6x + 21} dx$

2. Evaluate

(i) $\int \frac{x+1}{\sqrt{x^2 - x + 1}} dx$

(ii) $\int (6x+5)\sqrt{6 - 2x^2 + x}\, dx$

(iii) $\int x\sqrt{1 + x - x^2}\, dx$

(iv) $\int \frac{1}{(1+x)\sqrt{3 + 2x - x^2}} dx$ on $(-1, 3)$

(v) $\int \sqrt{\frac{5-x}{x-2}}\, dx$ on $(2, 5)$

(vi) $\int \frac{1}{(x+2)\sqrt{x+1}} dx$ on $(-1, \infty)$

(vii) $\int \frac{1}{(2x+3)\sqrt{x+2}} dx$ on $I \subset (-2, \infty) - \left(\frac{-3}{2}\right)$

3. Evaluate

(i) $\int \dfrac{9\cos x - \sin x}{4\sin x + 5\cos x}\,dx$

(ii) $\int \dfrac{1}{1+\sin x + \cos x}\,dx$

(iii) $\int \dfrac{1}{4+5\sin x}\,dx$

(iv) $\int \dfrac{1}{4\cos x + 3\sin x}\,dx$

(v) $\int \dfrac{1}{\sin x + \sqrt{3}\cos x}\,dx$

(vi) $\int \dfrac{1}{5+4\cos x}\,dx$

(vii) $\int \dfrac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5}\,dx$

26.6 INTEGRATION BY PARTS

In differentiation you have learnt that

$$\frac{d}{dx}(fg) = f\frac{d}{dx}(g) + g\frac{d}{dx}(f)$$

$$f\frac{d}{dx}(g) = \frac{d}{dx}(fg) - g\frac{d}{dx}(f)$$

or $$f\frac{d}{dx}(g) = \frac{d}{dx}(fg) - g\frac{d}{dx}(f)$$

Also you know that $f\,\frac{d}{dx}(fg)\,dx = fg$

Integrating (1), we have

$$\int f \frac{d}{dx}(g)\, dx = \int \frac{d}{dx}(fg)\, dx - \int g \frac{d}{dx}(f)\, dx$$

$$= fg - \int g \frac{d}{dx}(f)\, dx$$

if we take $f = u(x)$: $\frac{d}{dx}(g) = v(x)$

(2) becomes $\int u(x)\, v(x)\, dx$

$$= u(x) \, . \int v(x)\, dx - \int \left[\frac{d}{dx}(u(x)) \int v(x)\, dx \right] dx$$

= I function × integral of II function
A

$-\int$[differential coefficient of Ifunction × integral of II function]dx
B

Here the important factor is the choice of I and II function in the product of two functions because either can be I or II function. For that the indicator will be part 'B' of the result above.

The first function is to be chosen such that it reduces to a next lower term or to a constant term after subsequent differentiations.

Inequations of integration like

$$x \sin x,\ x \cos^2 x,\ x^2 e^x$$

(1) algebraic function should be taken as the first function.

(2) If there is no algebraic function then look for a function which simplifies the production in 'B' as above; the choice can be in order of preference like choosing first function

(i) an inverse function (ii) a logarithmic function

(iii) a trigonometric function (iv) an exponential function

The following example will give a practice to the concept of choosing first function.

	I function	II function
1. $\int x \cos x \, dx$	x (being algebraic)	$\cos x$
2. $\int x^2 e^x \, dx$	x^2 (being algebraic)	e^x
3. $\int x^2 \log x \, dx$	$\log x$	x^2
4. $\int \frac{\log x}{(1+x^2)} dx$	$\log x$	$\frac{1}{(1+x)^2}$
5. $\int x \sin^{-1} x \, dx$	$\sin^{-1} x$	x
6. $\int \log x \, dx$	$\log x$	1 (In single function of logarithm and inverse trigonometric we take unity as II function)
7. $\int \sin^{-1} x \, dx$	$\sin^{-1} x$	1

Example 26.47: Evaluate :

$$\int x \cos dx$$

Solution: Taking the polynomial (algebraic function) x as the first function and trigonometric function cos x as the second function, we get

$$\underset{\text{I II}}{\int x \cos x \, dx} = x \int \cos x \, dx - \int \left[\frac{d}{dx}(x) . \int \cos x \, dx \right] dx$$

(I function × Integral of II function $-\int [\frac{d}{dx}$ (I function). $\int$(II function) dx)]dx

$$= x \sin x - \int 1.\sin x \, dx$$

$$= x \sin x - [-\cos x] + c$$

$$= x \sin x + \cos x + c$$

Example 26.48: Evaluate :

$$\int x^2 \sin x \, dx$$

Notes

Solution: Taking algebraic function x^2 as I function and $\sin x$ as II function, we have,

$$\int \underset{\text{I}}{x^2} \underset{\text{II}}{\sin x}\, dx = x^2 \int \sin x - \int \left[\frac{d}{dx}(x^2) \int \sin x\, dx \right] dx$$

$$= -x^2 \cos x - 2\int x(-\cos x)\, dx$$

$$= -x^2 \cos x + 2\int x \cos x\, dx \qquad ..(1)$$

Again, $\int x \cos x\, dx = x \sin x + \cos x + C$

Substituting (2) in (1) we have

$$\int x^2 \sin x\, dx = x^2 \cos x + 2[x \sin x + \cos x] + C$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C$$

Example 26.49: Evaluate :

$$\int x^2 \log x\, dx$$

Solution: In order of preference $\log x$ is to be taken as I function.

$$\int \underset{\text{I}}{\log x}\, \underset{\text{II}}{x^2}\, dx = \frac{x^3}{3}\log x - \int \frac{1}{x}.\frac{x^3}{3} dx$$

$$= \frac{x^3}{3}\log x - \int \frac{x^2}{3} dx$$

$$= \frac{x^3}{3}\log x - \frac{1}{3}\left(\frac{x^3}{3}\right) + C$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C.$$

Example 26.50: Evaluate

$$\int \frac{\log x}{(1+x)^2} dx$$

Solution: $\int \frac{\log x}{(1+x)^2} dx = \int \log x \frac{1}{(1+x)^2} dx$

$$= \int \log x \left(-\frac{1}{1+x}\right) dx - \int \frac{1}{x} . \left(\frac{-1}{1+x}\right) dx$$

$$= \frac{-\log x}{1+x} + \int \frac{1}{x(1+x)} dx$$

$$= \frac{-\log x}{1+x} + \int \left[\frac{1}{x} - \frac{1}{1+x}\right] dx$$

$$= \frac{-\log x}{1+x} + \int \frac{1}{x} dx - \int \frac{1}{1+x} dx$$

$$= \frac{-\log x}{1+x} + \log|x| - \log|1+x| + C$$

$$= \frac{-\log x}{1+x} + \log\left|\frac{x}{1+x}\right| + C$$

Example 26.51: Evaluate :

$$\int x\, e^{2x}\, dx$$

Solution: $\int x\, e^{2x}\, dx = x \frac{e^{2x}}{x} - \int 1 . \frac{e^{2x}}{2} dx$

$$= x \frac{e^{2x}}{x} - \frac{1}{2}\left(\frac{e^{2x}}{2}\right) + C$$

$$= x \frac{e^{2x}}{x} - \frac{1}{4} e^{2x} + C$$

Example 26.52: Evaluate :

$$\int \sin^{-1} x\, dx$$

Solution: $\int \sin^{-1} x \, dx = \int \sin^{-1} x .1 . \, dx$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Let $1 - x^2 = t$

$\Rightarrow \quad -2x \, dx = dt$

$\Rightarrow \quad x \, dx = \frac{-1}{2} dt$

$\therefore \quad \int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \frac{dt}{\sqrt{t}}$

$$= -\sqrt{t} + C$$

$$= -\sqrt{1-x^2} + C$$

$\therefore \quad \int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} + C$

EXERCISE 26.6

1. (a) $\int x \sin x \, dx$ (b) $\int (1+x^2) \cos 2x \, dx$

   (c) $\int x \sin 2x \, dx$

2. (a) $\int x \tan^2 x \, dx$ (b) $\int x^2 \sin^2 x \, dx$

3. (a) $\int x^3 \log 2x \, dx$ (b) $(1-x^2) \log x \, dx$ (c) $\int (\log x)^2 \, dx$

4. (a) $\int \frac{\log x}{x^n} dx$ (b) $\int \frac{\log(\log x)}{x} dx$

5. (a) $\int x^2 e^{3x} \, dx$ (b) $\int x \, e^{3x} \, dx$

6. (a) $\int x (\log x)^2 \, dx$

7. (a) $\int \sec^{-1} x \, dx$ (b) $\int x \cot^{-1} x \, dx$

26.7 INTEGRAL OF THE FORM $\int e^x [f(x) + f'(x)]\, dx$

Where $f'(x)$ is the differentiation of $f(x)$. In such type of integration while integrating by parts the solution will be $e^x\ (f(x) + C$.

For example, consider

$$\int e^x [\tan x + \log \sec x]\, dx$$

Let $\qquad f(x) = \log \sec x$

then $\qquad f'(x) = \dfrac{\sec x \tan x}{\sec x} = \tan x$

So (1) can be rewritten as

$$\int e^x [f'(x) + f(x)]\, dx = e^x (f(x)) + C = e^x \log \sec x + C$$

Alternatively, you can evaluate it as under :

$$\int e^x [\tan x + \log \sec x]\, dx = \int \underset{\text{I}}{e^x} \underset{\text{II}}{\tan x}\, dx + \int e^x \log \sec x\, dx$$

$$= e^x \log \sec x - \int e^x \log \sec x\, dx + \int e^x \log \sec x\, dx$$

$$= \int e^x \log \sec x + C$$

Example 26.53: Evaluate the following :

(a) $\displaystyle\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$ $\qquad$ (b) $\displaystyle\int e^x \left(\frac{1 + x \log x}{x}\right) dx$

(c) $\displaystyle\int \frac{x e^x}{(x+1)^2}\, dx$ $\qquad$ (d) $\displaystyle\int e^x \left[\frac{1 + \sin x}{1 + \cos x}\right] dx$

Solution:

(a) $\displaystyle\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \int e^x \left[\frac{1}{x} + \frac{d}{dx}\left(\frac{1}{x}\right)\right] dx = e^x \left(\frac{1}{x}\right)$

(b) $\displaystyle\int e^x \left(\frac{1 + x \log x}{x}\right) dx = \int e^x \left(\frac{1}{x} + \log x\right) dx$

$$= \int e^x \left(\log x + \frac{d}{dx}(\log x) \right) dx$$

$$= e^x \log x + C$$

(c) $$\int \frac{x\, e^x}{(x+1)^2}\, dx = \int \frac{x+1-1}{(x+1)^2}\, e^x\, dx$$

$$= \int e^x \left(\frac{1}{x+1} - \frac{1}{(x+1)^2} \right) dx$$

$$= \int e^x \left(\frac{1}{x+1} + \frac{d}{dx}\left(\frac{1}{(x+1)} \right) \right) dx$$

$$= e^x \left(\frac{1}{x+1} \right) + C$$

(d) $$\int e^x \left[\frac{1+\sin x}{1+\cos x} \right] dx = \int e^x \left[\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} \right] dx$$

$$= \int e^x \left[\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2} \right] dx$$

$$= \int e^x \left[\tan\frac{x}{2} + \frac{d}{dx}\left(\tan\frac{x}{2} \right) \right] dx$$

$$= e^x \tan\frac{x}{2} + C.$$

EXERCISE 26.7

Evaluate :

1. (a) $\int e^x \sec x\,[1 + \tan x]\,dx$

   (b) $\int e^x \left[\sec x + \log|\sec x + \tan x| \right] dx$

2. (a) $\int \left(\frac{x-1}{x^2} \right) e^x\, dx$

(b) $\int e^x \left(\sin^{-1} x - \frac{1}{\sqrt{1-x^2}} \right) dx$

3. $\int e^x \frac{(x-1)}{(x+1)^3} dx$

4. $\int \frac{xe^x}{(x+1)^2} dx$

5. $\int \frac{x + \sin x}{1 + \cos x} dx$

6. $\int e^x \sin 2x \, dx$

26.8 INTEGRATION BY USING PARTIAL FRACTIONS

By now we are equipped with the various techniques of integration.

But there still may be a case like $\frac{4x+5}{x^2+x+6}$ where the substitution or the integration by parts may not be of much help. In this case, we take the help of another technique called **technique of integration using partial function**.

Any proper rational fraction $\frac{p(x)}{q(x)}$ can be expressed as the sum of rational functions, each having a single factor of $q(x)$. Each such fraction is known as **partial fraction** and the process of obtaining them is called decomposition or resolving of the given fraction into partial fractions.

For example, $\frac{3}{x+2} + \frac{5}{x-1} = \frac{8x+7}{(x+2)\,(x-1)} = \frac{8x+7}{x^2+x-2}$

Here $\frac{3}{x+2}, \frac{5}{x-1}$ are called partial fraction of $\frac{8x+7}{x^2+x-2}$

If $\frac{f(x)}{g(x)}$ is a proper fraction and $g(x)$ can be resolved into real factors then,

(a) correspnding to each non repeated linear factor $ax + b$, there is a partial fraction of the form $\frac{A}{ax+b}$

(b) for $(ax + b)^2$ we take the sum of two partial fractions as

$$\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2}$$

For $(ax+b)^3$ we take the sum of three partial fraction as

$$\frac{A}{(ax+b)}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$$

and so on.

(c) For a non - fractorisable quadratic polynomial ax^2+bx+c there is a partial fraction

$$\frac{Ax+B}{ax^2+bx+c}$$

Therefore, if $g(x)$ is a proper fraction $\frac{f(x)}{g(x)}$ and can be resolved into real factors, then $\frac{f(x)}{g(x)}$ can be written in the following form :

Factor in the denominator	**Corresponding parital fraction**
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{(ax+b)}+\frac{B}{(ax+b)^2}$
$(ax+b)^3$	$\frac{A}{(ax+b)}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^2$	$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{\left(ax^2+bx+c\right)^2}$

where A, B, C, D are arbitraty constants.

The rational functions which we shall consider for integration will be those whose denominators can be fracted into linear and quadratic factors.

Example 26.54: Evaluate :

$$\int\frac{2x+5}{x^2-x-2}dx$$

Solution: $$\frac{2x+5}{x^2-x-2} = \frac{2x+5}{(x-2)(x+1)}$$

Let $$\frac{2x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

Multiplying both sides by $(x-2)\ (x+1)$, we have

$$2x+5 = A(x+1) + B(x-2)$$

Putting $x = 2$, $\Rightarrow$ $9 = 3A$ or $A = 3$

Putting $x = -1$, $\Rightarrow$ $3 = -3B$ or $B = -1$

Substituting these values in (1), we have

$$\frac{2x+5}{(x-2)(x+1)} = \frac{3}{x-2} - \frac{1}{x+1}$$

$$\Rightarrow \int \frac{2x+5}{x^2-x-2}\,dx = \int \frac{3}{x-2}\,dx - \int \frac{1}{x+1}\,dx$$

$$= 3\log|x-2| - \log|x+1| + C$$

Example 26.55: Evaluate :

(a) $\displaystyle\int \frac{x^2-x-1}{x^3-x^2-6x}\,dx$ (b) $\displaystyle\int \frac{1}{(x^2-1)(x+1)}\,dx$

Solution: (a) $$\frac{x^2-x-1}{x^3-x^2-6x} = \frac{x^2-x-1}{x(x-3)\ (x+2)}$$

Let $$\frac{x^2-x-1}{x(x-3)\ (x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}$$

Multiplying both sides by $x(x-3)\ (x+2)$, we have

$$x^2-x-1 = A(x-3)(x+2) + Bx(x+2) + Cx(x-3)$$

Putting $x = 3$, we get $15B = 5$ or $B = \frac{1}{3}$

Putting $x = 0$, we get $-6A = -1$ or $A = \frac{1}{6}$

Putting $x = -2$, we get $10C = 5$ or $C = \frac{1}{2}$

Substituting these values in (1), we have

$$\frac{x^2 - x - 1}{x(x-3)(x+2)} = \frac{1}{6x} + \frac{1}{3(x+3)} + \frac{1}{2(x+2)}$$

$$\Rightarrow \int \frac{x^2 - x - 1}{x^3 - x^2 - 6x} dx = \int \frac{1}{6x} dx + \int \frac{1}{3(x-3)} dx + \int \frac{1}{2(x+2)} dx$$

$$= \frac{1}{6}\log|x| + \frac{1}{3}\log|x-3| + \frac{1}{2}\log|x+2| + C$$

(b) $$\frac{1}{(x^2-1)(x+1)} = \frac{1}{(x+1)(x-1)(x+1)} = \frac{1}{(x-1)(x+1)^2}$$

Let $$\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Multiplying both sides by $(x^2-1)(x+1)$, we have

$$1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

Putting $x = 1$, we get $A = \frac{1}{4}$

Putting $x = -1$, we get $C = -\frac{1}{2}$

$$0 = A + B$$

$$\Rightarrow \quad B = -\frac{1}{4}$$

$$\therefore \int \frac{1}{(x^2-1)(x+1)} dx = \int \frac{1}{4(x-1)} dx - \frac{1}{4}\int \frac{1}{x+1} dx - \frac{1}{2}\int \frac{1}{(x+1)^2} dx$$

$$= \frac{1}{4}\log|x-1| - \frac{1}{4}\log|x+1| - \frac{1}{2}\left(-\frac{1}{x+1}\right) + C$$

$$= \frac{1}{4}\log|x-1| - \frac{1}{4}\log|x+1| + \frac{1}{2(x+1)} + C$$

Example 26.56: Evaluate

$$\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\, d\theta$$

Solution:

Let $I = \int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\, d\theta = \int \frac{\tan\theta(1+\tan^2\theta)}{1+\tan^3\theta}\, d\theta$

$$= \int \frac{\tan\theta\sec^2\theta}{1+\tan^3\theta}\, d\theta$$

Let $\tan\theta = t$, then $\sec^2\theta\, d\theta = dt$

$\therefore$ $I = \int \frac{t\, dt}{1+t^3} = \int \frac{t\, dt}{(1+t)\,(1-t+t^2)}$

Let $\frac{t}{(1+t)\,(1-t+t^2)} = \frac{A}{1+t} + \frac{Bt + C}{1-t+t^2}$

Then $t = A(1-t+t^2) + (Bt+C)\ \ (1+t)$

Comparing the coefficients of t , we get

$$A + B = 0,\ -A + B + C = 1,\ A + C = 0$$

$$\Rightarrow\ A = -\frac{1}{3},\ B = \frac{1}{3},\ C = \frac{1}{3}$$

$$I = -\frac{1}{3}\int \frac{1}{1+t}\, dt + \frac{1}{3}\int \frac{t+1}{1-t+t^2}\, dt$$

$$= -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{2t+2}{t^2-t+1}\, dt$$

$$= -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{(2t-1)+3}{t^2-t+1}\, dt$$

$$= -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{(2t-1)}{t^2-t+1}\, dt + \frac{1}{2}\int \frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

$$= -\frac{1}{3}\log|1+\tan\theta| + \frac{1}{6}\log|\tan^2\theta-\tan\theta+1|$$

$$+ \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta-1}{\sqrt{3}}\right) + C$$

EXERCISE 26.8

Evaluate the following :

1. (a) $\int\frac{x+1}{(x-2)(x-3)}dx$ (b) $\int\frac{x}{x^2-16}dx$

2. (a) $\int\frac{x^3}{x^2-4}dx$ (b) $\int\frac{2x^2+x+1}{(x-1)^2(x+2)}dx$

3. $\int\frac{x^2+x+1}{(x-1)^3}dx$

4. (a) $\int\frac{\sin x}{\sin 4x}dx$ (b) $\int\frac{1-\cos x}{\cos x(1+\cos x)}dx$

26.9 REDUCTION FORMULAE

There are many functions whose integrals cannot be reduced to one or the other of the well known standard forms of integration. However, in some cases these integrals can be connected algebraically with integrals of other expressions which are directly integrable or which may be easier to integrate than the original functions. Such connecting algebraic relations are called reduction formulae. The formulae connect an integral with another which is of the same type, but is of lower degreee or order or at any rate relatively easier to integrate. In this section, we illustrate the method of integration by successive reduction.

26.9.1 Reduction formula for $\int x^n e^{ax} dx$ n being a positive integer

Let $I_n = \int x^n e^{ax} dx$

On using formula for integration by parts, we get

$$I_n = \frac{x^n e^{ax}}{a} - \int n x^{n-1} \frac{e^{ax}}{a} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

This is called a reduction formula for $\int x^n e^{ax} dx$. Now I_{n-1} in turn can be connected to I_{n-2}. By successive reduction of n, the original integral I_n finally depends on I_0 where

$$I_0 = \int e^{ax} dx = \frac{e^{ax}}{a}$$

Example 26.57:

$$\int x^3 e^{5x} dx$$

Sol: We take $a = 5$ and use the reduction formula for $n = 3, 2, 1$ in that order.

Then we have

$$I_3 = \int x^3 e^{5x} dx = \frac{x^3 e^{5x}}{5} - \frac{3}{5} I_2$$

$$I_2 = \frac{x^2 e^{5x}}{5} - \frac{2}{5} I_1$$

$$I_1 = \frac{x e^{5x}}{5} - \frac{1}{5} I_0$$

and $I_0 = \dfrac{e^{5x}}{5} + c$

Hence $I_3 = \dfrac{x^3 e^{5x}}{5} - \dfrac{3}{5^2} x^2 e^{5x} + \dfrac{6}{5^3} x e^{5x} - \dfrac{6}{5^4} e^{5x} + c.$

1. Theorem : Reduction formula for $\int \sin^n x \, dx$ for an integer $n \geq 2$

Proof: Let $I_n = \int \sin^n x \, dx$

$$I_n = \int \sin^{n-1} x \sin x \, dx$$

$$= \int \sin^{n-1} x \frac{d}{dx} (-\cos x) \, dx$$

$$= \sin^{n-1} x \; (-\cos x) - \int (n-1) \sin^{n-2} x \cos x \, (-\cos x) \, dx$$

$$= -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \, (1 - \sin^2 x) \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

Hence $I_n = \dfrac{-\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} I_{n-2}$

This is called reduction formula for $\int \sin^n x \, dx$

If n is even, after succesive reduction, we get

$$I_0 = \int (\sin x)^0 \, dx = x + c_1$$

If n is odd, after successive reduction, we get

$$I_1 = \int (\sin x)^1 \, dx = -\cos x + c_2$$

Example 26.58:

Evaluate $\int \sin^4 x \, dx$.

Sol: On using the reduction formula for $\int \sin^4 x\,dx$ with $n = 4$ and 2 in that order we have

$$I_4 = \int \sin^4 x\,dx$$

$$= -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} I_2$$

$$= -\frac{\sin^3 x \cos x}{4} + \frac{3}{4}\left[-\frac{\sin x \cos x}{2} + \frac{1}{2} I_0\right]$$

$$= -\frac{\sin^3 x \cos x}{4} - \frac{3}{8}\sin x \cos x + \frac{3}{8}x + c.$$

Notes :

(1) Reduction formula for $\int \text{Tan}^n x\,dx$ for an integer $n \geq 2$.

Let $I_n = \int \text{Tan}^n x = \frac{\text{Tan}^{n-1} x}{n-1} - I_{n-2}$

where n is even, I_n will finally depend on

$$I_0 = \int dx = x + c_1$$

when n is odd, I_n will finally depend on

$$I_1 = \int \text{Tan} x\,dx = \log|\sec x| + c_2$$

(2) Reduction formula for $\int \sec^n x\,dx$ for an integer $n \geq 2$

Let $I_n = \int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \,\text{Tan}\, x + \frac{n-2}{n-1} I_{n-2}.$

when n is even the last integral to which I_n can be reduced is $I_0 = \int dx = x + c_1$

When n is odd, the ultimate integral is I_1.

which is $I_1 = \int \sec x\,dx = \log|\sec x + \tan x| + c_2$

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Example 26.59:

$$\int \sec^5 x \, dx$$

Sol : On using $n = 5$ in the above reduction formula

$$I_5 = \int \sec^5 x \, dx = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} I_3$$

$$= \frac{\sec^3 x \tan x}{4} + \frac{3}{4} \frac{\sec x \operatorname{Tan} x}{2} + \frac{3}{8} I_1$$

$$= \frac{\sec^3 x \tan x}{4} + \frac{3}{8} \sec x \operatorname{Tan} x + \frac{3}{8} \log |\operatorname{Sec} x + \operatorname{Tan} x| + c.$$

EXERCISE 26.9

I. Evaluate the following integrals.

1. $\int x^2 e^{-3x} \, dx$

2. $\int x^3 e^{ax} \, dx$

II. Evaluate the following integrals.

1. $\int \operatorname{Tan}^4 x \, dx$

2. $\int \cos^4 x \, dx$

III 1.If $I_n = \int \cos^n x \, dx$. Then show that

$$I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$$

KEY WORDS

- Integration is inverse of differention
- Standard form of some indefinite integrals

(a) $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

(b) $\int \dfrac{1}{x} \, dx = \log |x| + C$

(c) $\int \sin x \, ax = -\cos x + C$

(d) $\int \cos x \, dx = \sin x + C$

(e) $\int \sec^2 x \, dx = \tan x + C$

(f) $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

(g) $\int \sec x \tan x \, dx = \sec x + C$

(h) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

(i) $\int \dfrac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$

(j) $\int \dfrac{1}{1+x^2} \, dx = \tan^{-1} x + C$

(k) $\int \dfrac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C$

(l) $\int e^x dx = e^x + C$

(m) $\int a^x dx = \dfrac{a^x}{\log a} + C \quad (a > 0 \text{ and } a \neq 1)$

- Properties of indefinite integrals

(a) $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

(b) $\int kf(x) \, dx = k \int f(x) \, dx$

(i) $\int (ax+b)^n \, dx = \dfrac{1}{a} \dfrac{(ax+b)^{n+1}}{n+1} + C(n \neq -1)$

(ii) $\int \dfrac{1}{ax+b} \, dx = \dfrac{1}{a} \log |ax+b| + C$

(iii) $\int \sin(ax+b)\,dx \;=\; \dfrac{-1}{a}\cos(ax+b) + C$

(iv) $\int \cos(ax+b)\,dx \;=\; \dfrac{1}{a}\sin(ax+b) + C$

(v) $\int \sec^2(ax+b)\,dx \;=\; \dfrac{1}{a}\tan(ax+b) + C$

(vi) $\int \text{cosec}^2(ax+b)\,dx \;=\; -\dfrac{1}{a}\cot(ax+b) + C$

(vii) $\int \sec(ax+b)\tan(ax+b)\,dx \;=\; \dfrac{1}{a}\sec(ax+b) + C$

(viii) $\int \text{cosec}(ax+b)\cot(ax+b)\,dx \;=\; -\dfrac{1}{a}\text{cosec}(ax+b) + C$

(xi) $\int e^{ax+b}\,dx = \dfrac{1}{a}e^{ax+b} + C$

$\int \dfrac{f'(x)}{f(x)}\,dx = \log|f(x)| + C$

- (i) $\int \tan x\,dx = -\log|\cos x| + C = \log|\sec| + C$

(ii) $\int \cot x\,dx = \log|\sin x| + C$

(iii) $\int \sec x\,dx = \log|\sec x + \tan x| + C$

(iv) $\int \text{cosec}\,x\,dx = \log|\text{cosec}\,x - \cot x| + C$

- (i) $\int \dfrac{1}{a^2-x^2}\,dx = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right| + C$

(ii) $\int \dfrac{1}{x^2-a^2}\,dx = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + C$

(iii) $\int \dfrac{1}{x^2-a^2}\,dx = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$

(iv) $\int \dfrac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\dfrac{x}{a} + C$

(v) $\int \dfrac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + C$

(vi) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$

- (i) $\int \frac{x^2+1}{\sqrt{x^4+1}}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + C$

(ii) $\int \frac{x^2-1}{\sqrt{x^4+1}}\,dx = \frac{1}{2\sqrt{2}}\tan\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right) + C$

(iii) $\int \frac{x^2}{\sqrt{x^4+1}}\,dx = \frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{2}}\right) + \log\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)\right] + C$

(iv) $\int \frac{1}{\sqrt{x^4+1}}\,dx = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$

- (i) $\int \frac{x^2+1}{\sqrt{x^4+x^2+1}}\,dx = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{3}}\right) + C$

(ii) $\int \frac{x^2-1}{\sqrt{x^4+x^2+1}}\,dx = \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$

(iii) $\int \frac{1}{\sqrt{x^4+x^2+1}}\,dx$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot x-1}{\sqrt{2}}\right)\right] + C$$

(iv) $\int\left(\sqrt{\tan x} + \cot x\right)dx = \sqrt{2}\sin^{-1}(\sin x - \cos x) + C$

- Integral of the product of two functions

  I function × Integral of II function $-\int$[Derivative of I function × Integral of II function] dx

  - $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$
  - $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + C$

  $$\int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| + C$$

  $$\int \sqrt{a^2 + x^2}\, dx = \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2} \log|x + \sqrt{a^2 - x^2}| + C$$

- Rational fractions are of following two types :

(1) Proper, where degree of variable of numerator < denominator

(2) Improper, where degree of variable of numerator $\geq$ denominator.

- If $g(x)$ is a proper fraction $\frac{f(x)}{g(x)}$ can be resolved into real factors, then $\frac{f(x)}{g(x)}$ can be written in the following form :

Factors in denominator	**Corresponding partial fraction**
$ax + b$	$\frac{A}{ax + b}$
$(ax + b)^2$	$\frac{A}{ax + b} + \frac{B}{(ax + b)^2}$
$(ax + b)^3$	$\frac{A}{ax + b} + \frac{B}{(ax + b)^2} + \frac{C}{(ax + b)^3}$
$ax^2 + bx + c$	$\frac{Ax + B}{ax^2 + bx + c}$
$(ax^2 + bx + c)^2$	$\frac{Ax + B}{ax^2 + bx + c} + \frac{Cx + D}{(ax^2 + bx + c)^2}$

where A, B, C, D are arbitrary constants.

- If $I_n = \int x^n e^{ax}\, dx$, Then

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1} \quad \text{For a positive inter } n.$$

- If $I_n = \int \sin^n x\, dx$ then

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}, \text{ for an inter } n \geq 2.$$

- If $I_n = \int \cos^n x\, dx$, Then

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}, \text{ for an integer } n \geq 2$$

- If $I_n = \int \text{Tan}^n x\, dx$, Then

$$I_n = \frac{\text{Tan}^{n-1} x}{n-1} - I_{n-2}, \text{ for an integer } n \geq 2.$$

SUPPORTIVE WEB SITES

- http://www.wikipedia.org
- http://mathworld.wolfram.com

PRACTICE EXERCISE

Integrate the following functions w.r.t. x :

1. $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$
2. $\sqrt{1+\sin 2x}$
3. $\dfrac{\cos 2x}{\cos^2 x \sin^2 x}$
4. $(\tan x - \cot x)^2$
5. $\dfrac{4}{1+x^2} - \dfrac{1}{\sqrt{1-x^2}}$
6. $\dfrac{2\sin^2 x}{1+\cos 2x}$
7. $\dfrac{2\cos^2 x}{1-\cos 2x}$
8. $\left(\sin \dfrac{x}{2} + \cos \dfrac{x}{2}\right)^2$

9. $\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2$

10. $\cos(7x-\pi)$

11. $\sin(3x+4)$

12. $\sec^2(2x+b)$

13. $\int\frac{dx}{\sin x - \cos x}$

14. $\int\frac{1}{(1+x^2)\tan^{-1}x}\,dx$

15. $\int\frac{\text{cosecx}}{\log\left(\tan\frac{x}{2}\right)}\,dx$

16. $\int\frac{\cot x}{3+4\log\sin x}\,dx$

17. $\int\frac{dx}{\sin 2x\log\tan x}\,dx$

18. $\int\frac{e^x+1}{e^x-1}\,dx$

19. $\int\sec^4 x\tan x\,dx$

20. $\int e^x\sin e^x\,dx$

21. $\int\frac{x\,dx}{\sqrt{2x^2+3}}$

22. $\int\frac{\sec^2 x}{\sqrt{\tan x}}\,dx$

23. $\int\sqrt{25-9x^2}\,dx$

24. $\int\sqrt{2ax-x^2}\,dx$

25. $\int\sqrt{3x^2+4}\,dx$

26. $\int\sqrt{1+9x^2}\,dx$

27. $\int\frac{x^2\,dx}{\sqrt{x^2-a^2}}$

28. $\int\frac{dx}{\sqrt{\sin^2 x+4\cos^2 x}}$

29. $\int\frac{dx}{2+\cos x}$

30. $\int\frac{dx}{x^2-6x+13}$

31. $\int\frac{dx}{1+3\sin^2 x}$

32. $\int\frac{x^2}{x^2-a^2}\,dx$

33. $\int\frac{dx}{x\sqrt{9+x^4}}$

34. $\int\frac{\sin x}{\sin 3x}\,dx$

35. $\frac{dx}{1-4\cos^2 x}$

36. $\int\sec^2(ax+b)\,dx$

37. $\int \frac{dx}{x(2+\log x)}$

38. $\int \frac{x^5}{1+x^6}\, dx$

39. $\int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx$

40. $\int \frac{\cot x}{\log \sin x}\, dx$

41. $\int \frac{\sec^2 x}{a+b\,\tan x}\, dx$

42. $\int \frac{\sin x}{1+\cos x}\, dx$

43. $\int \cos^2 x\, dx$

44. $\int \sin^2 x\, dx$

45. $\int \sin 5x \sin 3x\, dx$

46. $\int \sin^2 x \cos^3 x\, dx$

47. $\int \sin^4 x\, dx$

48. $\int \frac{1}{1+\sin x}\, dx$

49. $\int \tan^3 x\, dx$

50. $\int \frac{\cos x - \sin x}{1+\sin 2x}\, dx$

51. $\int \frac{\operatorname{cosec}^2 x}{1+\cot x}\, dx$

52. $\int \frac{1+x+\cos 2x}{x^2+\sin 2x+2x}\, dx$

53. $\int \frac{\sec\theta \operatorname{cosec}\theta\, d\theta}{\log \tan\theta}$

54. $\int \frac{\cot\theta\, d\theta}{\log \sin\theta}$

55. $\int \frac{dx}{1+4x^2}$

56. $\int \frac{1-\tan\theta}{1+\tan\theta}\, d\theta$

57. $\int \frac{1}{x^2} e^{-\frac{1}{x}}\, dx$

58. $\int \frac{\sin x \cos x\, dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

59. $\int \frac{dx}{\sin x + \cos x}$

60. $\int e^x \left(\cos^{-1} x - \frac{1}{\sqrt{1-x^2}} \right) dx$

61. $\int e^x \left(\frac{\sin x + \cos x}{\cos^2 x} \right) dx$

62. $\int \tan^{-1} \sqrt{\frac{1-\cos x}{1+\cos x}}\, dx$

63. $\int \cos\left[2\cot^{-1}\left(\sqrt{\frac{1-x}{1+x}} \right) \right] dx$

64. $\int \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}}\, dx$

65. $\int \sqrt{x}\, \log x\, dx$

66. $\int e^x (1+x) \log(x e^x)\, dx$

67. $\int \frac{\log x}{(1+x)^2} dx$

68. $\int e^x \sin^2 x \, dx$

69. $\int \cos(\log x) \, dx$

70. $\int \log(x+1) \, dx$

71. $\int \frac{x^2+1}{(x-1)^2 (x+3)} dx$

72. $\int \frac{\sin\theta \cos\theta}{\cos^2\theta - \cos\theta - 2} d\theta$

73. $\int \frac{dx}{x(x^5+1)}$

74. $\int \frac{x^2+1}{(x^2+2)(2x^2+1)} dx$

75. $\int \frac{\log x}{x(1+\log x)(2+\log x)} dx$

76. $\int \frac{dx}{1-e^x}$

ANSWERS

EXERCISE 26.1

1. (a) $\frac{x^6}{6} + C$ (b) $\sin x + C$ (c) 0

2. (a) $\frac{x^4}{4} + C$ (b) $\frac{x^{-6}}{-6} + C$ (c) $\frac{3}{5} x^{\frac{5}{3}} + C$

   (d) $2\sqrt{x} + C$ (e) $\frac{3}{7} x^{\frac{7}{3}} + C$ (f) $9x^{\frac{1}{9}} + C$

3. (a) $-\frac{1}{\sin\theta} + C$ (b) $\tan\theta + C$

EXERCISE 26.2

1. (i) $\frac{x^2}{2} + \frac{1}{2}x + C$ (ii) $x - \tan^{-1} x + C$

   (iii) $\frac{2}{3}x^{3/2} + 4\sqrt{x} + C$ (iv) $6\sqrt{x} - 2\log x - \frac{1}{3x} + C$

2. (i) $\frac{1}{2}\tan x + C$ (ii) $\tan x - x + C$

(iii) $-\sec x + C$ (iv) $\sqrt{2}\sin x + C$

EXERCISE 26.3

1. (i) $\frac{\cos(4-5x)}{5}+C$ (ii) $\frac{\tan(2+3x)}{3}+C$

(iii) $\log\left|\sec\left(x+\frac{\pi}{4}\right)+\tan\left(x+\frac{\pi}{4}\right)\right|+C$ (iv) $\frac{1}{4}\sin(4x+5)+C$

(v) $-\frac{1}{5}\operatorname{cosec}(3+5x)+C$

2. (i) $\frac{1}{12(3-4x)^3}+C$ (ii) $\frac{(x+1)^5}{5}+C$

(iii) $-\frac{(4-7x)^{11}}{77}+C$ (iv) $\frac{1}{3}\log|3x-5|+C$

(v) $-\frac{2}{9}\sqrt{5-9x}+C$ (vi) $\frac{(2x+1)^3}{6}+C$

(vii) $\log|x+1|+C$

3. (i) $\frac{e^{2x+1}}{2}+C$ (ii) $-\frac{1}{4e^{7+4x}}+C$

4. $\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+C$

EXERCISE 26.4

1. (i) $\frac{1}{6}\log|3x^2-2|+C$

(ii) $\log|x^2+9x+30|+C$

(iii) $\frac{1}{3}\log|x^2+3x+3|+C$

(iv) $\log|8+\log x|+C$

2. (i) $\frac{1}{b}\log|a+be^x|+C$

(ii) $\text{Tan}^{-1}(e^x)+C$

EXERCISE 26.5

1. (i) $\tan^{-1}(\sin x+2)+C$

(ii) $\sinh^{-1}\left(\frac{x+1}{3}\right)+C$

(iii) $\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+C$

(iv) $\frac{(3x-4)}{6}\sqrt{3+8x-3x^2}+\frac{25}{6\sqrt{3}}\sin^{-1}\left(\frac{3x-4}{5}\right)+C$

(v) $\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$

(vi) $\frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C$

2. (i) $\sqrt{x^2-x+1}+\frac{3}{2}\sinh^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$

(ii) $-(6-2x^2+x)^{3/2}+\frac{637}{32\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)$

$+\frac{13}{16}(4x-1)\sqrt{6-2x^2+x}+C$

(iii) $-\frac{1}{3}(1+x-x^2)^{3/2}+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+\frac{2x-1}{8}\sqrt{1+x-x^2}+c$

(iv) $\frac{-1}{2}\sqrt{\frac{3-x}{1+x}}+C$

(v) $\sqrt{7x-x^2-10}+\frac{3}{2}\sin^{-1}\left(\frac{2x-7}{3}\right)+C$

(vi) $2\tan^{-1}(\sqrt{x+1})+C$

(vii) $\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2x+4}-1}{\sqrt{2x+4}+4}\right|+C$

3. (i) $x+\log|4\sin x+5\cos x|+C$

(ii) $\log\left|1+\tan\frac{x}{2}\right|+C$

(iii) $\frac{1}{3}\log\left|\frac{2\tan\frac{x}{2}+1}{2\left(\tan\frac{x}{2}+2\right)}\right|+C$

(iv) $\frac{1}{5}\log\left|\frac{2\tan\frac{x}{2}+1}{2\tan\frac{x}{2}-4}\right|+C$

(v) $\frac{1}{2}\log\left|\frac{\sqrt{3}\tan\frac{x}{2}+1}{\sqrt{3}\tan\frac{x}{2}-3}\right|+C$

(vi) $\frac{2}{3}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right)+C$

(vii) $\frac{1}{25}\log|3\sin x+4\cos x+5|+\frac{18}{25}x-\frac{4}{5\left(\tan\frac{x}{2}+3\right)}+C$

EXERCISE 26.6

1. (a) $-x \cos x + \sin x + C$

 (b) $\frac{1}{2}(1+x^2) \sin 2x + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4} + C$

 (c) $\frac{-x \cos 2x}{2} + \frac{1}{2} \frac{\sin 2x}{2} + C$

2. (a) $x \tan x - \log | \sec x | -x + C$

 (b) $\frac{1}{6} x^3 - \frac{1}{4} x^2 \sin 2x - \frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + C$

3. (a) $\frac{x^4 \log 2x}{4} - \frac{x^4}{16} + C$ (b) $\left(x - \frac{x^3}{3}\right) \log x - x + \frac{x^3}{9} + C$

 (c) $x (\log x)^2 - 2x \log x + 2x + C$

4. (a) $\frac{x^{1-n}}{1-n} \log x - \frac{x^{1-n}}{(1-n)^2} + C$ (b) $\log x . [\log (\log x) -1] + C$

5. (a) $e^{3x}\left[\frac{x^2}{3} - \frac{2x}{9} + \frac{2}{27}\right] + C$ (b) $x \frac{e^{4x}}{4} - \frac{e^{4x}}{16} + C$

6. $\frac{x^2}{2}\left[(\log x)^2 - \log x + \frac{1}{2}\right] + C$

7. (a) $x \sec^{-1} x - \log \left| x + \sqrt{x^2 - 1} \right| + C$

 (b) $\frac{x^2}{2} \cot^{-1} x + \frac{x}{2} + \frac{1}{2} \cot^{-1} x + C$

EXERCISE 26.7

1. (a) $e^x \sec x + C$ (b) $e^x \log | \sec x + \tan x | + C$

2. (a) $\frac{1}{x} e^x + C$ (b) $e^x \sin^{-1} x + C$

3. (a) $\dfrac{e^x}{(1+x)^2}$ + C   4. $\dfrac{e^x}{1+x}$ + C

5. $x \tan \dfrac{x}{2}$ + C

6. $\dfrac{1}{5} e^x (\sin 2x - 2\cos 2x)$ + C

EXERCISE 26.8

1. (a) $4 \log | x - 3 | -3 \log | x - 2 | + C$

   (b) $\dfrac{1}{2} \log |x-4+\log | x+4| + C$

2. (a) $\dfrac{x^2}{2} - 2[\log |x-2| + \log | x+2|] + C$

   (b) $\dfrac{11}{9} \log |x-1| + \dfrac{7}{9} \log (x+2) - \dfrac{4}{3(x-1)} + C$

3. $\log | x-1| - \dfrac{3}{(x-1)} - \dfrac{3}{2(x-1)^2} + C$

4. (a) $\dfrac{1}{8} \log |1-\sin x| - \dfrac{1}{8} |1 + \sin x|$

   $-\dfrac{1}{4\sqrt{2}} \log |1-\sqrt{2}\sin x| + \dfrac{1}{4\sqrt{2}} \log|1 + \sqrt{2} \sin x| + C$

   (b) $\log | \sec x + \tan x | - 2 \tan \dfrac{x}{2} + C$

EXERCISE 26.9

I (1) $-\dfrac{e^{-3x}}{27} (9x^2 + 6x + 2) + c$

(2) $\dfrac{e^{ax}}{a^4} (a^3 x^3 - 3a^2 x^2 + 6ax - 6) + c$

II(1) $-\frac{\sin^3 x \cos x}{4} - \frac{3}{8}\sin x \cos x + \frac{3}{8}x + c$

(2) $\frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + c$

PRACTICE EXERCISE

1. $\sec x - \operatorname{cosec} x + C$
2. $\sin x - \cos x + C$
3. $-\cot x - \tan x + C$
4. $\tan x - \cot x - 4x + C$
5. $4\tan^{-1} x - \sin^{-1} x + C$
6. $\tan x - x + C$
7. $-\cot x - x + C$
8. $x - \cos x + C$
9. $x + \cos x + C$
10. $\frac{\sin(7x-\pi)}{7} + C$
11. $\frac{-\cos(3x+4)}{3} + C$
12. $\frac{\tan(2x+b)}{2} + C$
13. $\frac{1}{\sqrt{2}}\log\left|\operatorname{cosec}\left(x-\frac{\pi}{4}\right) - \cot\left(x-\frac{\pi}{4}\right)\right| + C$
14. $\log|\tan^{-1} x| + C$
15. $\log\left|\log\tan\frac{x}{2}\right| + C$
16. $\frac{1}{4}\log|3+4\log\sin x| + C$
17. $\frac{1}{2}\log|\log\tan x| + C$
18. $2\log\left|e^{\frac{x}{2}} - e^{\frac{-x}{2}}\right| + C$
19. $\frac{1}{4}\sec^4 x + C$
20. $-\cos e^x \mid + C$

21. $\dfrac{\sqrt{2x^2+3}}{2}+\text{C}$

21. $2\sqrt{\tan x}+\text{C}$

23. $\dfrac{1}{6}x\sqrt{(25-9x^2)}\,\dfrac{25}{6}\sin^{-1}\left(\dfrac{3}{5}x\right)+\text{C}$

24. $\dfrac{1}{2}(x-a)\sqrt{2ax-x^2}+\dfrac{1}{2}a^2\sin^{-1}\left(\dfrac{x-a}{a}\right)+\text{C}$

25. $\dfrac{x\sqrt{3x^2+4}}{2}+\dfrac{2}{\sqrt{3}}\log\left|\dfrac{\sqrt{3}x+\sqrt{x^2+4}}{2}\right|+\text{C}$

26. $\dfrac{x\sqrt{9x^2+1}}{2}+\dfrac{1}{6}\log\left|3x+\sqrt{1+9x^2}\right|+\text{C}$

27. $\left[\dfrac{1}{2}x\sqrt{x^2-a^2}+\dfrac{1}{2}a^2\log\left|x+\sqrt{x^2-a^2}\right|\right]+\text{C}$

28. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{\tan x}{2}\right)+\text{C}$

29. $\dfrac{2}{\sqrt{3}}\tan^{-1}\left[\dfrac{\tan\left(\dfrac{x}{2}\right)}{\sqrt{3}}\right]+\text{C}$

30. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x-3}{2}\right)+\text{C}$

31. $\dfrac{1}{2}\tan^{-1}(2\tan x)+\text{C}$

32. $x+\dfrac{a}{2}\log\left|\dfrac{x-a}{x+a}\right|+\text{C}$

33. $\dfrac{1}{12}\log\left|\dfrac{\sqrt{9+x^4}-3}{\sqrt{9+x^4}+3}\right|+\text{C}$

34. $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + C$

35. $\frac{1}{2\sqrt{2}}\log\left|\frac{\tan x-\sqrt{2}}{\tan x+\sqrt{2}}\right| + C$

36. $\frac{1}{a}\tan(ax+b) + C$

37. $\log|(2+\log x)| + C$

38. $\frac{1}{6}\log(1+x^6) + C$

39. $\log|\sin x + \cos x| + C$

40. $\log|\log(\sin x)| + C$

41. $\frac{1}{b}\log|a+b\tan x| + C$

42. $-\log|1 + \cos x| + C$

43. $\frac{1}{2}\frac{\sin 2x}{2}+\frac{1}{2}x + C$

44. $-\cos x+\frac{\cos^3 x}{3} + C$

45. $\frac{1}{2}\frac{\sin 2x}{2}-\frac{1}{2}\frac{\sin 8x}{8} + C$

46. $\frac{1}{3}\sin^3 x-\frac{\sin^5 x}{5} + C$

47. $\frac{1}{32}[12x-8\sin 2x+\sin 4x] + C$

48. $\tan x - \sec x + C$

49. $\frac{\tan^2 x}{2}+\log|\cos x| + C$

50. $\dfrac{-1}{\cos x+\sin x}+C$

51. $\log\left|\dfrac{1}{1+\cot x}\right|+C$

52. $\dfrac{1}{2}\log\left|x^2+\sin 2x+2x\right|+C$

53. $\log|\tan\theta|+C$

54. $\log|\log\sin\theta|+C$

55. $\dfrac{1}{2}\tan^{-1}2x$

56. $\log|\cos\theta+\sin\theta|+C$

57. $e^{-\frac{1}{x}}+C$

58. $\dfrac{1}{2(a^2-b^2)}\log|a^2\sin^2 x+b^2\cos^2 x|+C$

59. $\dfrac{1}{\sqrt{2}}\log\left|\sec\left(x-\dfrac{\pi}{4}\right)+\tan\left(x-\dfrac{\pi}{4}\right)\right|+C$

60. $e^x\cos^{-1}x+C$

61. $e^x\sec x+C$

62. $\dfrac{1}{4}x^2+C$

63. $-\dfrac{1}{2}x^2+C$

64. $\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}+\dfrac{1}{2}\log|1-x^2|+C$

65. $\dfrac{2}{3}x^{\frac{3}{2}}\left(\log x-\dfrac{2}{3}\right)+C$

66. $xe^x\left[\log(xe^x)-1\right]$ + C

67. $-\dfrac{1}{1+x}\log|x|+\log|x|-\log|x+1|$+ C

68. $\dfrac{1}{2}e^x-\dfrac{e^x}{10}(2\sin 2x+\cos 2x)$+ C

69. $\dfrac{x}{2}[\cos(\log x)+\sin(\log x)]$ + C

70. $x\log|x+1|-x+\log|x+1|$ + C

71. $\dfrac{3}{8}\log|x-1|-\dfrac{1}{2(x-1)}+\dfrac{5}{8}\log|x+3|$+ C

72. $-\dfrac{2}{8}\log|\cos\theta-2|-\dfrac{1}{3}\log|\cos\theta+1|$+ C

73. $\dfrac{1}{5}\log\left|\dfrac{x^5}{x^5+1}\right|$ + C

74. $\dfrac{1}{3\sqrt{2}}\left[\tan^{1}x\left(\dfrac{x}{\sqrt{2}}+\tan^{-1}(\sqrt{2}x)\right)\right]$ + C

75. $\log\left|\dfrac{(2+\log x)^2}{1+\log x}\right|$ + C

76. $\log\left|\dfrac{e^x}{1-e^x}\right|$ + C

Chapter 27

DEFINITE INTEGRALS

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- define and interpret geometrically the definite integral as a limit of sum;
- evaluate a given definite integral using above definition;
- state fundamental theorem of integral calculus;
- state and use the following properties for evaluating definite integrals :

(i) $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

(ii) $\int_a^c f(x)\, dx = \int_b^b f(x)\, dx + \int_b^c f(x)\, dx$

(iii) $\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a-x)\, dx$

(iv) $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

(v) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(vi) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a - x) = f(x)$

$= 0$ if $f(2a - x) = -f(x)$

(vii) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ x if f is an even function of x.

$= 0$ if f is an odd function of x.

- apply definite integrals to find the area of a bounded region.

PREREQUISITES

- Knowledge of integration
- Area of a bounded region

INTRODUCTION

We recall from elementary calculus that to find the area of the region under the graph of a positive and continuous function f definition $[a, b]$, we subdivide the interval $[a, b]$ into a finite number of subintervals, say n, the k^{th} subinterval having length Δx_k, and we consider sums of the form $\sum_{k=1}^{n} f(t_k)\,\Delta x_k$, where t_k is some point in the k^{th} subinterval. Such a sum is an approximation to the area by means of the sum of the areas of rectangles. Suppose we make subdivisions finer and finer. It so happens that the sequence of the corresponding sums tends to a limits as $n \to \infty$. Thus, roughly speaking, this is Riemann's definition of the definite integeral $\int_a^b f(x)dx$, (A precise definition is given below).

27.1 DEFINITE INTEGRAL AS A LIMIT OF SUM

In this section we shall discuss the problem of finding the areas of regions whose boundary is not familiar to us. (See Fig. 27.1)

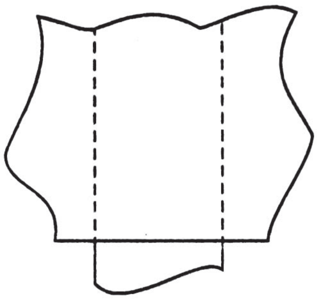


Fig. 27.1

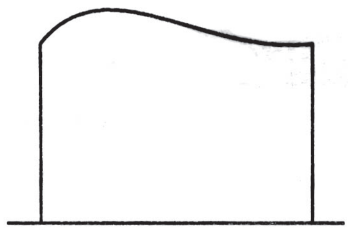


Fig. 27.2

Let us restrict our attention to finding the areas of such regions where the boundary is not familiar to us is on one side of x-axis only as in Fig. 27.2.

This is because we expect that it is possible to divide any region into a few subregions of this kind, find the areas of these subregions and finally add up all these areas to get the area of the whole region. (See Fig. 27.1)

Now, let $f(x)$ be a continuous function defined on the closed interval $[a, b]$. For the present, assume that all the values taken by the function are non-negative, so that the graph of the function is a curve above the x-axis (See. Fig.27.3).

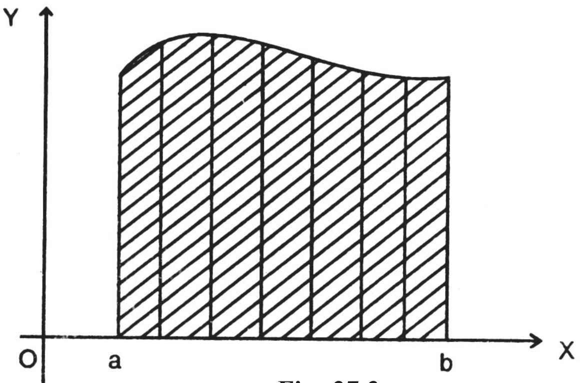


Fig. 27.3

Consider the region between this curve, the x-axis and the ordinates $x = a$ and $x = b$, that is, the shaded region in Fig.27.3. Now the problem is to find the area of the shaded region.

In order to solve this problem, we consider three special cases of $f(x)$ as rectangular region, triangular region and trapezoidal region.

The area of these regions = base × average height

In general for any function $f(x)$ on $[a, b]$

Area of the bounded region (shaded region in Fig. 27.3)

= base × average height

The base is the length of the domain interval $[a, b]$. The height at any point x is the value of $f(x)$ at that point. Therefore, the average height is the average of the values taken by f in $[a, b]$. (This may not be so easy to find because the height may not vary uniformly.) Our problem is how to find the average value of f in $[a, b]$.

27.1.1 Average Value of a Function in an Interval

If there are only finite number of values of f in $[a, b]$, we can easily get the average value by the formula.

$$\text{Average value of } f \text{ in } [a, b] = \frac{\text{Sum of the values of } f \text{ in } [a,b]}{\text{Numbers of values}}$$

But in our problem, there are infinite number of values taken by f in $[a, b]$. How to find the average in such a case? The above formula does not help us, so we resort to estimate the average value of f in the following way:

First Estimate : Take the value of f at ‘a’ only. The value of f at a is $f(a)$. We take this value, namely $f(a)$, as a rough estimate of the average value of f in $[a, b]$.

Average value of f in $[a, b]$ (first estimate) = $f(a)$ (i)

Second Estimate : Divide $[a, b]$into two equal parts or sub-intervals

Let the length of each sub-interval be h, $h = \dfrac{b-a}{2}$.

Take the values of f at the left end points of the sub-intervals. The values are $f(a)$ and $f(a + h)$. (Fig. 27.4)

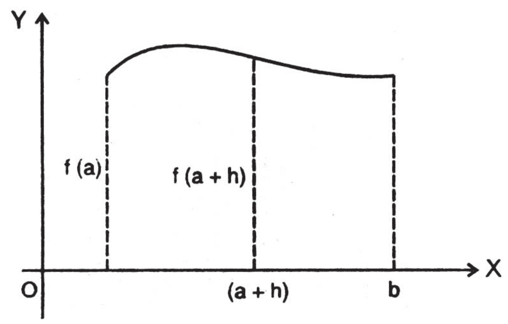


Fig. 27.4

Take the average of these two values as the average of f in $[a, b]$.

Average value of f in $[a, b]$ (Second estimate)

$$= \frac{f(a) + f(a+h)}{2}, \; h = \frac{b-a}{2} \qquad \text{...(ii)}$$

This estimate is expected to be a better estimate than the first.

Proceeding in a similar manner, divide the interval $[a, b]$ into n subintervals of length h (Fig. 27.5) $h = \dfrac{b-a}{n}$.

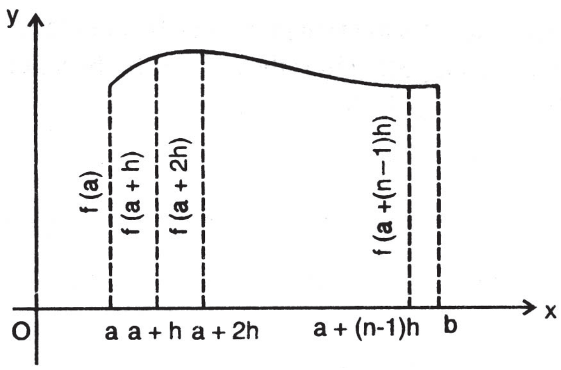


Fig. 27.5

Take the values of f at the left end points of the n subintervals.

The values are $f(a), f(a+h), \ldots\ldots, f[a+(n-1)h]$. Take the average of these n values of f in $[a, b]$.

Average value of f in $[a, b]$ (n^{th} estimate)

$$= \frac{f(a) + f(a+h) + \ldots. + f[a+(n-1)h]}{n}, \; h = \frac{b-a}{n} \qquad \text{...(iii)}$$

For larger values of n, (iii) is expected to be a better estimate of what we seek as the average value of f in $[a, b]$

Thus, we get the following sequence of estimates for the average value of f in $[a, b]$:

$f(a)$

$$\frac{1}{2}[f(a)+f(a+h)],\ h=\frac{b-a}{2}$$

$$\frac{1}{3}[f(a)+f(a+h)+f(a+2h)],\quad h=\frac{b-a}{3}$$

....

....

$$\frac{1}{n}[f(a)+f(a+h)+........+f\{a+(n-1)h\}],\quad h=\frac{b-a}{n}$$

As we go farther and farther along this sequence, we are going closer and closer to our destination, namely, the average value taken by f in $[a, b]$. Therefore, it is reasonable to take the limit of these estimates as the average value taken by f in $[a, b]$. In other words,

Average value of f in $[a, b]$

$$\lim_{n\to\infty}\frac{1}{n}\{f(a)+f(a+h)+f(a+2h)+....+f[a+(n-1)h]\},$$

$$h=\frac{b-a}{n} \qquad \text{(iv)}$$

It can be proved that this limit exists for all continuous functions f on a closed interval $[a, b]$.

Now, we have the formula to find the area of the shaded region in Fig. 27.3, The base is $(b-a)$ and the average height is given by (iv). The area of the region bounded by the curve $f(x)$, x-axis, the ordinates $x = a$ and $x = b$.

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\{f(a)+f(a+h)+f(a+2h)+....+f[a+(n-1)h]\},$$

$$\lim_{n \to 0} \frac{1}{n} [f(a) + f(a+h) + ... + f\{a+(n-1)h]\}, \quad h = \frac{b-a}{n} \qquad \text{(v)}$$

We take the expression on R.H.S. of (v) as the definition of a **definite integral.** This integral is denoted by

$$\int_a^b f(x)\, dx$$

read as integral of $f(x)$ from a to b'. The numbers a and b in the symbol $\int_a^b f(x)\, dx$ are called respectively the lower and upper limits of integration, and $f(x)$ is called the inte grand.

Note : In obtaining the estimates of the average values of f in [a, b], we have taken the left end points of the subintervals. Why left end points?

Why not right end points of the subintervals? We can as well take the right end points of the subintervals throughout and in that case we get

$$\int_a^b f(x)\, dx = (b-a) \lim_{n \to \infty} \frac{1}{n} \{f(a+h) + f(a+2h) + ... + f(b)\}, \quad h = \frac{b-a}{n}$$

$$= \lim_{h \to 0} h[f(a+h) + f(a+2h) + ... + f(b)] \qquad \text{(vi)}$$

Example 27.1: Find $\int_1^2 x\, dx$ as the limit of sum.

Solution: By definition,

$$\int_a^b f(x)\, dx = (b-a) \lim_{n \to \infty} \frac{1}{n} [f(a) + f(a+h) + ... + f\{a+(n-1)h\}],$$

$$h = \frac{b-a}{n}$$

Here $a = 1,\ b = 2,\ f(x) = x$ and $h = \frac{1}{n}$

$$\therefore \quad \int_1^2 x\,dx = \lim_{n\to\infty} \frac{1}{n}\left[f(1) + f\left(1+\frac{1}{n}\right) + ... + f\left(1+\frac{n-1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[1 + \left(1+\frac{1}{n}\right) + \left(1+\frac{2}{n}\right) .... + \left(1+\frac{n-1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\underbrace{1+1+...+1}_{n\ times} + \left(\frac{1}{n}+\frac{2}{n}+ .... + \frac{n-1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[n + \frac{1}{n}(1+2) + ...+ (n-1))\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[n + \frac{(n-1).n}{n.2}\right]$$

$$\left[\sin ce\ 1+2+3+...+(n-1) = \frac{(n-1).n}{2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{3n-1}{2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{3}{2} - \frac{1}{2n}\right] = \frac{3}{2}$$

Example 27.2: Find $\int_1^2 e^x\,dx$ as the limit of sum.

Solution: By definition,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + ... + f\{a+(n-1)h\}]$$

Where $h = \dfrac{b-a}{n}$

Here $a = 0,\ b = 2,\ f(x) = e^x$ and $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

$$\int_0^2 e^x\,dx = \lim_{h\to 0} h[f(0) + f(h) + f(2h) + ... + f\{(n-1)h]$$

$$= \lim_{h \to 0} h[e^0 + e^h + e^{2h} + ... + e^{(n-1)h}]$$

$$= \lim_{h \to 0} h\left[e^0\left(\frac{(e^h)^n - 1}{e^h - 1}\right)\right]$$

$$\left[\text{Since } a + ar + ar^2 + ... + ar^{n-1} = a\left(\frac{r^n - 1}{r - 1}\right)\right]$$

$$= \lim_{h \to 0} h\left[\frac{e^{nh} - 1}{e^h - 1}\right] \lim_{h \to 0} \frac{h}{h}\left[\frac{e^2 - 1}{\left(\frac{e^h - 1}{h}\right)}\right] \qquad (\because \quad nh = 2)$$

$$= \lim_{h \to 0} \frac{e^2 - 1}{\frac{e^h - 1}{h}} = \frac{e^2 - 1}{1}$$

$$= e^2 - 1 \qquad \left[\because \lim_{h \to 0} \frac{e^h - 1}{h} = 1\right]$$

In examples 27.1 and 27.2 we observe that finding the definite integral as the limit of sum is quite difficult. In order to overcome this difficulty we have the fundamental theorem of integral calculus which states that

Theorem 1 : If f is continuous in$[a, b]$ and F is an antiderivative of f in $[a, b]$ then

$$\int_a^b f(x)\, dx = F(b) - F(a) \qquad ...(1)$$

The difference $F(b) - F(a)$ is commonly denoted by $[F(x)]_a^b$ so that (1) can be written as

$$\int_a^b f(x)\, dx = F(x)]_a^b \text{ or } [F(x)]_a^b$$

In words, the theorem tells us that

$$\int_a^b f(x)\, dx = \text{(Value of antiderivative at the upper limit b)}$$
$$- \text{(Value of the same antiderivative at the lower limit a)}$$

MODULE - V
Calculus

Notes

Example 27.3: Find $\int_1^2 x\, dx$

Solution: $\int_1^2 x\, dx = \left[\frac{x^2}{2}\right]_1^2$

$$= \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

Example 27.4: Evaluate the following.

(a) $\int_0^{\pi/2} \cos x\, dx$ (b) $\int_0^2 e^{2x}\, dx$

Solution: We know that

$$\int \cos x\, dx = \sin x + c$$

$$\therefore \quad \int_0^{\pi/2} \cos x\, dx = [\sin x]_0^{\pi/2}$$

$$= \sin \frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

(b) $\int_0^2 e^{2x}\, dx = \left[\frac{e^{2x}}{2}\right]_0^2$, $\left[\because \int e^x dx = e^x\right]$

$$= \left(\frac{e^4 - 1}{2}\right).$$

Theorem 2 : If f and g are continuous functions defined in $[a, b]$ and c is a constant then,

(i) $\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

(ii) $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

(iii) $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Example 27.5: Evaluate $\int_0^2 (4x^2 - 5x + 7)\,dx$

Solution: $\int_0^2 (4x^2 - 5x + 7)\,dx = \int_0^2 4x^2\,dx - \int_0^2 5x\,dx + \int_0^2 7\,dx$

$$= 4\int_0^2 x^2\,dx - 5\int_0^2 x\,dx + 7\int_0^2 1\,dx$$

$$= 4.\left[\frac{x^3}{3}\right]_0^2 - 5\left[\frac{x^2}{2}\right]_0^2 + 7\,[x]_0^2$$

$$= 4.\left(\frac{8}{3}\right) - 5\left(\frac{4}{2}\right) + 7(2)$$

$$= \frac{32}{3} - 10 + 14$$

$$= \frac{44}{3}.$$

EXERCISE 27.1

1. Find $\int_0^5 (x+1)\,dx$ as the limit of sum.

2. Find $\int_{-1}^1 e^x.\,dx$ as the limit of sum.

3. Evaluate (a) $\int_0^{\pi/4} \sin x\,dx$ (b) $\int_0^{\pi/2} (\sin x + \cos x)\,dx$

(c) $\int_0^1 \frac{1}{1+x^2}\,dx$ (d) $\int_1^2 (4x^3 - 5x^2 + 6x + 9)\,dx$

27.2 EVALUATION OF DEFINITE INTEGRAL BY SUBSTITUTION

The principal step in the evaluation of a definite integral is to find the related indefinite integral. In the preceding lesson we have discussed several methods for finding the indefinite integral. One of the important methods for finding indefinite integrals is the method of substitution. When we use substitution method for evaluation the definite integrals, like

$$\int_2^3 \frac{x}{1+x^2}\,dx,\ \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx,$$

the steps could be as follows :

(i) Make appropriate substitution to reduce the given integral to a known form to integrate. Write the integral in terms of the new variable.

(ii) Integrate the new integrand with respect to the new variable.

(iii) Change the limits accordingly and find the difference of the values at the upper and lower limits.

Note : If we don't change the limit with respect to the new variable then after integrating resubstitute for the new variable and write the answer in original variable. Find the values of the answer thus obtained at the given limits of the integral.

Example 27.6: Evaluate $\int_2^3 \frac{x}{1+x^2}\,dx$

Solution : Let $1+x^2=t$

$$2x\ dx = dt \quad \text{or} \quad x\ dx = \frac{1}{2}\ dt$$

When $x = 2,\ t = 5$ and $x = 3,\ t = 10$. Therefore, 5 and 10 are the limits when t is the variable

Thus $$\int_2^3 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_5^{10} \frac{1}{t}\,dt$$

$$= \frac{1}{2}\left[\log t\right]_5^{10}$$

$$= \frac{1}{2}\left[\log 10 - \log 5\right]$$

$$= \frac{1}{2}\log 2$$

Example 27.7: Evaluate the following

(a) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$ (b) $\displaystyle\int_0^{\pi/2} \frac{\sin 2\theta}{\sin^2\theta + \cos^4\theta}\,d\theta$

(c) $\displaystyle\int_0^{\pi/2} \frac{dx}{5+4\cos x}$

Solution : (a) Let $\cos x = t$ then $\sin x\, dx = -\,dt$

when $x = 0$ and $x = \frac{\pi}{2}$, $t = 0$. As x varies from 0 to $\frac{\pi}{2}$ t varies from 1 to 0.

$$\therefore \quad \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_1^0 \frac{1}{1+t^2}\,dt = -\left[\tan^{-1} t\right]_1^0$$

$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$

$$= -\left[0 - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}$$

(b) $\displaystyle I = \int_0^{\pi/2} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta}\,d\theta$

$$= \int_0^{\pi/2} \frac{\sin 2\theta}{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\sin 2\theta}{1 - 2\sin^2\theta\cos^2\theta}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\sin 2\theta\, d\theta}{1 - 2\sin^2\theta(1-\sin^2\theta)}$$

Let $\sin^2\theta = t$.

Then $2\sin\theta\cos\theta\, d\theta = dt$ i.e, $\sin 2\theta\, d\theta = dt$

when $\theta = 0$, $t = 0$ and $\theta = \frac{\pi}{2}$, $t = 1$. As θ varies from 0 to $\frac{\pi}{2}$, the new variable t varies from 0 to 1.

$$\therefore \qquad I = \int_0^1 \frac{1}{1-2t(1-t)}\,dt$$

$$= \int_0^1 \frac{1}{2t^2-2t+1}\,dt$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{t^2-t+\frac{1}{4}+\frac{1}{4}}\,dt$$

$$I = \frac{1}{2}\int_0^1 \frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\,dt$$

$$= \frac{1}{2}\cdot\frac{\frac{1}{1}}{\frac{1}{2}}\left[\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$= \left[\tan^{-1}1 - \tan^{-1}(-1)\right]$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

(c) We know that $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\therefore \qquad \int_0^{\pi/2} \frac{1}{5+4\cos x}\,dx = \int_0^{\pi/2} \frac{1}{5+\dfrac{4\left(1-\tan^2\left(\frac{x}{2}\right)\right)}{\left(1+\tan^2\left(\frac{x}{2}\right)\right)}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sec^2\left(\frac{x}{2}\right)}{9+\tan^2\left(\frac{x}{2}\right)} dx \qquad (1)$$

Let $\tan \frac{x}{2} = t$

Then $\sec^2 \frac{x}{2} dx = 2dt$ when $x = 0$, $t = 0$, when $x = \frac{\pi}{2}$, $t = 1$

$$\therefore \int_0^{\pi/2} \frac{1}{5+4\cos x} dx = 2\int_0^1 \frac{1}{9+t^2} dt \qquad \text{[From (1)]}$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^1 = \frac{2}{3}\left[\tan^{-1}\frac{1}{3}\right]$$

27.3 SOME PROPERTIES OF DEFINITE INTEGRALS

The definite integral of $f(x)$ between the limits a and b has already been defined as

$$\int_a^b f(x)\,dx + \text{F}(b) - \text{F}(a), \text{ Where } \frac{d}{dx}[\text{F}(x)] = f(x),$$

where a and b are the lower and upper limits of integration respectively. Now we state below some important and useful properties of such definite integrals.

(i) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ (ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$ where $a < c < b$

(iv) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(v) $\int_a^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^c f(2a-x)\,dx$

(vi) $\int_a^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(vii) $\int_0^{2a} f(x)\,dx = \begin{cases} 0, & \text{if } f(2a-x) = -f(x) \\ 2\int_0^{a} f(x)\,dx, & \text{if } f(2a-x) = f(x) \end{cases}$

(viii) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function of } x \\ 2\int_0^{a} f(x)\,dx, & \text{if } f(x) \text{is an even function of } x \end{cases}$

Many of the definite integrals may be evaluated easily with the help of the above stated properties, which could have been very difficult otherwise.

The use of these properties in evaluating definite integrals will be illustrated in the following examples.

Example 27.8: Show that

(a) $\int_0^{\pi/2} \log|\tan x|\,dx = 0$

(b) $\int_0^{\pi} \frac{x}{1+\sin x}\,dx = \pi$

Solution: (a) Let $I = \int_0^{\pi/2} \log|\tan x|\,dx$

Using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx,$

$$I = \int_0^{\pi/2} \log\left(\tan\left(\frac{\pi}{2} - x\right)\right) dx$$

$$= \int_0^{\pi/2} \log(\cot x)\,dx$$

$$= \int_0^{\pi/2} \log(\tan x)^{-1}\,dx$$

$$= -\int_0^{\pi/2} \log \tan x \, dx$$

$$= -1 \qquad \text{[Using (i)]}$$

$\therefore \quad 2I = 0$

i.e. $\quad I = 0$ or $\int_0^{\pi/2} \log|\tan x| \, dx = 0$

(b) $\int_0^{\pi} \frac{x}{1+\sin x} dx$

Let $\quad I = \int_0^{\pi} \frac{x}{1+\sin x} dx \qquad$ .... (i)

$$\therefore \quad I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx \qquad \text{.... (ii)}$$

Adding (i) and (ii)

$$2I = \int_0^{\pi} \frac{x + \pi - x}{1+\sin x} dx = \pi \int_0^{\pi} \frac{1}{1+\sin x} dx$$

or $$2I = \pi \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x} dx$$

$$= \pi \int_0^{\pi} \left(\sec^2 x - \tan x \sec x\right) dx$$

$$= \pi \, [\tan x - \sec x]_0^{\pi}$$

$$= \pi \, [(\tan x - \sec \pi) - (\tan \theta - \sec \theta)]$$

$$= \pi \, [(0 \, (-1) - (0 - 1)]$$

$\therefore \quad I = \pi$

MODULE - V
Calculus

Notes

Example 27.9: Evaluate

(a) $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}$ (b) $\displaystyle\int_0^{\pi/2} \frac{\sin x-\cos x}{1+\sin x\cos x}\,dx$

Solution: (a) $\displaystyle I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

Also $\displaystyle I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx$

(Using the property $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$)

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \quad \text{.... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} 1.dx$$

$$= [x]_0^{\pi/2} = \frac{\pi}{2}$$

$\therefore \quad I = \dfrac{\pi}{4}$

i.e, $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}$

(b) Let $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x}\,dx$ (i)

Then $I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\,dx$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\cos x \sin x}\,dx$ (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{1+\sin x \cos x} + \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin x \cos x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin x - \cos x + \cos x - \sin x}{1+\sin x \cos x}\,dx$$

$= 0$

$I = 0$

Example 27.10: (a) $\int_{-a}^{a} \frac{x e^{x^2}}{1+x^2}\,dx$ (b) $\int_{-3}^{3} |x+1|\,dx$

Solution: (a) $f(x) = \frac{xe^{x^2}}{1+x^2}$

$f(-x) = -\frac{xe^{x^2}}{1+x^2}$

$= -f(x)$.

$\therefore$ $f(x)$ is an odd function of x.

$\int_{-a}^{a} \frac{x e^{x^2}}{1+x^2}\,dx = 0$.

(b) $\int_{-3}^{3} |x+1|\ dx$

$$|x+1| = \begin{cases} x+1, \text{ if } x \geq -1 \\ -x-1, \text{ if } x < -1 \end{cases}$$

$$\therefore \int_{-3}^{3} |x+1|\ dx = \int_{-3}^{-1} |x+1|\ dx + \int_{-1}^{3} |x+1|\ dx, \text{ using property (iii)}$$

$$= \int_{-3}^{-1} (-x-1)\ dx + \int_{-1}^{3} (x+1)\ dx$$

$$= \left[\frac{-x^2}{2} - x\right]_{-3}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{3}$$

$$= -\frac{1}{2} + 1 + \frac{9}{2} - 3 + \frac{9}{2} + 3 - \frac{1}{2} + 1 = 10$$

Example 27.11: Evaluate $\int_{0}^{\pi/2} \log(\sin x)\ dx$

Solution: Let $I = \int_{0}^{\pi/2} \log(\sin x)\ dx$

Also $\quad I = \int_{0}^{\pi/2} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx$, [Using property (iv)]

$$= \int_{0}^{\pi/2} \log(\cos x)\ dx$$

Adding (i) and (ii), we get

$$2I = \int_{0}^{\pi/2} [\log(\sin x) + \log(\cos x)\, dx$$

$$= \int_{0}^{\pi/2} \log(\sin x \cos x)\ dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log(\sin 2x)\, dx - \int_0^{\pi/2} \log(2)\, dx$$

$$= \int_0^{\pi/2} \log(\sin 2x)\, dx - \frac{\pi}{2} \log 2$$

Again, let $I_1 = \int_0^{\pi/2} \log(\sin 2x)\, dx$

Put $2x = t \quad \Rightarrow dx = \frac{1}{2} dt$

When $x = 0,\ t = 0$ and $x = \frac{\pi}{2},\ t = \pi$

$$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\, dt$$

$$= \frac{1}{2}\cdot 2 \int_0^{\pi/2} \log(\sin t)\, dt, \qquad \text{[using property (vi)]}$$

$$= \frac{1}{2}\cdot 2 \int_0^{\pi/2} \log(\sin x)\, dx \qquad \text{[using property (i)]}$$

$\therefore \quad I_1 = I,$ [from (i)] ...(iv)

Putting this value in (iii), we get

$$2I = I - \frac{\pi}{2} \log 2 \qquad \Rightarrow I = -\frac{\pi}{2} \log 2$$

Hence, $\int_0^{\pi/2} \log(\sin x)\, dx = -\frac{\pi}{2} \log 2$

EXERCISE 27.2

Evaluate the following integrals :

1. $\int_0^1 x e^{x^2} dx$ 2. $\int_0^{\pi/2} \frac{dx}{5+4\sin x}$ 3. $\int_0^1 \frac{2x+3}{5x^2+1} dx$

4. $\int_{-5}^5 |x+2|\, dx$ 5. $\int_0^2 x\sqrt{2-x}\, dx$ 6. $\int_0^{\pi/2} \frac{\sin x}{\cos x+\sin x} dx$

7. $\int_0^{\pi/2} \log|\cos x|\, dx$ 8. $\int_{-a}^{a} \frac{x^3 e^{x^4}}{1+x^2} dx$ 9. $\int_0^{\pi/2} \sin 2x \log|\tan x|\, dx$

10. $\int_0^{\pi/2} \frac{\cos x}{1+\sin x+\cos x} dx$

27.4 APPLICATIONS OF INTEGRATION

Suppose that f and g are two continuous functions on an interval [a, b] such that $[a, b]$ such that $f(x) \le g(x)$ for $x \in [a, b]$ that is, the curve $y = f(x)$ does not cross under the curve $y = g(x)$ over $[a, b]$.

Now the question is how to find the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by $x = a$, $x = b$.

Again what happens when the upper curve $y = f(x)$ intersects the lower curve $y = g(x)$ at either the left hand boundary $x = a$, the right hand boundary $x = b$ or both?

27.4.1 Area Bounded by the Curve, x-axis and the Ordinates

Let AB be the curve $y = f(x)$ and CA, DB the two ordinates at $x = a$ and $x = b$ respectively.

Suppose $y = f(x)$ is an increasing function of x in the interval $a \le x \le b$

Let $P(x, y)$ be any point on the curve and $Q(x + \delta x, y + \delta y)$ a neighbouring point on it.

Draw their ordinates PM and QN.

Here we observe that as x changes the area (ACMP) also changes. Let

A = Area (ACMP)

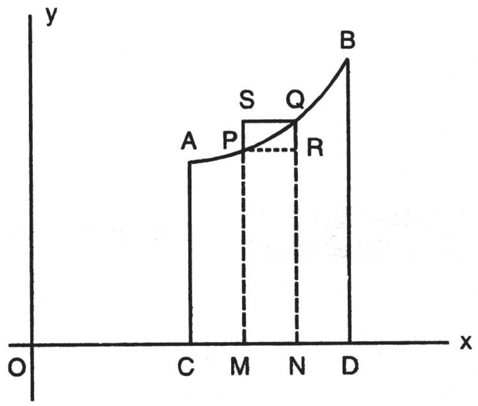


Fig.27.6

Then the area (ACNQ) = A + δA.

The area (PMNQ)=Area (ACNQ)– Area (ACMP)

$$= A + \delta A - A = \delta A.$$

Complete the rectangle PRQS. Then the area (PMNQ) lies between the areas of rectangles PMNR and SMNQ, that is

δA lies between $y\ \delta x$ and $(y + \delta y)\ \delta x$

$\Rightarrow \dfrac{\delta A}{\delta x}$ lies between y and $(y + \delta y)$

In the limiting case when $Q \to P$, $\delta x \to 0$ and $\delta y \to 0$

$\therefore \lim\limits_{\delta y \to 0} \dfrac{\delta A}{\delta x}$ lies between y and $\lim\limits_{\delta y \to 0} (y + \delta y)$

$$\therefore \qquad \frac{dA}{dx} = y$$

Integrating both sides with respect to x, from $x = a$ to $x = b$, we have

$$\int_a^b y\, dx = \int_a^b \frac{dA}{dx},\ dx = [A]_a^b$$

$$= (\text{Area when } x = b) - (\text{Area when } x = a)$$

$= \text{Area (ACDB)} - 0$

$= \text{Area (ACDB)}$

Hence Area (AC BD) $= \int_a^b f(x)\, dx$

The area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a,\ x = b$ is

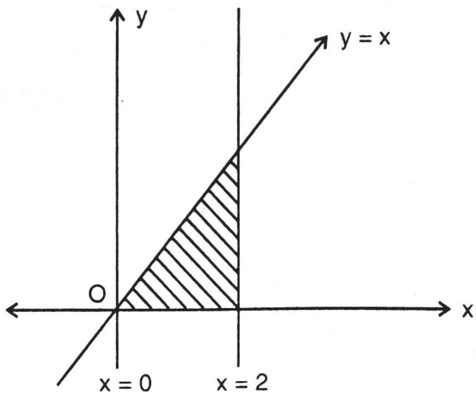
$$\int_a^b f(x)\, dx \quad \text{or} \quad \int_a^b y\, dx\ .$$

where $y = f(x)$ is a continuous single valued function and y does not change sign in the interval $a \le x \le b$.

Example 27.12 : Find the area bounded by the curve $y = x$, x-axis and the lines $x = 0,\ x = 2$.

Solution: The given curve is $y = x$

$\therefore$ Required area bounded by the curve, x-axis and the ordinates $x = 0$, $x = 2$ (as shown in Fig. 27.7)



Fing 27.7

is

$$\int_0^2 x\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$= 2 - 0 = 2$ Square units.

Example 27.13: Find the area bounded by the curve $y = e^x$, x-axis and the ordinates $x = 0$ and $x = a > 0$.

Solution: The given curve is $y = e^x$

Required area bounded by the curve, x-axis and the ordinates $x = 0$, $x = a$ is

$$\int_0^a e^x \, dx$$

$$= \left[e^x\right]_0^a$$

$$= \left(e^a - 1\right) \text{ square units.}$$

Example 27.14: Find the area bounded by the curve $y = c \cos\left(\frac{x}{c}\right)$ x-axis and the ordinates $x = 0, x = a \; 2a \le c.\pi$.

Solution: The given curve is $y = c \cos\left(\frac{x}{c}\right)$

$\therefore$ Required area $= \int_0^a y \, dx$

$$= \int_0^a c\cos\left(\frac{x}{c}\right) dx$$

$$= c^2\left[\sin\left(\frac{x}{c}\right)\right]_0^a$$

$$= c^2\left(\sin\left(\frac{a}{c}\right) - \sin 0\right)$$

$$= c^2 \sin\left(\frac{a}{c}\right) \text{square units.}$$

Example 27.15: Find the area enclosed by the circle, $x^2 + y^2 = a^2$, and x-axis in the first quadrant.

Solution: The given curve is $x^2 + y^2 = a^2$, which is a circle whose centre and radius are (0, 0) and a respectively. Therefore, we have to find the areaenclosed by the circle $x^2 + y^2 = a^2$, the x-axis and the ordinates $x = 0$ and $x = a$.

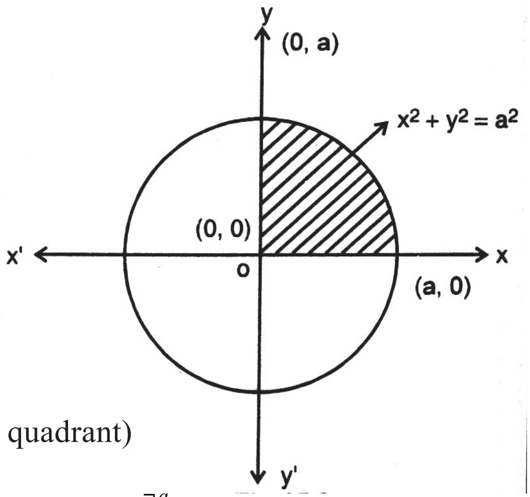


Fig 27.8

$$\therefore \text{ Required area } = \int_0^a y\, dx$$

$$= \int_0^a \sqrt{a^2 - x^2}\, dx,$$

($\because$ y is positive in the first quadrant)

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1} 1 - 0 - \frac{a^2}{2}\sin^{-1} 0$$

$$= \frac{a^2}{2} \cdot \frac{\pi}{2} \quad \left(\because \sin^{-1} 1 = \frac{\pi}{2},\ \sin^{-1} 0 = 0\right)$$

$$= \frac{\pi a^2}{4} \text{ square units.}$$

Example 27.16: Find the area bounded by the x-axis, ordinates and the following curves :

(i) $xy = c^2,\ x = a,\ x = b,\ a > b > 0$

(ii) $y = \log_e x,\ x = a,\ x = b,\ b > a > 1$

Solution: Here we have to find the area bounded by the x-axis, the ordinates $x = a$, $x = b$ and the curve

$$xy = c^2 \qquad \text{or} \qquad y = \frac{c^2}{x}$$

$$\therefore \text{ Area} = \int_b^a y\, dx \qquad (\because a > b \text{ given})$$

$$= \int_b^a \frac{c^2}{x}\, dx$$

$$= c^2 \left[\log x\right]_b^a$$

$$= c^2 (\log a - \log b)$$

$$= c^2 \log\left(\frac{a}{b}\right)$$

(ii) Here $y = \log_e x$

$$\therefore \text{ Area} = \int_a^b \log_e x \, dx \qquad (\because b > a > 1)$$

$$= \left[x \log_e x\right]_a^b - \int_a^b x \cdot \frac{1}{x} \, dx$$

$$= b \log_e b - a \log_e a - \int_a^b dx$$

$$= b \log_e b - a \log_e a - \left[x\right]_a^b$$

$$= b \log_e b - a \log_e a - b + a$$

$$= b (\log_e b - 1) - a (\log_e a - 1)$$

$$= b \log_e \left(\frac{b}{e}\right) - a \log_e \left(\frac{a}{e}\right) \qquad (\because \log_e e = 1)$$

EXERCISE 27.3

1. Find the area bounded by the curve $y = x^2$ x-axis and the lines $x = 0,\ x = 2$.
2. Find the area bounded by the curve $y = 3x$ x, x-axis and the lines $x = 0$ and $x = 3$.
3. Find the area bounded by the curve $y = e^{2x}$ x-axis and the ordinates, $x = 0,\ x = a,\ a > 0$.
4. Find the area bounded by the x-axis, the curve $y = c\sin\left(\frac{x}{c}\right)$ and the ordinates $x = 0,\ x = 2a \le c\pi$.

27.4.2 Area Bounded by the Curve $x = f(y)$ between y-axis and the Lines $y = c$, $y = d$

Let AB be the curve $x = f(y)$ and let CA, DB be the abscissae at $y = c$, $y = d$ respectively.

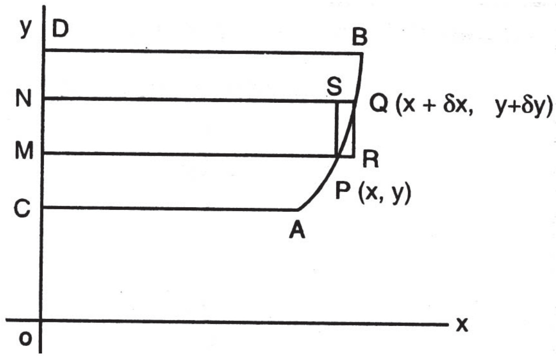


Fig 27.9

Let $p(x, y)$ be any point on the curve and let $Q(x + \delta x, y + \delta y)$ be a neighbouring point on it. Draw PM and QN perpendiculars on y-axis from P and Q respectively. As y changes, the area (ACMP) also changes and hence clearly a function of y. Let A denote the area (ACMP), then the area (ACNQ) will be $A + \delta A$.

The area (PMNQ) = Area (ACNQ) – Area (ACMP) = $A + \delta A - A = \delta A$

Complete the rectangle PRQS. Then the area (PMNQ) lies between the area (PMNS) and the area (RMNQ), that is,

δA lies between $x \;\delta y$ and $(x + \delta x)\;\delta y$

$$\Rightarrow \frac{\delta A}{\delta y} \text{ lies between } x \text{ and } x, x + \delta x :\bullet$$

In the limiting position when $Q \to P$, $\delta x \to 0$ and $\delta y \to 0$

$$\therefore \lim_{\delta y \to 0} \frac{\delta A}{\delta y} \text{ lies between x and } \lim_{\delta x \to 0} (x + \delta x)$$

$$\Rightarrow \frac{dA}{dy} = x$$

Integrating both sides with respect to y, between the limits c to d, we get

$$\int_c^d x\, dy = \int_c^d \frac{dA}{dy} . dy$$

$$= [A]_c^d$$

= (Area when $y = d$) – (Area when $y = c$)

= Area (ACDB) – 0

= Area(ACDB)

Hence area (ACDB) $= \int_c^d x\, dy = \int_c^d f(y)\, dy$

The area bounded by the curve $x = f(y)$ the y-axis and the lines $y = c$ and $y = d$ is

$$\int_c^d x\, dy \qquad \text{or} \qquad \int_c^d f(y)\, dy$$

where $x = f(y)$ is a continuous single valued function and x does not change sign in the interval $c \le y \le d$.

Example 27.17: Find the area bounded by the curve $x = y$, y-axis and the lines $y = 0, y = 3$.

Solution: The given curve is $x = y$

$\therefore$ Required area bounded by the curve, y-axis and the lines $y = 0$, $y = 3$ is

$$= \int_0^3 x\, dy$$

$$= \int_0^3 y\, dy$$

$$= \left[\frac{y^2}{2}\right]_0^3$$

$$= \frac{9}{2} - 0$$

$$= \frac{9}{2} \text{ square units}$$

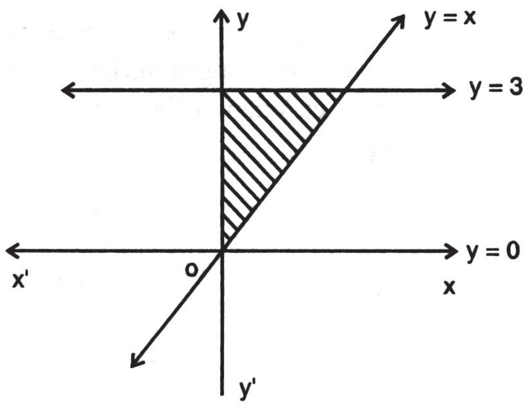


Fig. 27.10

Example 27.18: Find the area bounded by the curve $x = y^2$ y-axis and the lines, $y = 0$, $y = 2$.

Solution: The equation of the curve is $x = y^2$

$\therefore$ Required area bounded by the curve, y-axis and the lines $y = 0$, $y = 2$

$$= \int_0^2 y^2 \, dy = \left[\frac{y^3}{3}\right]_0^2$$

$$= \frac{8}{3} - 0$$

$$= \frac{8}{3} \text{ square units.}$$

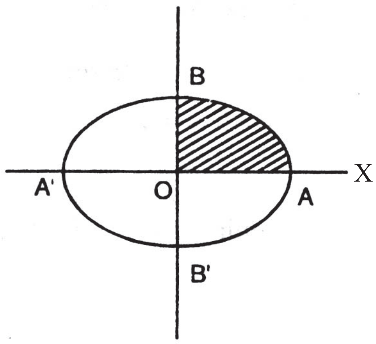


Fig. 27.13

Example 27.19: Find the area enclosed by t axis in the first quadrant.

Solution: The given curve is $x^2 + y^2 = a^2$ v is (0, 0) and radius a. Therefore, we have to find the area enclosed by the circle $x^2 + y^2 = a^2$, the y-axis and the abscissae $y = 0$, $y = a$.

$$\therefore \text{ Required area } = \int_0^a x \, dy$$

$$= \int_0^a \sqrt{a^2 - y^2} \, dy$$

(because x is positive in first quadrant)

$$= \left[\frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)\right]_0^a$$

$$= 0 + \frac{a^2}{2}\sin^{-1} 1 - 0 - \frac{a^2}{2}\sin^{-1} 0$$

$$= \frac{\pi a^2}{4} \text{ square units}$$

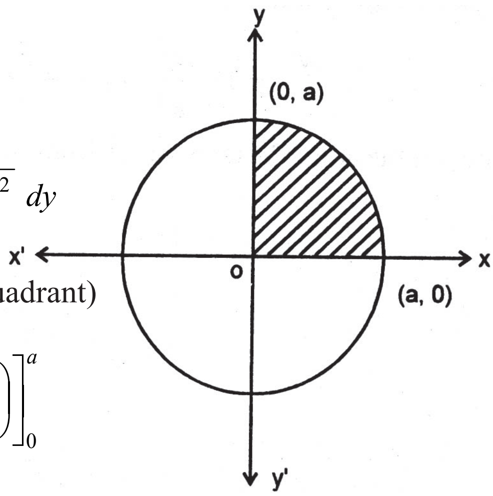


Fig. 37.11

$$\left(\because \sin^{-1} 0 = 0, \ \sin^{-1} 1 = \frac{\pi}{2} \right)$$

Note : The area is same as in Example 39.14, the reason is the given curve is symmetrical about both the axes. In such problems if we have been asked to find the area of the curve, without any restriction we can do by either method

Example 27.20: Find the whole area bounded by the circle $x^2 + y^2 = a^2$.

Solution: The equation of the curve is $x^2 + y^2 = a^2$.

The circle is symmetrical about both the axes, so the whole area of the circle is four times the area os the circle in the first quadrant, that is,

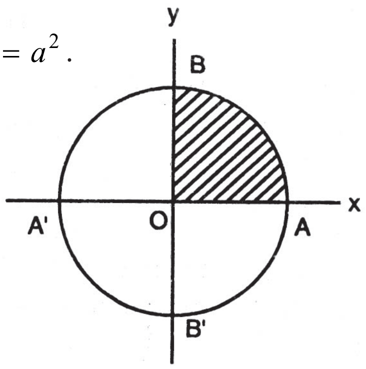


Fig. 27.12

Area of circle = 4 $\times$ area of OAB

$$= 4 \times \frac{\pi a^2}{4} \quad \text{(From Example 39.15 and 39.19)}$$

$= \pi \, a^2$ square units.

Example 27.21: Find the whole area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution: The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The ellipse is symmetrical about both the axes and so the whole area of the ellipse is four times the area in the first quadrant, that is,

Whole area of the ellipse = 4 $\times$ area (OAB)

In the first quadrant,

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \quad \text{or} \quad y = \frac{b}{a}\sqrt{a^2 - x^2}$$

Now for the area (OAB), x varies from0 to a

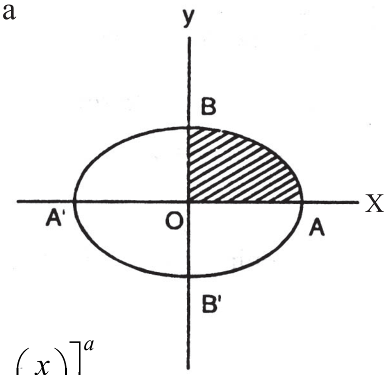


Fig. 27.13

$$\therefore \quad \text{Area(OAB)} = \int_0^a y\, dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}1 - 0 - \frac{a^2}{2}\sin^{-1}0\right]$$

$$= \frac{ab\pi}{4}$$

Hence the whole area of the ellipse

$$= 4 \times \frac{ab\pi}{4}$$

$= \pi\, ab$ square units.

27.4.3 Area between two Curves

Suppose that $f(x)$ and $g(x)$ are two continuous and non-negative functions

on an interval $[a, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$ that is, the curve $y = f(x)$ does not cross under the curve $y = g(x)$ for $x \in [a, b]$. We want to find the area bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by $x = a$ and $x = b$.

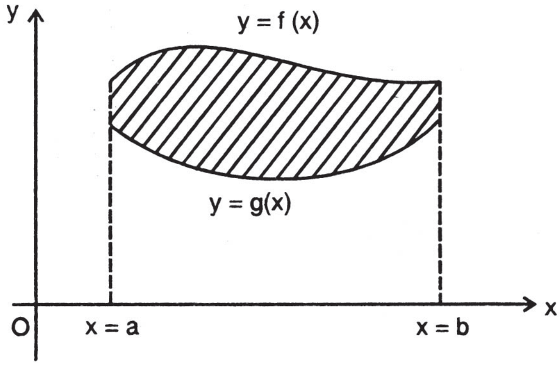


Fig. 27.14

Let A= [Area under $y = f(x)$]   [Area under $y = g(x)$] .....(1)

Now using the definition for the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$, we have

Area under

$$y = f(x) = \int_a^b f(x)\,dx \qquad ...(2)$$

Similarly,Area under $y = g(x) = \int_a^b g(x)\,dx$ ...(3)

Using equations (2) and (3) in (1), we get

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx \qquad ....(4)$$

What happens when the function g has negative values also? This formula can be extended by translating the curves $f(x)$ and $g(x)$ upwards until both are above the x-axis. To do this let-m be the minimum value of $g(x)$ on $[a, b]$ (see Fig. 27.15).

Since $g(x) \geq -m \quad \Rightarrow \quad g(x) + m \geq 0$

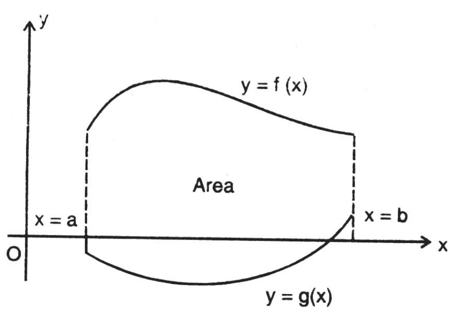


Fig. 27.15

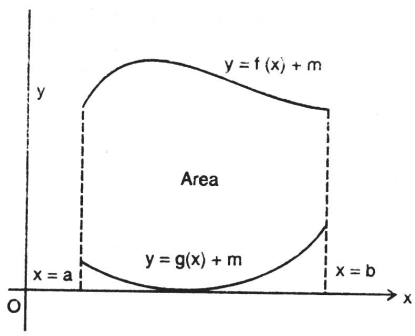


Fig. 27.16

Now, the functions $g(x)$ + m and $f(x) + m$ are non-negative on $[a, b]$ (see Fig. 27.16). It is intuitively clear that the area of a region is unchanged by translation, so the area A between f and g is the same as the area between $g(x)+m$ and $f(x)+m$. Thus,

A = [area under $y = [f(x)+m]$] – [area under $y = [g(x)+m]$] .....(5)

Now using the definitions for the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$, we have

$$\text{Area under } y = f(x)+m = \int_a^b [f(x)+m]\,dx \quad ...(6)$$

and $$\text{Area under } y = g(x)+m = \int_a^b [g(x)+m]\,dx \quad ...(7)$$

The equations (6), (7) and (5) give

$$A = \int_a^b [f(x)+m]\,dx - \int_a^b [g(x)+m]\,dx$$

$$= \int_a^b [f(x)-g(x)]\,dx$$

which is same as (4) Thus,

If $f(x)$ and $g(x)$ are coare continuous functions on the interval $[a, b]$, and $f(x) \geq g(x)$, $\forall\ x \in [a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by $x = a$ and on the right by $x = b$ is

$$= \int_a^b [f(x)-g(x)]\,dx$$

Example 27.22: Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$.

Solution: $y = x + 6$ is the equation of the straight line and $y = x^2$ is the equation of the parabola which is symmetric about the y-axis and origin the vertex. Also the region is bounded by the lines $x = 0$ and $x = 2$.

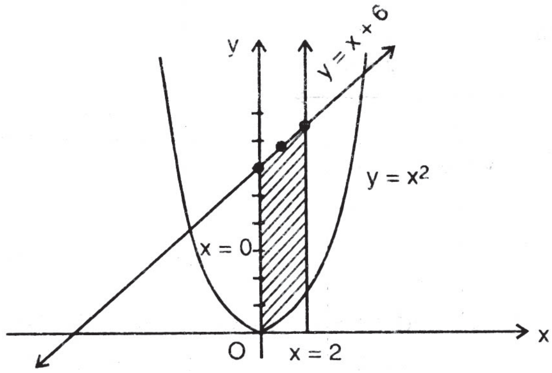


Fig. 27.17

Thus, $$A = \int_0^2 (x+6)\,dx - \int_a^2 x^2\,dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{34}{3} - 0 = \frac{34}{3} \text{ square units.}$$

Example 27.23: Find the area of the region enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution: We know that $y = x^2$ is the equation of the parabola which is symmetric about the y-axis and vertex is origin and $y = x + 6$ is the equation of the straight line which makes an angle 45° with the x-axis and having the intercepts of −6 and 6 with the x and y axes respectively. (See Fig. 27.18).

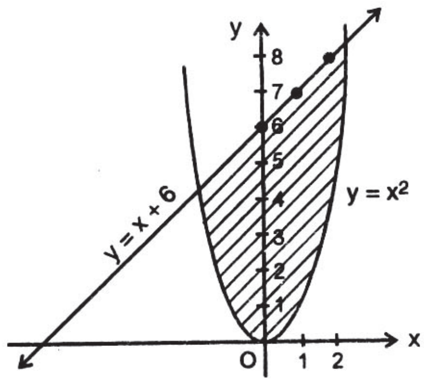


Fig. 27.18

A sketch of the region shows that the lower boundary is $y = x^2$ and the upper boundary is $y = x + 6$. These two curves intersect at two points, say A and B. Solving these two equations we get

$$x^2 = x + 6 \Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = 3, -2$$

When $x = 3$, $y = 9$ and when $x = -2$, $y = 4$

$$\therefore \text{ The required area } = \int_{-2}^{3} \left[(x+6) - x^2 \right] dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x}{3} \right]_{-2}^{3}$$

$$= \frac{27}{2} - \left(-\frac{22}{3} \right)$$

$$= \frac{125}{6} \text{ square units.}$$

Example 27.24: Find the area of the region enclosed between the curves $y = x^2$ and $y = x$.

Solution: We know that $y = x^2$ is the equation of the parabola which is symmetric about the y-axis and vertex is origin. $y = x$ is the equation of the straight line passing through the origin and making an angle of 45° with the x-axis (see Fig. 27.19).

A sketch of the region shows that the lower boundary is $y = x^2$ and the upper boundary is the line $y = x$. These two curves intersect at two points O and A. Solving these two equations, we get

$$x^2 = x$$

$$\Rightarrow x(x - 1) = 0$$

$$\Rightarrow x = 0.1$$

Here $f(x) = x$. $g(x) = x^2$.

$a = 0$ and $b = 1$

Therefore, the required area

$$= \int_{0}^{1} (x - x^2)\, dx$$

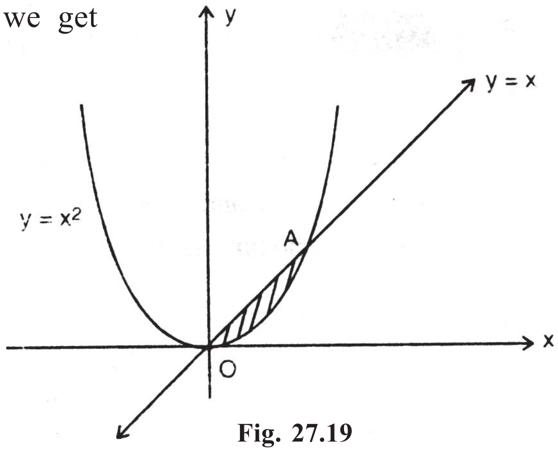


Fig. 27.19

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ square units.}$$

Example 27.25: Find the area bounded by the curves $y^2 = 4x$, and $y = x$.

Solution: We know that $y^2 = 4x$ the equation of the parabola which is symmetric about the x-axis and origin is the vertex $y = x$ is the equation of the straight line passing through origin and making an angle of 45° with the x-axis (see Fig. 27.20).

A sketch of the region shows that the lower boundary is $x^2 = 4ay$ and the upper boundary is $y^2 = 4ax$. These two curves intersect at two points O and A. Solving these two equations, we get

$$\frac{y^2}{4} - y = 0$$

$$\Rightarrow \quad y(y-4) = 0$$

$$\Rightarrow \quad y = 0.4$$

when $y = 0$, $x = 0$ and when $y = 4$, $x = 4$

Here $f(x) = (4x)^{1/2}$. $g(x) = x.a = 0.b = 4$

Therefore, the required area is

$$= \int_0^4 (2x^{\frac{1}{2}} - x)\, dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^4$$

$$= \frac{32}{3} - 8$$

$$= \frac{8}{3} \text{ square units.}$$

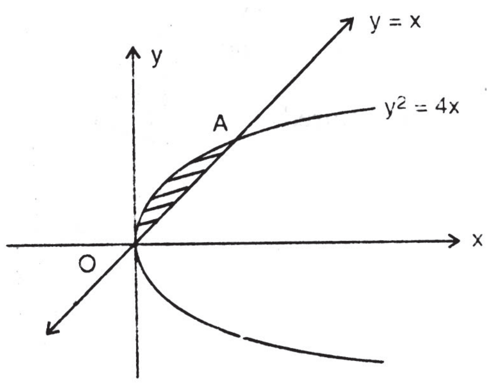


Fig. 27.20

MODULE - V
Calculus

Notes

Example 27.26: Find the area common to two parabolas $x^2 = 4ay$ and $y^2 = 4ax$.

Solution: We know that $y^2 = 4ax$ and $x^2 = 4ay$ •are the equations of the parabolas, which are symmetric about the x-axis and y-axis respectively.

Also both the parabolas have their vertices at the origin (see Fig. 27.19).

A sketch of the region shows that the lower boundary is $x^2 = 4ay$ and the upper boundary is $y^2 = 4ax$. These two curves intersect at two points O and A. Solving these two equations, we have

$$\frac{x^4}{16a^2} = 4ax$$

$$\Rightarrow \quad x(x^3 - 64\ a^3) = 0$$

$$\Rightarrow \quad x = 0,\ 4a$$

Hence the two parabolas intersect at point (0, 0) and $(4a, 4a)$.

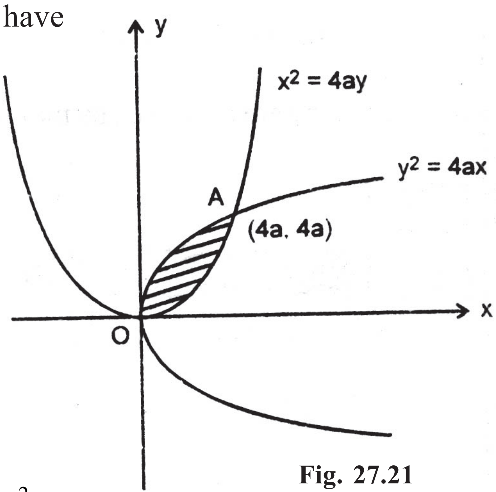


Fig. 27.21

Here $f(x) = \sqrt{4ax}, g(x) = \frac{x^2}{4a}$, $a = 0$ and $b = 4a$

Therefore, required area

$$= \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a} \right] dx$$

$$= \left[\frac{2.2\sqrt{a}x^{\frac{3}{2}}}{3} - \frac{x^3}{12a} \right]_0^{4a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16}{3} a^2 \text{ square units.}$$

EXCERISE 27.4

1. Find the area of the circle $x^2 + y^2 = 9$.

2. Find the area of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

3. Find the area of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

4. Find the area bounded by the curves $y^2 = 4ax$ and $y = \frac{x^2}{4a}$.

5. Find the area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$.

6. Find the area enclosed by the curves $y = x^2$ and $y = x + 2$.

27.5 REDUCTION FORMULAE

In this section, we derive some reduction formulae for the evaluation of definite integrals of $\sin^n x, \cos^n x, \sin^m x, \cos^n x$ between 0 and $\frac{\pi}{2}$ for positive integers m, n.

The following theorem gives a useful formula to evaluate the definite integral of $\sin^n x$ between 0 and $\frac{\pi}{2}$, when n is an integer $n \geq 2$.

27.5.1 Theorem

Let n be an integer greater than or equal to 2. Then

$$\int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \cdots\cdots \frac{1}{2} \frac{\pi}{2}, & \text{If } n \text{ is even} \\ \frac{n-1}{n} \frac{n-3}{n-2} \cdots\cdots \frac{2}{3}, & \text{If } n \text{ is odd} \end{cases}$$

Note: 1. $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$

Example 27.27: Find

(1) $\int_0^{\pi/2} \sin^4 x\, dx$ (ii) $\int_0^{\pi/2} \sin^7 x\, dx$ (iii) $\int_0^{\pi/2} \cos^8 x\, dx$

Sol: (i) $\int_0^{\pi/2} \sin^4 x\, dx = \frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2}$

$$= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{16}\pi$$

(ii) $\int_0^{\pi/2} \sin^7 x\, dx = \frac{7-1}{7} \cdot \frac{7-3}{7-2} \cdot \frac{7-5}{7-4}$

$$= \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{16}{35}.$$

(iii) $\int_0^{\pi/2} \cos^8 x\, dx = \frac{8-1}{8} \cdot \frac{8-3}{8-2} \cdot \frac{8-5}{8-4} \cdot \frac{8-7}{8-6} \cdot \frac{\pi}{2}$

$$= \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{35}{256}\pi.$$

27.5.2 Theorem : If m and n are positive integers then

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx =$$

$$\begin{cases} \frac{1}{m+1}, & \text{if } n = 1 \\ \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}, & \text{if } 1 \neq n \text{ is odd} \\ \frac{n-1}{m+n}, \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdots \frac{1}{2} \cdot \frac{\pi}{2}, & \text{if } n \text{ is even and } m \text{ is even} \\ \frac{n-1}{m+n}, \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdots \frac{2}{3}, & \text{if } n \text{ is even and } 1 \neq m \text{ is odd} \\ \frac{1}{n+1}, & \text{if } m = 1 \end{cases}$$

Example 27.28:

Evaluate the following definite integrals

(i) $\int_0^{\pi/2} \sin^4 x \cos^5 x\, dx$

(ii) $\int_0^{\pi/2} \sin^5 x \cos^4 x\, dx$

(iii) $\int_0^{\pi/2} \sin^6 x \cos^4 x\, dx$

Sol :

(i) $$\int_0^{\pi/2} \sin^4 x \cos^5 x\, dx = \frac{5-1}{4+5} \cdot \frac{5-3}{4+5-2} \cdot \frac{1}{4+1}$$

$$= \frac{4}{9} \cdot \frac{2}{7} \cdot \frac{1}{5} = \frac{8}{315}$$

(ii) $$\int_0^{\pi/2} \sin^5 x \cos^4 x\, dx = \frac{4-1}{5+4} \cdot \frac{4-3}{5+4-2} \cdot \frac{5-1}{5} \cdot \frac{5-3}{5-2}$$

$$= \frac{3}{9} \cdot \frac{1}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{315}$$

(iii) $$\int_0^{\pi/2} \sin^6 x \cos^4 x\, dx = \frac{4-1}{6+4} \cdot \frac{4-3}{6+4-2} \cdot \frac{6-1}{6} \cdot \frac{6-3}{6-2} \cdot \frac{6-5}{6-4} \cdot \frac{\pi}{2}$$

$$= \frac{3}{10} \cdot \frac{1}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3}{512} \pi.$$

Example 27.29:

Find $\int_0^{2\pi} \sin^4 x \ \cos^6 x\, dx$

Sol: Let $f(x) = \sin^4 x \cos^6 x$

Since $f(2\pi - x) = f(\pi - x) = f(x)$, it follows from previous theorem

$$\int_0^{2\pi} f(x)\, dx = 2\int_0^{\pi} \sin^4 x \ \cos^6 x\, dx$$

$$= 4\int_0^{\pi/2} \sin^4 x \ \cos^6 x\, dx$$

$$= 4 . \frac{6-1}{4+6} . \frac{6-3}{4+6-2} . \frac{6-5}{4+6-4} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$$

$$= \frac{3}{128}\pi.$$

Example 27.30:

Find $\int_{-\pi/2}^{\pi/2} \sin^2 x \ \cos^4 x\, dx$

Sol: Let $f(x) = \sin^2 x \ \cos^4 x$ since f is even we have

$$\int_{-\pi/2}^{\pi/2} f(x)\, dx = 2\int_0^{\pi/2} f(x)\, dx$$

Hence $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^4 x \ dx = 2\int_0^{\pi/2} \sin^2 x \cos^4 x \ dx$

$$= 2 . \frac{4-1}{2+4} . \frac{4-3}{2+4-2} . \frac{1}{2} . \frac{\pi}{2}$$

$$= 2 . \frac{3}{6} . \frac{1}{4} . \frac{\pi}{4} = \frac{\pi}{16}.$$

EXERCISE 29.5

I. Find the values of the following integrals.

1. $\int_0^{\pi/2} \sin^{10} x \, dx$

2. $\int_0^{\pi/2} \sin^4 x \cos^4 x \, dx$

3. $\int_0^{\pi/2} \cos^7 x \sin^2 x \, dx$

4. $\int_0^{\pi/4} \sin^3 \theta \, d\theta$

KEY WORDS

- If f is continuous in $[a, b]$ and F is an anti derivative of f in $[a, b]$, then $[a, b]$ then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

- If f and g are continuous in $[a, b]$ and c is a constant, then

(i) $\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx$

(ii) $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b f(x)\, dx$

(iii) $\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$

- The area bounded by the curve $y = f(x)$, the x-axis and the ordinates

$$x = a, x = b \text{ is } \int_a^b f(x)\,dx \text{ or } \int_a^b y\ dx$$

where $y = f(x)$ is a continuous single valued function and y does not change sign in the interval $a \leq x \leq b$.

- If $f(x)$ and $g(x)$ $[a, b]$ are continuous functions on the interval $[a, b]$ and $f(x) \geq g(x)$, for all $x \in [a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = f(x)$, below by $y = g(x)$, on the left by $x = a$ and on the right by $x = b$ is

$$\int_a^b [f(x) - g(x)]\ dx$$

- If n is an integer $n \geq 2$, then

$$\int_0^{\pi/2} \sin^n x\ dx = \begin{cases} \dfrac{n-1}{n}\,\dfrac{n-3}{n-2} \ldots\ldots \dfrac{1}{2} \cdot \dfrac{\pi}{2}, & \text{If } n \text{ is even} \\ \dfrac{n-1}{n}\,\dfrac{n-3}{n-2} \ldots\ldots \dfrac{2}{3}. & \text{If } n \text{ is odd} \end{cases}$$

- $\int_0^{\pi/2} \sin^3 x\ dx = \int_0^{\pi/2} \cos^n x\ dx$, n is a positive integer.

- If m and n are positive intgers then

$$\int_0^{\pi/2} \sin^m x \cos^n x\ dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}.$$

If $1 \neq n$ is odd

$$\frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdots\cdot \frac{1}{2}\,\frac{\pi}{2}$$

If n is even and m is even

$$\frac{n-1}{m+n}\cdot\frac{n-3}{m+n-2}\cdots\frac{1}{m+2}\cdot\frac{m-1}{m}\cdots\frac{2}{3}$$

If n is even and $1 \neq m$ is odd.

SUPPORTIVE WEB SITES

- *http:// www.wikipedia.org*
- *http:// mathworld.wolfram.com*

PRACTICE EXERCISE

1. Evaluate the following integrals (1 to 5) as the limit of sum.

1. $\int_a^b x\,dx$ 2. $\int_a^b x^2\,dx$ 3. $\int_a^b \sin x\,dx$

4. $\int_a^b \cos x\,dx$ 5. $\int_0^2 (x^2+1)x\,dx$

2. Evaluate the following integrals (6 to 25)

6. $\int_0^2 \sqrt{a^2-x^2}\,dx$ 7. $\int_0^{\pi/2} \sin 2x\,dx$

8. $\int_{\pi/4}^{\pi/2} \cot x\,dx$ 9. $\int_0^{\pi/2} \cos^2 x\,dx$

10. $\int_0^1 \sin^{-1} x\,dx$ 11. $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

12. $\int_3^4 \frac{1}{x^2-4}\,dx$ 13. $\int_0^{\pi} \frac{1}{5+3\cos\theta}\,d\theta$

14. $\int_0^{\pi/4} 2\tan^3 x\,dx$ 15. $\int_0^{\pi/2} \sin^3 x\,dx$

16. $\int_0^2 x\sqrt{x+2}\,dx$ 17. $\int_0^{\pi/2} \sqrt{\sin\theta}\cos^5\theta\,d\theta$

18. $\int_0^{\pi} x \log \sin x \, dx$

19. $\int_0^{\pi} \log(1 + \cos x)\, dx$

20. $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

21. $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$

22. $\int_0^{\pi/4} \log(1 + \tan x)\, dx$

23. $\int_0^{\pi/8} \sin^5 2x \cos 2x \, dx$

24. $\int_0^{2} x(x^2 + 1)^3 \, dx$

ANSWERS

EXERCISE 27.1

1. $\frac{35}{2}$
2. $e - \frac{1}{e}$
3. (a) $\frac{\sqrt{2}-1}{\sqrt{2}}$ (b) 2 (c) $\frac{\pi}{4}$ (d) $\frac{64}{3}$

EXERCISE 27.2

1. $\frac{e-1}{2}$
2. $\frac{2}{3}\tan^{-1}\frac{1}{3}$
3. $\frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\frac{1}{3}$
4. 29
5. $\frac{24\sqrt{2}}{15}$
6. $\frac{\pi}{4}$
7. $-\frac{\pi}{2}\log 2$
8. 0
9. 0
10. $\frac{1}{2}\left[\frac{\pi}{2} - \log 2\right]$

EXERCISE 27.3

1. $\frac{8}{3}$ sq.units
2. $\frac{27}{2}$ sq.units
3. $\frac{e^{2a}-1}{2}$ sq.units
4. $c^2\left(1-\cos\frac{a}{c}\right)$

EXERCISE 27.4

1. 9π sq.units
2. 6π sq.units.
3. 20π sq.units
4. $\frac{16}{3}a^2$ sq.units
5. $\frac{16}{3}$ sq.units
6. $\frac{9}{2}$ sq.units

EXERCISE 27.5

(1) $\frac{63}{512}\pi$

(2) $\frac{3}{256}\pi$

(3) $\frac{16}{315}$

(4) $\frac{2}{3}-\frac{5}{6\sqrt{2}}$

PRACTICE EXERCISE

1. $\frac{b^2 - a^2}{2}$
2. $\frac{b^3 - a^3}{3}$
3. $\cos a - \cos b$
4. $\sin b - \sin a$
5. $\frac{14}{3}$
6. $\frac{\pi a^2}{4}$
7. 1
8. $\frac{1}{2}\log 2$
9. $\frac{\pi}{4}$
10. $\frac{\pi}{2} - 1$
11. $\frac{\pi}{2}$
12. $\frac{1}{4}\log\frac{5}{3}$
13. $\frac{\pi}{4}$
14. $1 - \log 2$
15. $\frac{2}{3}$
16. $\frac{16}{15}\left(2 + \sqrt{2}\right)$
17. $\frac{64}{231}$
18. $-\frac{\pi^2}{2}\log 2$
19. $-\pi\log 2$
20. $\frac{\pi^2}{4}$
21. $\frac{1}{\sqrt{2}}\log\left(1+\sqrt{2}\right)$
22. $\frac{\pi}{8}\log 2$
23. $\frac{1}{96}$
24. 78

Chapter 28

DIFFERENTIAL EQUATIONS

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- define a differential equation, its order and degree;
- determine the order and degree of a differential equation;
- form differential equation from a given situation;
- illustrate the terms "general solution" and "particular solution" of a differential equation through examples;
- solve differential equations of the following types :

(i) $\frac{dy}{dx} = f(x)$ (ii) $\frac{dy}{dx} = f(x)\, g(y)$ (iii) $\frac{dy}{dx} = \frac{f(x)}{g(y)}$

(iv) $\frac{dy}{dx} + p(x)\, y = Q(x)$ (v) $\frac{d^2 y}{dx^2} = f(x)$

- find the particular solution of a given differential equation for given conditions.

PREREQUISITES

- Integration of algebraic functions, rational functions and trigonometric functions.

INTRODUCTION

In pervious lesson we studied the concept of differentiation and integration. Now in differential equations have application in many branches of physics, physical chemistry etc.

The present section as aimed at defining an ordinary differential equations forming such an equation from a given firmly of curves or surfaces. We also define two concepts, namely order and degree of an ordinary differential equation.

If a differential equation contains only one independent variable. Then it is called an ordinary differential equation and if it contains more than one independent variable, then it is called a partial differential equation. Hence ordinary differential equation contains only ordinary derivatives where as a partial differential equation contains partial derivatives.

Since derivatives as a rate of change, it is only natural that differential equations write in the description of change in state or motion. Differential equations occurs in problems of radio active decay, Newton's law of cooling, the motion of a particle of a planet, chemical reactions.

28.1 DIFFERENTIAL EQUATIONS

As stated in the introduction, many important problems in Physics, Biology and Social Sciences, when formulated in mathematical terms, lead to equations that involve derivatives. Equations which involve one or more differential coefficients such as $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ (or differentials) etc. and independent and dependent variables are called differential equations.

(i) $\frac{dy}{dx} = \cos x$ (ii) $\frac{d^2y}{dx^2} + y = 0$ (iii) $xdx + ydy = 0$

(iv) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0$ (iv) $y = \frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

28.2 ORDER AND DEGREE OF A DIFFERENTIAL EQUATION

Order : It is the order of the highest derivative occurring in the differential equation.

Degree : It is the degree of the highest order derivative in the differential equation after the equation is free from negative and fractional powers of the derivatives. For example,

	Differential Equation	Order	Degree
(i)	$\frac{dy}{dx} = \sin x$	One	One
(ii)	$\left(\frac{dy}{dx}\right)^2 + 3y^2 = 5x$	One	Two
(iii)	$\left(\frac{d^2s}{dt^2}\right)^2 + t^2\left(\frac{ds}{dt}\right)^4 = 0$	Two	Two
(iv)	$\frac{d^3v}{dr^3} + \frac{2}{r}\frac{dv}{dr} = 0$	Three	One
(v)	$\left(\frac{d^4y}{dx^4}\right)^2 + x^3\left(\frac{d^3y}{dx^3}\right)^5 = \sin x$	Four	Two

Example 28.1: Find the order and degree of the differential equation :

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = 0$$

Solution: The given differential equation is

$$\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = -\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$ has fractional index. Therefore, we first square both sides to remove fractional index.

Squaring both sides, we have

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

Hence each of the order of the diferential equation is 2 and the degree of the differential equation is also 2.

Note : Before finding the degree of a differential equation, it should be free from radicals and fractions as far as derivatives are concerned.

28.3 LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS

A differential equation in which the dependent variable and all of its derivatives occur only in the first degree and are not multiplied together is called a **linear differential equation**. A differential equation which is not linear is called non-linear differential equation . For example, the differential equations

$$\frac{d^2y}{dx^2} + y = 0 \text{ and } \cos^2 x.\frac{d^3y}{dx^3} + x^3.\frac{dy}{dx} + y = 0 \text{ are linear.}$$

The differential equation $\left(\frac{dy}{dx}\right)^2 + \frac{y}{x} = \log x$ is non-linear as degree of $\frac{dy}{dx}$ is two.

Further the differential equation $y\frac{d^2y}{dx^2} - 4 = x$ is non-linear because the dependent variable $\frac{d^2y}{dx^2}$ are multiplied together.

28.4 FORMATION OF A DIFFERENTIAL EQUATION

Consider the family of all straight lines passing through the origin

(see Fig. 28.1).

This family of lines can be represented by

$$y = mx \qquad ...(1)$$

Differentiating both sides, we get

$$\frac{dy}{dx} = m \qquad ...(2)$$

From (1) and (2), we get

$$y = x\frac{dy}{dx} \qquad ...(3)$$

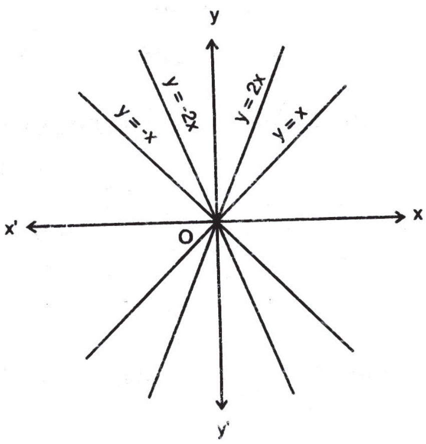


Fig. 28.1

So $y = mx$, and $y = x\frac{dy}{dx}$ represent the same family. Clearly equation (3) is a differential equation.

Working Rule : To form the differential equation corresponding to an equation involving two variables, say x and y and some arbitrary constants, say, a, b, c, etc.

(1) Differentiate the equation as many times as the number of arbitrary constants in the equation.

(2) Eliminate the arbitrary constants from these equations.

Remark : If an equation contains n arbitrary constants then we will obtain a differential equation of n^{th} order.

Example 28.2 Form the differential equation representing the family of curves.

$$y = ax^2 + bx \qquad ...(1)$$

Solution: Differentiating both sides, we get

$$\frac{dy}{dx} = 2ax + b \qquad ...(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = 2a \qquad ...(3)$$

$$\Rightarrow \ a = \frac{1}{2}\frac{d^2y}{dx^2} \qquad ...(4)$$

(The equation (1) contains two arbitrary constants. Therefore, we differentiate this equation two times and eliminate 'a' and 'b').

On putting the value of 'a' in equation (2), we get

$$\frac{dy}{dx} = x\frac{d^2y}{dx^2} + b$$

$$\Rightarrow \ b = \frac{dy}{dx} - x\frac{d^2y}{dx^2} \qquad ...(5)$$

Substituting the values of 'a' and 'b' given in (4) and (5) above in equation (1), we get

$$y = x^2\left(\frac{1}{2}\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)$$

or $$y = \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2}$$

or $$y = x\frac{dy}{dx} - \frac{x^2}{2}\frac{d^2y}{dx^2}$$

or $$\frac{x^2}{2}\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

which is the required differential equation.

Example 28.3 : Form the differential equation representing the family of curves

$$y = a \cos (x + b).$$

Solution: $y = a \cos (x + b)$ ...(1)

Differentiating both sides, we get

$$\frac{dy}{dx} = -a\sin(x+b) \qquad ...(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -a\cos(x+b) \qquad ...(3)$$

From (1) and (3), we get

$$\frac{d^2y}{dx^2} = -y \quad \text{or} \quad \frac{d^2y}{dx^2} + y = 0$$

which is the required differential equation.

Example 28.4 : Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.

Solution: As the centre lies on the x-axis, its coordinates will be (a, 0).

Since each circle passes through the origin, its radius is a.

Then the equation of any circle will be

$$(x - a)^2 + y^2 = a^2 \qquad ...(1)$$

To find the corresponding differential equation, we differentiate equation (1) and get

$$2(x-a) + 2y\frac{dy}{dx} = 0$$

or $$x - a + y\frac{dy}{dx} = 0$$

or $$a = y\frac{dy}{dx} + x$$

$$\left(x - y\frac{dy}{dx} - x\right)^2 + y^2 = \left(y\frac{dy}{dx} + x\right)^2$$

or $$\left(y\frac{dy}{dx}\right)^2 + y^2 = x^2 + \left(y\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

or $$y^2 = x^2 + 2xy\frac{dy}{dx}$$

which is the required differential equation.

Remark : If an equation contains one arbitrary constant then the corresponding differential equation is of the first order and if an equation contains two arbitrary constants then the corresponding differential equation is of the second order and so on.

Example 28.5 Assuming that a spherical rain drop evaporates at a rate proportional to its surface area, form a differential equation involving the rate of change of the radius of the rain drop.

Solution: Let r(t) denote the radius (in mm) of the rain drop after *t* minutes. Since the radius is decreasing as t increases, the rate of change of *r* must be negative. If V denotes the volume of the rain drop and S its surface area, we have

$$V = \frac{4}{3}\pi r^3 \qquad ...(1)$$

and

$$S = 4\pi r^2 \qquad ...(2)$$

It is also given that

$$\frac{dV}{dt} \propto S$$

or

$$\frac{dV}{dt} = -KS$$

or

$$\frac{dV}{dr}\cdot\frac{dr}{dt} = -KS \qquad ...(3)$$

Using (1), (2) and (3) we have

$$4\pi r^2 . \frac{dr}{dt} = -4K\pi r^2$$

or

$$\frac{dr}{dt} = K$$

which is the required differential equation.

EXERCISE 28.1

1. Find the order and degree of the differential equation

$$y = x\frac{dy}{dx} + \frac{1}{\left(\frac{dy}{dx}\right)} .$$

2. Write the order and degree of each of the following differential equations.

(a) $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$ (b) $y = 2x\frac{dy}{dx} + x\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

(c) $\sqrt{1-x^2}\,dx + \sqrt{1-y^2}\,dy = 0$ (d) $\left(\frac{d^2s}{dt^2}\right)^2 + 3\left(\frac{ds}{dt}\right)^3 + 4 = 0$

3. State whether the following differential equations are linear or non-linear.

(a) $(xy^2 - x)\,dx + (y - x^2y)\,dy = 0$ (b) $dx + dy = 0$

(c) $\frac{dy}{dx} = \cos x$ (d) $\frac{dy}{dx} + \sin^2 y = 0$

4. Form the differential equation corresponding to

$(x - a)^2 + (y - b)^2 = r^2$ by eliminating 'a' and 'b'.

5. Form the differential equation corresponding to

(a) $y^2 = m(a^2 - x^2)$

(b) Form the differential equation corresponding to

$y^2 - 2ay + x^2 = a^2$, where a is an arbitrary constant.

(c) Find the differential equation of the family of curves

$y = Ae^{2x} + Be^{-3x}$ where A and B are arbitrary constants.

(d) Find the differential equation of all straight lines passing through the point (3,2).

(e) Find the differential equation of all the circles which pass through origin and whose centres lie on y-axis.

28.5 GENERAL AND PARTICULAR SOLUTIONS

Finding solution of a differential equation is a reverse process. Here we try to find an equation which gives rise to the given differential equation through the process of differentiations and elimination of constants. The equation so found is called the primitive or the solution of the differential equation.

Remarks

1. If we differentiate the primitive, we get the differential equation and if we integrate the differential equation, we get the primitive.
2. Solution of a differential equation is one which satisfies the differential equation.

Example 28.6 : Show that $y = C_1 \sin x + C_2 \cos x$, where C_1 and C_2 are arbitrary

constants, is a solution of the differential equation:

$$\frac{d^2y}{dx^2} + y = 0$$

Solution: We are given that

$$y = c_1 \sin x + c_2 \cos x \qquad ...(1)$$

Differentiating both sides of (1), we get

$$\frac{dy}{dx} = c_1 \cos x - c_2 \sin x \qquad ...(2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -c_1 \sin x - c_2 \cos x \qquad ...(3)$$

Substituting the values of $\frac{d^2y}{dx^2}$ and y in the given differential equation, we get

$$\frac{d^2y}{dx^2} + y = c_1 \sin x + c_2 \cos x + (-c_1 \sin x - c_2 \cos x)$$

or $$\frac{d^2y}{dx^2} + y = 0$$

In integration, the arbitrary constants play important role. For different values of the constants we get the different solutions of the differential equation.

A solution which contains as many as arbitrary constants as the order of the differential equation is called the **General Solution** or complete primitive.

If we give the particular values to the arbitrary constants in the general solution of differential equation, the resulting solution is called a **Particular Solution**.

Remark

General Solution contains as many arbitrary constants as is the order of the differential equation.

Example 28.7 : Show that $y = cx + \frac{a}{c}$ (where c is a constant) is a solution of the differential equation.

$$y = x\frac{dy}{dx} + a\frac{dx}{dy}$$

Solution: We have $y = cx + \frac{a}{c}$ ... (1)

Differentiating (1), we get

$$\frac{dy}{dx} = c \Rightarrow \frac{dx}{dy} = \frac{1}{c}$$

On substituting the values of $\frac{dy}{dx}$ and $\frac{dx}{dy}$ in R.H.S of the differential equation, we have

$$x(c)\ a\left(\frac{1}{c}\right) = cx + \frac{a}{c} = y$$

$\Rightarrow$ R.H.S. = L.H.S.

Hence $y = cx + \frac{a}{c}$ is a solution of the given differential equation.

Example 28.8: If $y = 3x^2 + c$ is the general solution of the differential equation $\frac{dy}{dx} - 6x = 0$, then find the particular solution when $y = 3, x = 2$.

Solution: The general solution of the given differential equation is given as

$$y = 3x^2 + c \qquad ...(1)$$

Now on substituting $y = 3, x = 2$ in the above equation , we get

$$3 = 12 + c \quad \text{or} \quad C = -9$$

By substituting the value of C in the general solution (1), we get

$$y = 3x^2 - 9$$

which is the required particular solution.

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Calculus

Notes

28.6 TECHNIQUES OF SOLVING A DIFFERENTIAL EQUATION

28.6.1 When Variables are Separable

(i) Differential equation of the type $\frac{dy}{dx} = f(x)$

Consider the differential equation of the type $\frac{dy}{dx} = f(x)$

or $dy = f(x)\ dx$

On integrating both sides, we get

$$\int dy = \int f(x)\,dx$$

$$y = \int f(x)\,dx + c$$

where c is an arbitrary constant. This is the general solution.

Note: It is necessary to write c in the general solution, otherwise it will become a particular solution.

Example 28.9: Solve

$$(x+2)\frac{dy}{dx} = x^2 + 4x - 5$$

Solution: The given differential equation is $(x+2)\frac{dy}{dx} = x^2 + 4x - 5$

or $\frac{dy}{dx} = \frac{x^2+4x-5}{x+2}$ or $\frac{dy}{dx} = \frac{x^2+4x+4-4-5}{x+2}$

or $\frac{dy}{dx} = \frac{(x+2)^2}{x+2} - \frac{9}{x+2}$ or $\frac{dy}{dx} = x+2-\frac{9}{x+2}$

or $dy = \left(x+2-\frac{9}{x+2}\right)dx$

On integrating both sides of (1), we have

$$\int dy = \int\left(x+2-\frac{9}{x+2}\right)dx \text{ or } y = \frac{x^2}{2} + 2x - 9\log|x+2| + c,$$

where c is an arbitrary constant, is the required general solution.

Example 28.10: Solve

$$\frac{dy}{dx} = 2x^3 - x$$

given that $y = 0$ when $x = 0$.

Solution: The given differential equation is $\frac{dy}{dx} = 2x^3 - x$

or $\quad dy = (2x^3 - x)dx$

On integrating both sides of (1), we get

$$\int dy = \int \left(2x^3 - x\right)dx \quad \text{or} \quad y = 2.\frac{x^4}{4} - \frac{x^2}{2} + C$$

or $$y = \frac{x^4}{2} - \frac{x^2}{2} + C \qquad ...(1)$$

where C is an arbitrary constant.

Since $y = 1$ when $x = 0$, therefore, if we substitute these values in (2) we will get

$$1 = 0 - 0 + C \qquad \Rightarrow \quad C = 1$$

Now, on putting the value of C in (2), we get

$$y = \frac{1}{2}(x^4 - x^2) + 1 \quad \text{or} \quad y = \frac{1}{2}x^2(x^2 - 1) + 1$$

which is the required particular solution.

(ii) Differential equations of the type $\frac{dy}{dx} = f(x).g(y)$

Consider the differential equation of the type

$$\frac{dy}{dx} = f(x).g(y)$$

or $$\frac{dy}{g(y)} = f(x)\,dx \qquad ...(1)$$

In equation (1), x’s and y’s have been separated from one another. Therefore, this equation is also known differential equation with variables separable.

To solve such differential equations, we integrate both sides and add an arbitrary constant on one side.

To illustrate this method, let us take few examples.

Example 28.11: Solve

$$(1 + x^2)dy = (1 + y^2)\, dx$$

Solution: The given differential equation

$$(1 + x^2)dy = (1 + y^2)\, dx$$

can be written as $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$ (Here variables have been seperated)

On integrating both sides of (1), we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

or $$\tan^{-1}y = \tan^{-1}x + c$$

where C is an arbitrary constant.

This is the required solution.

Example 28.12: Solve

$$\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$$

Solution: The given differential equation

$$\left(x^2 - yx^2\right)\frac{dy}{dx} + y^2 + xy^2 = 0$$

can be written as $x^2(1-y)\,\frac{dy}{dx} + y^2(1+x) = 0$

or $$\frac{(1-y)}{y^2}dy = \frac{-(1+x)}{x^2}dx \quad \text{(Variables separated)} \qquad ...(1)$$

If we integrate both sides of (1), we get

or $$\int\left(\frac{1}{y^2} - \frac{1}{y}\right)dy = \int\left(-\frac{1}{x^2} - \frac{1}{x}\right)dx$$

where C is an arbitrary constant.

or $$-\frac{1}{y}-\log|y|=\frac{1}{x}-\log|x|+C$$

or $$\log\left|\frac{x}{y}\right|=\frac{1}{x}+\frac{1}{y}+C$$

Which is the required general solution.

Example 28.13: Solve $\frac{dy}{dx}=(3x+y+4)^2$

Solution: Put $3x + y + 4 = t$ then

$$\frac{dy}{dx}=\frac{dt}{dx}-3$$

So that the given equation becomes

$$\frac{dt}{dx}-3=t^2$$

$$\frac{dt}{t^2+3}=dx$$

Hence $$\int\frac{dt}{t^2+3}=\int dx$$

$$\Rightarrow \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+x+C$$

$$\Rightarrow \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{3x+y+4}{\sqrt{3}}\right)=x+C$$

EXERCISE 28.2

1. Solve the following differential equations.

(i) $\frac{dy}{dx}=e^{y-x}$ (ii) $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$

(iii) $\frac{dy}{dx}=e^{x-y}+x^x e^{-y}$ (iv) $(e^x + 1)y\,dy + (y + 1)\,dx = 0$

2. Solve the following differential equations.

(i) $(xy^2 + x)dy + (yx^2 + y)dy = 0$

(ii) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

(iii) $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

(iv) $\frac{dy}{dx} = \tan^2(x+y)$

28.6.2 Homogeneous Differential Equations

Consider the following differential equations :

(i) $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$ (ii) $(x^3+y^3)dx - 3xy^2dy = 0$

(iii) $\frac{dy}{dx} = \frac{x^3+xy^2}{y^2x}$.

In equation (i) above, we see that each term except $\frac{dy}{dx}$ is of degree 2.
[as degree of y^2 is 2, degree of x^2 is 2 and degree of xy is 1 + 1 =2]

In equation (ii) each term except $\frac{dy}{dx}$ is of degree 3.

In equation(iii) each term except $\frac{dy}{dx}$ is of degree 3, as it can be rewritten

as $y^2x\frac{dy}{dx} = x^2 + xy^2$

Such equations are called **homogeneous equations**.

Remarks

Homogeneous equations do not have constant terms.

For example, differential equation

$$(x^2 + 3yx)\ dx - (x^3 + x)dy = 0$$

is not a homogeneous equation as the degree of the function except $\frac{dy}{dx}$ in each term is not the same. [degree of x^2 is 2, that of $3yx$ is 2, of x^3 is 3, and of x is 1]

28.6.3 Solution of Homogeneous Differential Equation :

To solve such equations, we proceed in the following manner :

(1) write one variable = v. (the other variable).
(i.e. either $y = vx$ or $x = vy$)

(2) reduce the equation to separable form

(3) solve the equation as we had done earlier.

Example 28.14: Show that $f(x,y) = x - y\log y + y\log x$ is a homogeneous function of x and y.

Solution: Now, for $k > 0$

$$\begin{aligned} f(kx, ky) &= kx - ky\log(ky) + ky\log(kx) \\ &= k[x - y\log(ky) + y\log(kx)] \\ &= k[x - y\log k - y\log y + y\log k + y\log x] \\ &= k[x - y\log y + y\log x] = kf(x,y) \end{aligned}$$

so that $f(x,y)$ is a homogeneous function of degree 1.

Example 28.15: Express $(1+e^{x/y})dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$ in the form $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$.

Solution : The given equation can be written as $\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y}-1\right)}{1+e^{x/y}} = F\left(\frac{x}{y}\right)$ which is in the required form.

Example 28.16: Express $(x\sqrt{x^2+y^2} - y^2)dx + xy\,dy = 0$ in the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

Solution: From the given equation

$$\frac{dy}{dx} = \frac{y^2 - x\sqrt{x^2+y^2}}{xy} = \frac{\frac{y^2}{x^2} - \frac{x\sqrt{x^2+y^2}}{x^2}}{\frac{xy}{x^2}}$$

$$= \frac{\left(\frac{y}{x}\right)^2 - \sqrt{1+\left(\frac{y}{x}\right)^2}}{\left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right).$$

Example 28.17: Express $\frac{dy}{dx} = \frac{y}{x + ye^{\frac{-2x}{y}}}$ in the form $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$.

Solution: From the given equation

$$\frac{dx}{dy} = \frac{x + ye^{\frac{-2x}{y}}}{y}$$

$$= \frac{x}{y} + e^{-2(x/y)} = F\left(\frac{x}{y}\right).$$

Example 28.18: Solve $\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - xy}$.

Solution: The given equation is a homogeneous equation, since both the numerator and denominator are homogeneous functions each of degree 2.

Now put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

so that the given equation becomes $v + x\frac{dv}{dx} = \frac{v^2 - 2v}{1 - v}$.

Hence, $x\frac{dv}{dx} = \frac{2v^2 - 3v}{1 - v}$ so that $\frac{1 - v}{2v^2 - 3v}dv = \frac{dx}{x}$.

Therefore, $\int \frac{1 - v}{2v^2 - 3v}dv = \int \frac{dx}{x}$.

Hence, $-\frac{1}{3}\int\left(\frac{1}{v} + \frac{1}{2v - 3}\right)dv = \log x - \log c$

so that $-\frac{1}{3}\left[\log v + \frac{1}{2}\log(2v - 3)\right] = \log x - \log c$

that is, $-\frac{1}{3}\log(v\sqrt{2v - 3}) = \log x - \log c$

that is, $\log(v\sqrt{2v-3}) = -3\log x + 3\log c = -\log x^3 + \log c^3$

that is, $\log(x^3 v\sqrt{2v-3}) = \log c^3$.

Hence $x^3 v(\sqrt{2v-3}) = c^3$.

Put $v = \frac{y}{x}$. Then $x^3 \frac{y}{x}\sqrt{\frac{2y}{x}-3} = c^3$

that is, $x^2 y\sqrt{\frac{2y}{x}-3} = c^3$ (or) $xy\sqrt{2xy-3x^2} = c^3$.

This is the general solution of the given equation.

Example 28.19: Solve $(x^2 + y^2)dx = 2xy\,dy$.

Solution : The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} \quad \text{... (1)}$$

which is a homogeneous equation, since the numerator and denominator on the right are homogeneous functions each of degree 2. Put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Therefore, (1) becomes

$$v + x\frac{dv}{dx} = \frac{x^2(1+v^2)}{2x^2 v} = \frac{1+v^2}{2v} \text{ so that } x\frac{dv}{dx} = \frac{1-v^2}{2v}.$$

Hence, $\frac{2v}{1-v^2}dv = \frac{dx}{x}$ so that $\int \frac{2v}{1-v^2}dv = \int \frac{dx}{x}$

that is, $-\log(1-v^2) = \log x + \log c$

so that $\log[xc(1-v^2)] = 0 = \log 1$.

Hence $xc(1-v^2) = 1$

that is, $c(x^2 - y^2) = x$ $\left(\text{since } v = \frac{y}{x}\right)$

which is the general solution of the given equation.

Example 28.20: Solve $xy^2dy - (x^3 + y^3)dx = 0.$

Solution : The given equation can be written as

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2} \qquad \text{... (1)}$$

which is a homogeneous equation.

Put $y = vx$. Then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Therefore, (1) becomes $v + x\dfrac{dv}{dx} = \dfrac{1+v^3}{v^2}$

so that $x\dfrac{dv}{dx} = \dfrac{1}{v^2}$ (or) $v^2\, dv = \dfrac{dx}{x}$.

Therefore, $\int v^2 dv = \int \dfrac{dx}{x}$ so that $\dfrac{v^3}{3} = \log x + \log c$

that is, $\dfrac{y^3}{3x^3} = \log x + \log c$ (or) $y^3 = 3x^3 \log (cx)$

which is the general solution of the given equation.

Example 28.21: Solve $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2x^2}$. ... (1)

Solution : The given equation is a homogeneous equation. Put $y = vx$. Then

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ so that (1) becomes $v + x\dfrac{dv}{dx} = \dfrac{1+v^2}{2}$,

that is, $2x\, dv = (1 + v^2 - 2v)dx$. Separating variables, we have

$$\frac{2dv}{(v-1)^2} = \frac{dx}{x}.$$

Integrating, we get

$$\frac{-2}{v-1} = \log x + c.$$

But $v = \dfrac{y}{x}$, so

$$-\frac{2}{v-1} = \frac{-2}{\frac{y}{x}-1} = \frac{-2x}{y-x} = \frac{2x}{x-y}.$$

Hence $\quad \dfrac{2x}{x-y} = \log x + c$

so that $2x = (x-y)(\log x + c)$ which is the general solution of the given equation.

Example 28.22: Solve $(x^3 - 3xy^2)dx + (3x^2y - y^3)dy = 0.$

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y}. \qquad \text{... (1)}$$

Therefore, the given equation is a homogeneous equation.

Put $y = vx$. Then $v + x\dfrac{dv}{dx} = \dfrac{dy}{dx}$ so that (1) becomes

$$v + x\frac{dv}{dx} = \frac{1-3v^2}{v^3 - 3v}. \qquad \text{... (2)}$$

Therefore, $x\dfrac{dv}{dx} = \dfrac{1-v^4}{v^3-3v} = \dfrac{v^4-1}{3v-v^3}$ so that $\dfrac{3v-v^3}{(v+1)(v-1)(v^2+1)}dv = \dfrac{dx}{x}$

that is $\left[\dfrac{1}{2(v+1)} + \dfrac{1}{2(v-1)} - \dfrac{2v}{v^2+1}\right]dv = \dfrac{dx}{x}$ (by partial fractions).

Integrating, we get

$$\frac{1}{2}\log(v+1) + \frac{1}{2}\log(v-1) - \log(v^2+1) = \log x + \log c$$

that is, $\quad \log\left[\dfrac{\sqrt{v+1}\,\sqrt{v-1}}{v^2+1}\right] = \log(cx)$ so that $\dfrac{\sqrt{v^2-1}}{v^2+1} = cx$

(or) $\quad \dfrac{v^2-1}{(v^2+1)^2} = c^2x^2.$

Since $y = \dfrac{v}{x}$, $\; y^2 - x^2 = c^2(y^2+x^2)$

which is the required general solution.

EXERCISE 28.3

1. Express $\left(x - y\tan^{-1}\frac{y}{x}\right)dx + x\tan^{-1}\frac{y}{x}dy = 0$ in the form $F\left(\frac{y}{x}\right) = \frac{dy}{dx}$.

2. Solve the following differential equation.

   (i) $\frac{dy}{dx} = \frac{x-y}{x+y}$

   (ii) $(x^2 + y^2)\ dy = 2xy\ \ dx$

   (iii) $y^2dx + (x^2 - xy)\ dy = 0$

   (iv) $\frac{dy}{dx} = \frac{(x+y)^2}{2x^2}$

   (v) $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

   (vi) $(x^2 - y^2)\ \frac{dy}{dx} = xy$

3. Solve the following differential equations.

   (i) $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

   (ii) $x\sin\frac{y}{x} \cdot \frac{dy}{dx} = y\sin\frac{y}{x} - x$

   (iii) $\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x}$ where $x > 0,\ y > 0$ and which passes through the point $\left(1, \frac{\pi}{4}\right)$

28.6.4 Non-Homogeneous Differential Equations

Differential equations of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} \qquad \dots (1)$$

where a, b, c, a', b', c' are constants and c and c' are not both zero are called **non-homogeneous equations**. We reduce (1) to a homogeneous equation by suitable substitutions for x and y.

We explain three methods (in case (i), case (ii) and case (iii)) of solving (1) depending on the nature of coefficients of x and y in the numerator and denominator of the R.H.S. of (1).

Case(i)

Suppose that $b = -a'$. Then (1) becomes $\dfrac{dy}{dx} = \dfrac{ax - a'y + c}{a'x + b'y + c'}$.

Therefore, $(a'x + b'y + c')dy - (ax - a'y + c)dx = 0$

that is, $a'(x\,dy + y\,dx) + b'y\,dy - ax\,dx + c'dy - c\,dx = 0$

that is, $a'd(xy) + b'd\left(\dfrac{y^2}{2}\right) - ad\left(\dfrac{x^2}{2}\right) + c'\,dy - c\,dx = 0.$

Integrating, we get $a'\,xy + b'\dfrac{y^2}{2} - a\dfrac{x^2}{2} + c'y - cx = k$

which is the required solution.

Note : In the above case solution can be obtained by integrating each term after regrouping.

Example 28.23 : Let us solve $\dfrac{dy}{dx} = \dfrac{3x - y + 7}{x - 7y - 3}$.

Here $b = -1 = -a'$. Hence we can solve by case(i). Now $(x - 7y - 3)dy - (3x - y + 7)dx = 0$.

Therefore, $(x\,dy + y\,dx) - 7y\,dy - 3\,dy - 3x\,dx - 7dx = 0$

that is, $d(xy) - 7d\left(\dfrac{y^2}{2}\right) - 3dy - 3d\left(\dfrac{x^2}{2}\right) - 7dx = 0.$

Integrating, we get

$$xy - \frac{7y^2}{2} - 3y - 3\frac{x^2}{2} - 7x = c$$

(or) $\quad 2xy - 7y^2 - 6y - 3x^2 - 14x = 2c$

which is the required solution.

Case(ii) : Suppose that $\frac{a}{a'} = \frac{b}{b'} = m(\text{say})$.

Then (1) becomes

$$\frac{dy}{dx} = \frac{ax+by+c}{\frac{1}{m}(ax+by)+c'} \qquad \text{... (2)}$$

Put $ax+by=v$. Then $a+b\frac{dy}{dx} = \frac{dv}{dx}$.

Therefore, $\quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$

so that (2) becomes $\quad \frac{1}{b}\left(\frac{dv}{dx} - a\right) = \frac{v+c}{\frac{v}{m}+c'}$.

Therefore, $\quad \frac{dv}{dx} = \frac{bm(v+c)}{v+c'm} + a$

that is, $\quad \frac{v+c'm}{bm(v+c)+a(v+c'm)}dv = dx$

which can be solved by variables separable method.

Example 28.24 : We shall solve $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$.

Here $a=1,\ b=-1\ a'=2,\ b'=-2$ and hence

$$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{2}.$$

Therefore, we can solve the equation by case(ii).

Put $x-y=v$. Then $1-\frac{dy}{dx} = \frac{dv}{dx}$

so that the given equation becomes

$$1-\frac{dv}{dx} = \frac{v+3}{2v+5}$$

that is, $\frac{dv}{dx} = \frac{v+2}{2v+5}$

so that $dx = \frac{2v+5}{v+2} dv = \left(2 + \frac{1}{v+2}\right) dv$.

Integrating, we get

$$x = 2v + \log(v+2) + c$$

that is, $x = 2(x-y) + \log(x-y+2) + c$

which is the required solution.

Note : If $b = -a'$ with $\frac{a}{a'} = \frac{b}{b'}$, then the given equation can be solved easily by using case (i) rather than case (ii).

Case(iii) : Suppose that $b \neq -a'$ and $\frac{a}{a'} \neq \frac{b}{b'}$.

Then taking $x = X + h,\ y = Y + k$, where X and Y are variables and h and k are constants, we get $\frac{dy}{dx} = \frac{dY}{dX}$.

Hence (1) becomes

$$\frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{a'(X+h) + b'(Y+k) + c'}$$

that is, $$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \quad \text{... (i)}$$

Now choose constants h and k such that

$$ah + bk + c = 0 \quad \text{... (ii)}$$

and $$a'h + b'k + c' = 0 \quad \text{... (iii)}$$

Since $\frac{a}{a'} \neq \frac{b}{b'}$, we can solve (ii) and (iii) for h and k. Hence (1) becomes

$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$$

which is a homogeneous equation in X and Y and hence can be solved by homogeneous equation method, that is by putting Y = VX.

Example 28.25 : We shall solve $(2x + y + 3)dx = (2y + x + 1)dy$.

The given equation can be written as

$$\frac{dy}{dx} = \frac{2x+y+3}{2y+x+1} \quad \text{... (i)}$$

Here $a=2,\ b=1,\ a'=1,\ b'=2$. Hence, $b \neq -a'$ and $\frac{a}{a'} \neq \frac{b}{b'}$.

Therefore, the given equation can be solved by case (iii).

Put $x = X+h,\ y = Y+k$ in (i).

Then $\frac{dy}{dx} = \frac{dY}{dX}$ and $\frac{dY}{dX} = \frac{2X+Y+2h+k+3}{2Y+X+2k+h+1}$ ... (ii)

Now choose h and k such that

$$2h+k+3=0 \text{ and } h+2k+1=0.$$

Solving them for h and k, we get $h = -\frac{5}{3},\ k = \frac{1}{3}$.

Hence (ii) becomes $\frac{dY}{dX} = \frac{2X+Y}{2Y+X}$ ... (iii)

which is a homogeneous equation.

Put $Y = VX$. Then $\frac{dY}{dX} = V + X\frac{dV}{dX}$.

Therefore, (iii) becomes $V + X\frac{dV}{dX} = \frac{2+V}{2V+1}$

that is, $X\frac{dV}{dX} = \frac{2(1-V^2)}{2V+1}$ and hence $\frac{2V+1}{(1+V)(1-V)}dV = \frac{2dX}{X}$.

that is, $\frac{3}{2(1-V)}dV - \frac{1}{2(1+V)}dV = \frac{2dX}{X}$.

Integrating, we get $-\frac{3}{2}\log(1-V) - \frac{1}{2}\log(1+V) = 2\log X - \log c$

that is, $3\log(1-V) + \log(1+V) + 4\log X = 2\log c$

(or) $\log[(1-V)^3\,(1+V)X^4] = \log c^2$

so that $X^4(1-V)^3(1+V) = c^2$.

Since $V=\frac{Y}{X}$, we get $X^4\left(1-\frac{Y}{X}\right)^3\left(1+\frac{Y}{X}\right)=c^2$

that is, $(X+Y)(X-Y)^3=c^2$.

Substituting for X and Y, we get,

$$\left(x+\frac{5}{3}+y-\frac{1}{3}\right)\left(x+\frac{5}{3}-y+\frac{1}{3}\right)^3=c^2$$

(or) $\left(x+y+\frac{4}{3}\right)(x-y+2)^3=c^2$

which is the required solution.

EXERCISE 28.4

I. Solve the following differential equations.

(i) $\frac{dy}{dx}=\frac{-3x-2y+5}{2x+3y-5}$

(ii) $\frac{dy}{dx}=\frac{x-y+2}{x+y-1}$

II. Solve the following differential equations.

(i) $(2x+2y+3)\frac{dy}{dx}=x+y+1$

(ii) $\frac{dy}{dx}=\frac{4x+6y+5}{3y+2x+4}$

III. Solve the following differential equations.

(i) $(x-y-2)dx+(x-2y-3)dy=0$

(ii) $(x-y)dy=(x+y+1)dx$

28.6.5 Linear Differential Equation

Consider the equation

$$\frac{dy}{dx}+Py=Q \qquad ...(1)$$

where P and Q are functions of x. This is linear equation of order one.

To solve equation (1), we first multiply both sides of equation (1) by $e^{\int P dx}$

$$e^{\int P dx}\frac{dy}{dx} + Py\, e^{\int P dx} = Qe^{\int P dx}$$

or
$$\frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx} \qquad ...(2)$$

$$\left[\because \frac{d}{dx}\left(ye^{\int P dx}\right) = e^{\int P dx}\frac{dy}{dx} + Py.e^{\int P dx}\right]$$

On integrating, we get

$$ye^{\int P dx} = \int Qe^{\int P dx}dx + C$$

where C is an arbitrary constant,

or
$$y = e^{-\int P dx}\left[\int Qe^{\int P dx}dx + C\right]$$

Note: $e^{\int P dx}$ is called the integrating factor of the equation and is written as I.F in short.

Remarks

(i) We observe that the left hand side of the linear differential equation (1) has become $\frac{d}{dx}\left(ye^{\int P dx}\right)$ after the equation has been multiplied by the factor $e^{\int P dx}$.

(ii) The solution of the linear differential equation

$$\frac{dy}{dx} + Py = Q$$

P and Q being functions of x only is given by

$$ye^{\int P dx} = \int Q\left(e^{\int P dx}\right)dx + C$$

(iii) The coefficient of $\frac{dy}{dx}$ if not unity, must be made unity by dividing the equation by it throughout.

(iv) Some differential equations become linear differential equations if y is treated as the independent variable and x is treated as the dependent variable.

For example, $\frac{dy}{dx} + Px = Q$ where P and Q are functions of y only, is also a linear differential equation of the first order.

In this case $\quad I.F = e^{\int P dy}$, P & Q are functions of y.

and the solution is given by

$$x(\text{I.F}) = \int Q\,(\text{I.F.})dy + C.$$

Example 28.26: Find Integrating Facter of the differenital equation

$(\cos x)\frac{dy}{dx} + y\sin x = \tan x.$

Solution : The above equation can be written as $\frac{dy}{dx} + (\tan x)y = \sec x.\tan x.$

Therefore, $P = \tan x$ and hence $\int P dx = \int \tan x\,dx = \log \sec x$ so that

$$\text{I.F} = e^{\int P dx} = e^{\log \sec x} = \sec x.$$

Example 28.27: Solve $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$.

Solution : The given equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}. \qquad ...(1)$$

Here $\quad P = \frac{2x}{1+x^2}, \quad Q = \frac{4x^2}{1+x^2}$

Hence $\quad \text{I.F.} = e^{\int P dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$.

Multiplying both sides of (1) by $1+x^2$, we get

$$\frac{d}{dx}\left[(1+x^2)\,y\right] = 4x^2.$$

Integrating, we get

$$(1+x^2)y = \frac{4x^3}{3} + c$$

that is, $3y(1+x^2) = 4x^3 + 3c$ which is the required solution.

Example 28.28: Solve $\frac{1}{x}\frac{dy}{dx} + ye^x = e^{(1-x)e^x}$.

Solution : The given equation can be written as

$$\frac{dy}{dx} + xe^x y = x\, e^{(1-x)e^x} \quad \ldots (1)$$

Here $\quad P = xe^x$ and $Q = x\; e^{(1-x)e^x}$.

Therefore, $\quad \text{I.F.} = e^{\int x\, e^x dx} = e^{(x-1)e^x}$.

Multiplying both sides of (1) by $e^{(x-1)e^x}$ and then integrating, we get

$$y\, e^{(x-1)e^x} = \int x dx + c$$

that is, $\quad y\, e^{(x-1)e^x} = \frac{x^2}{2} + c \quad$ (or) $\quad 2y\, e^{(x-1)e^x} = x^2 + 2c$

which is the required solution.

Example 28.29: Solve $(1+y^2)dx = (\text{Tan}^{-1}y - x)\, dy$

Solution : The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\text{Tan}^{-1}y}{1+y^2} \quad \ldots (1)$$

which is linear in x.

Here $\quad P = \frac{1}{1+y^2}$, $Q = \frac{\text{Tan}^{-1}y}{1+y^2}$ so that

$$\text{I.F.} = e^{\int \frac{dy}{1+y^2}} = e^{\text{Tan}^{-1}y}.$$

Multiplying both sides of (1) by $e^{\text{Tan}^{-1}y}$ and then integrating, we get

$$x\, e^{\text{Tan}^{-1}y} = \int e^{\text{Tan}^{-1}y}\, \frac{\text{Tan}^{-1}y}{1+y^2}\, dy + c \quad \ldots (2)$$

Now put $\text{Tan}^{-1}y = t$. Then $\frac{1}{1+y^2}\, dy = dt$.

Hence (2) becomes

$$xe^t = \int t\, e^t\, dt + c = e^t\,(t-1) + c$$

so that $\quad x\, e^{\text{Tan}^{-1}y} = e^{\text{Tan}^{-1}y}\,(\text{Tan}^{-1}y - 1) + c$

which is the required solution.

Example 28.30: Solve

$$\frac{dy}{dx}+\frac{y}{x}=e^{-x}$$

Solution: Here $\mathrm{P}=\frac{1}{x}$, $\mathrm{Q}=e^{-x}$ (Note that both P an Q are functions of x)

I.F. (Integrating Factor) $e^{\int \mathrm{P}dx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x(x>0)$

On multiplying both sides of the equation by I.F., we get

$$x.\frac{dy}{dx}+y=x.e^{-x} \quad \text{or} \quad \frac{d}{dx}(y\,.\,x)=x.e^{-x}$$

Integrating both sides, we have

$$yx=\int xe^{-x}\,dx+\mathrm{C}$$

where C is an arbitrary constant

or $$xy=-xe^{-x}+\int e^{-x}dx+\mathrm{C}$$

or $$xy=-xe^{-x}-e^{-x}+\mathrm{C}$$

or $$xy=-e^{-x}(x+1)+\mathrm{C}$$

or $$y=-\left(\frac{x+1}{x}\right)e^{-x}+\frac{\mathrm{C}}{x}$$

Note: In the solution x > 0.

Example 28.31: Solve :

$$\sin x\frac{dy}{dx}+y\cos x=2\sin^2 x\cos x$$

Solution: The given differential equation is

$$\sin x\frac{dy}{dx}+y\cos x=2\sin^2 x\cos x$$

or $$\frac{dy}{dx}+y\cot x=2\sin x\cos x \quad ...(1)$$

Here $P = \cot x, \quad Q = 2 \sin x \cos x$

$$\text{I.F.} = e^{\int P dx} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$$

On multiplying both sides of equation (1) by I.F., we get ($\sin x > 0$)

$$\frac{d}{dx}(y \sin x) = 2\sin^2 x.\cos x$$

Further on integrating both sides, we have

$$y \sin x = \int 2\sin^2 x.\cos x dx + C$$

where C is an arbitrary constant ($\sin x > 0$)

or $$y \sin x = \frac{2}{3}\sin^3 x + C,$$ which is the required solution.

Example 28.32: Solve $\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$

Solution: The given differential equation is

$$\left(1+y^2\right)\frac{dx}{dy} = \tan^{-1} y - x$$

or $$\frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$$

or $$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \qquad ...(1)$$

which is of the form $\frac{dx}{dy} + Px = Q$ where P and Q are the functions of y only.

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Multiplying both sides of equation (1) by I.F., we get

$$\frac{d}{dy}(xe^{\tan^{-1} y}) = \frac{\tan^{-1} y}{1+y^2}(e^{\tan^{-1} y})$$

On integrating both sides, we get

or $$(e^{\tan^{-1} y})x = \int e^t.tdt + C$$

where C is an arbitrary constant and $t = \tan^{-1} y$ and $dt = \frac{1}{1+y^2}dy$

or $$(e^{\tan^{-1} y})x = te^t - \int e^t + C$$

or $$(e^{\tan^{-1} y})x = te^t - e^t + C$$

or $$(e^{\tan^{-1} y})x = \tan^{-1} y\, e^{\tan^{-1} y} - e^{\tan^{-1} y} + C$$ (on putting $t = \tan^{-1}y$)

or $$x = \tan^{-1} y - 1 + Ce^{\tan^{-1} y}.$$

EXERCISE 28.5

1. And the I.F of the following differential equation by transforming then into linear form.

   (i) $x\frac{dx}{dy} - y = 2x^2 \sec^2 2x$

   (ii) $y\frac{dx}{dy} - x = 2y^3$

2. Solve the following differential equation.

   (i) $\frac{dx}{dy} + y\tan x = \cos^3 x$

   (ii) $\frac{xdy}{dx} + 2y = \log x$

   (iii) $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x}$

   (iv) $\frac{dy}{dx} + \frac{2y}{x} = 2x^2$

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Notes

3. Solve the differential equations

(i) $x \log x \cdot \dfrac{dy}{dx} + y = 2\log x$

(ii) $(x + y + 1)\dfrac{dy}{dx} = 1$

KEY WORDS

- A diffferential equation is an equation involving independent variable, dependent variable and the derivatives of dependent variable (and differentials) with respect to independent variable.
- The order of a differential equation is the order of the highest derivative occurring in it.
- The degree of a differential equation is the degree of thehighest derivative after it has been freed from radicals and fractions as far as the derivatives are concerned.
- A differential equation in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together is called a linear differential equation.
- A linear differential equation is always of the first degree.
- A general solution ofa differential equation is that solution which ontains as many as the number of arbitrary constants as the order of the differential equation.
- A general solution becomes a particular solution when particular values of the arbitrary constants are determined satisfying the given conditions.
- The solution of the diffferential equation of the type $\dfrac{dy}{dx} = f(x)$ is obtained by integrating both sides.
- The solution of the differential equation of the type $\dfrac{dy}{dx} = f(x), g(y)$ is obtained after separting the variables and integrating both sides.

- The differential equation M(x, y) dx + N(x, y) dy = 0 is called homogeneous if M(x, y) and N(x, y) are homogeneous and are the same degree.
- The solution of a homogeneous differential equation is obtained by substituting $y= vx$ or $x= vy$ and then separating the variables.
- The solution of the first order linear equation $\frac{dy}{dx}+Py=Q$ is

  $$y\,e^{\int Pdx}=\int Q\left(e^{\int Pdx}\right)dx+C,$$ where C is an arbitrary constant.

  The expression $e^{\int Pdx}$ is called the integrating factor of the differential equation and is written as I.F. in short.

SUPPORTIVE WEB SITES

http : //www.wikipedia.org

http:// math world . wolfram.com

PRACTICE EXERCISE

1. Find the order and degree of the differential equation :

(a) $\left(\frac{d^2y}{dx^2}\right)^2+x^2\left(\frac{dy}{dx}\right)^4=0$ (b) $xdx+ydy=0$

(c) $\frac{d^4y}{dx^4}-4\frac{dy}{dx}+4y=5\cos 3x$ (d) $\frac{dy}{dx}=\cos x$

(e) $x^2\frac{d^2y}{dx^2}-xy\frac{dy}{dx}=y$ (f) $\frac{d^2y}{dx^2}+y=0$

(g) $y=x\frac{dy}{dx}+a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ (h) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}=a\frac{d^2y}{dx^2}$

2. Find which of the following equations are linear and which are non-linear

(a) $\frac{dy}{dx}=\cos x$ (b) $\frac{dy}{dx}+\frac{y}{x}=y^2\log x$

(c) $\left(\frac{d^2y}{dx^2}\right)^3 + x^2\left(\frac{dy}{dx}\right)^2 = 0$ (d) $x\frac{dy}{dx} - 4 = x$

(e) $dx + dy = 0$

3. From the differential equation corresponding to $y^2 - 2ay + x^2 = a^2$ by eliminating a.

4. Find the differential equation by eliminating a, b, c form $y = ax^2 + bx + c$ Write its order and degree.

5. How many constants are contained in the general solution of

   (a) Second order differential equation.

   (b) Differential equation of order three.

   (c) Differential equation of order five.

6. Show that $y = a\cos(\log x) + b\sin(\log x)$ is a solution of the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

7. Solve the following differential equations :

   (a) $\sin^2 x\frac{dy}{dx} = 3\cos x + 4$ (b) $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$

   (c) $\frac{dy}{dx} + \frac{\cos x \sin y}{\cos y} = 0$ (d) $dy + xydx = xdx$

   (e) $\frac{dy}{dx} + y\tan x = x^m \cos mx$ (f) $(1+y^2)\frac{dx}{dy} = \tan^{-1} y - x$

ANSWERS

EXERCISE 28.1

1. Order is 1 and degree is 2.

2. (a) Order 2, degree 1 (b) Order 2, degree 1

   (c) Order 1, degree 1 (d) Order 2, degree 2

3. (a) Non - linear (b) Linear (c) Linear (d) Non - linear

4. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$

5. (a) $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx}=0$

(b) $(x^2-2y^2)\left(\frac{dy}{dx}\right)^2 - 4xy\frac{dy}{dx} - x^2 = 0$

(c) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0$

(d) $y=(x-3)\frac{dy}{dx}+2$

(e) $(x^2-y^2)\frac{dy}{dx}-2xy=0$

EXERCISE 28.2

1. (i) $e^{-y} = e^{-x} + C$

(ii) $\tan^{-1}y = \tan^{-1}x + C$

(iii) $e^y = e^x + \frac{x^3}{3} + C$

(iv) $e^y = k(y + 1)(1 + e^{-x})$

2. (i) $(x^2 + 1)(y^2 + 1) = C$

(ii) $\tan(x + y) - \sec(x + y) = x + C$

(iii) $\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) = C$

(iv) $x-y-\frac{1}{2}\sin\left[2(x+y)\right]=C$

EXERCISE 28.3

1. $$\frac{dy}{dx}=\frac{\frac{y}{x}\tan^{-1}\left(\frac{y}{x}\right)-1}{\tan^{-1}\left(\frac{y}{x}\right)}$$

2. (i) $x^2 - 2xy - y^2 = A$

   (ii) $k(x^2 - y^2) = y$

   (iii) $ky = e^{y/x}$

   (iv) $\log\left(\frac{x+y}{c}\right)=\frac{-2xy}{(x+y)^2}$

   (v) $y - 2x = kx^2y$

   (vi) $x^2 + 2y^2(c + \log y) = 0$

3. (i) $x + ye^{x/y} = k$

   (ii) $kx = e^{\cos(y/x)}$

   (iii) $\tan\left(\frac{y}{x}\right)=1-\log x$

EXERCISE 28.4

1. (i) $4xy + 3(x^2 + y^2) - 10(x + y) = k$

   (ii) $y^2 - x^2 + 2xy - 2y - 4x = c$

2. (i) $6y - 3x + \log(3x + 3y + 4) = c$

   (ii) $y-2x+\frac{3}{8}\log(24y+16x+23)=k$

3. (i) $x^2 - 2y^2 - 2x - 4y - 2 = c\left[\dfrac{x - y\sqrt{2} - \sqrt{2} - 1}{x + y\sqrt{2} + \sqrt{2} - 1}\right]^{1/\sqrt{2}}$

   (ii) $2\tan^{-1}\left(\dfrac{2y+1}{2x+1}\right) = \log\left|c^2\left(x^2 + y^2 + x + y + \dfrac{1}{2}\right)\right|$

EXERCISE 28.5

1. (i) $\dfrac{1}{x}$ (ii) $\dfrac{1}{y}$

2. (i) $2y = x \cos x + \sin x \cos^2 x + c \cos x$

   (ii) $yx^2 = \dfrac{x^2}{2}\log x - \dfrac{x^2}{4} + c$

   (iii) $2y\, e^{\tan^{-1} x} = e^{2\tan^{-1} x} + c$

   (iv) $yx^2 = \dfrac{2x^5}{5} + c$

3. (i) $y \log x = (\log x)^2 + c$

   (ii) $x = ke^y - (y+2)$

PRACTICE EXERCISE

1. (a) order 2, degree 3
   (b) Order 1, degree 1
   (c) Order 4, degree 1
   (d) Order 1, degree 1
   (e) Order 2, degree 1
   (f) Order 2, degree 1
   (g) Order 1, degree 2
   (h) Order 2, degree 1

2. (a), (d), (e) are linear;

   (b), (c) are non-linear

3. $(x^2 - 2y^2)\left(\dfrac{dy}{dx}\right)^2 - 4xy\left(\dfrac{dy}{dx}\right) - x^2 = 0$

4. $\frac{d^3y}{dx^3}=0$, Order 3, degree 1.

5. (a) Two (b) Three (c) Five

7. (a) $y + 3 \operatorname{cosec} x + 4 \cot x = C$ (b) $e^y = e^x + \frac{x^3}{3} + C$

(c) $\sin y = Ce^{-\sin x}$ (d) $\log(1-y) + \frac{x^2}{2} = C$

(e) $y = \frac{x^{m+1}}{m+1}\cos x + C\cos x$ (f) $x = \tan^{-1} y - 1 + Ce^{-\tan^{-1} y}$

Chapter 29

MEASURES OF DISPERSION

LEARNING OUTCOMES

After studying this lesson, student will be able to:

- Explain the purpose of measures of dispersion;
- Compute and explain the various measures of dispersion-range, mean deviation, variance and standard deviation;
- Compute mean deviation from the mean and median of ungrouped data and grouped data;
- Calculate variance and standard deviation for grouped and ungrouped data
- Demonstrate the properties of variance and standard deviation.

PREREQUISITES

- Mean of grouped and raw data
- Median of grouped and ungrouped data

INTRODUCTION

The measures of central tendency are not adequate to describe data. Two data sets can have the same mean but they can be entirely different.

In order to understand it, let us consider an example.

The daily income of the workers in two factories are :

Factory A :	35	45	50	65	70	90	100
Factory B :	60	65	65	65	65	65	70

Here we observe that in both the groups the mean of the data is the same, namely, 65

(i) In group A, the observations are much more scattered from the mean.

(ii) In group B, almost all the observations are concentrated around the mean.

Certainly, the two groups differ even though they have the same mean.

Thus, there arises a need to differentiate between the groups. We need some other measures which concern with the measure of scatteredness (or spread).

To do this, we study what is known as **measures of dispersion.**

29.1 MEANING OF DISPERSION

To explain the meaning of dispersion, let us consider an example.

Two sections of 10 students each in class X in a certain school were given a common test in Mathematics (40 maximum marks). The scores of the students are given below :

Section A :	6	9	11	13	15	21	23	28	29	35
Section B :	15	16	16	17	18	19	20	21	23	25

The average score in section A is 19.

The average score in section B is 19.

Let us construct a dot diagram, on the same scale for section A and section B (see Fig. 29.1)

The position of mean is marked by an arrow in the dot diagram.

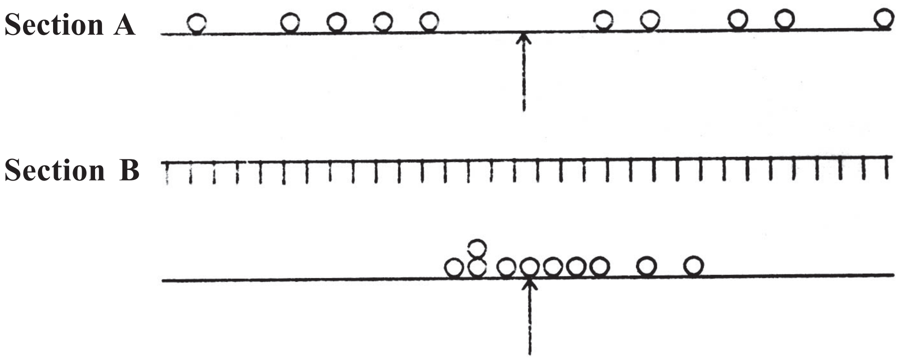

Fig. 29.1

Clearly, the extent of spread or dispersion of the data is different in section A from that of B. The measurement of the scatter of the given data about the average is said to be a measure of dispersion or scatter.

In this lesson, you will read about the following measures of dispersion :

(a) Range

(b) Mean deviation from mean

(c) Variance

(d) Standard deviation

29.2 DEFINITION OF VARIOUS MEASURES OF DISPERSION

(a) **Range :** In the above cited example, we observe that

(i) the scores of all the students in section A are ranging from 6 to 35;

(ii) the scores of the students in section B are ranging from 15 to 25.

The difference between the largest and the smallest scores in section A is 29(35 − 6)

The difference between the largest and smallest scores in section B is 10 (25 − 15).

Thus, the difference between the largest and the smallest value of a data, is termed as the range of the distribution.

(b) **Mean Deviation from Mean :** In Fig. 29.1, we note that the scores in section B cluster around the mean while in section A the scores are spread away from the mean. Let us take the deviation of each observation from the mean and add all such deviations. If the sum is 'large', the dispersion is 'large'. If, however, the sum is 'small' the dispersion is small.

Let us find the sum of deviations from the mean, i.e., 19 for scores in section A.

Observations (x_i)	**Deviations from the mean ($x_i - \bar{x}$)**
6	-13
9	-10
11	-8
13	-6
15	-4
21	2
23	4
28	9
29	10
35	16
190	0

Here, the sum is zero. It is neither 'large' nor 'small'. Is it a coincidence?

Let us now find the sum of deviations from the mean, i.e., 19 for scores in section B.

Observations (x_i)	Deviations from the mean ($x_i - \bar{x}$)
15	-4
16	-3
17	-2
18	-1
19	0
20	1
21	2
23	4
25	6
190	0

Again, the sum is zero. Certainly it is not a coincidence. In fact, we have proved earlier that **the sum of the deviations taken from the mean is always zero for any set of data.** Why is the sum always zero ?

On close examination, we find that the signs of some deviations are positive and of some other deviations are negative. Perhaps, this is what makes their sum always zero. In both the cases, deviations. But this can be avoided if we take only the **absolute value of the deviations** and then take their sum.

If we follow this method, we will obtain a measure (descriptor) called the mean deviation from the mean.

The mean deviation is the sum of the absolute values of the deviations from the **mean divided by the number of items,** (i.e., the sum of the frequencies).

(c) **Variance :** In the above case, we took the absolute value of the deviations taken from mean to get rid of the negative sign of the deviations. Another method is to square the deviations. Let us, therefore, square the deviations from the mean and then take their sum. If we divide this sum by the number of observations (i.e., the sum of the frequencies), we obtain the average of deviations, which is called variance. **Variance is usually denoted by** σ^2.

(d) **Standard Deviation :** If we take the positive square root of the variance, we obtain the root mean square deviation or simply called standard deviation and is denoted by σ.

29.3 MEAN DEVIATION FROM MEAN OF RAW AND GROUPED DATA

$$\text{Mean Deviation from mean of raw data} = \frac{\sum_{i=1}^{n} |x_i - \overline{x}|}{N}$$

$$\text{Mean deviation from mean of grouped data} = \frac{\sum_{i=1}^{n} [f_i |x_i - \overline{x}|]}{N}$$

where $N = \sum_{i=1}^{n} f_i, \quad \overline{x} = \frac{1}{N}\sum_{i=1}^{n} f\,|x_i|$

The following steps are employed to calculate the mean deviation from mean.

Step 1 : Make a column of deviation from the mean, namely $x_i - \overline{x}$ (In case of grouped data take x_i as the mid value of the class.)

Step2 : Take absolute value of each deviation and write in the column headed $|x_i - \overline{x}|$.

For calculating the mean deviation from the mean of raw data use

$$\text{Mean deviation of Mean} = \frac{\sum_{i=1}^{n} |x_i - \overline{x}|}{N}$$

For grouped data proceed to step 3.

Step 3 : Multiply each entry in step 2 by the corresponding frequency. We obtain $f_i(x_i - \overline{x})$ and write in the column headed $f_i |x_i - \overline{x}|$.

Step 4 : Find the sum of the column in step 3. We obtain $\sum_{i=1}^{n} [f_i |x_i - \overline{x}|]$

Step 5: Divide the sum obtained in step 4 by N.

Now let us take few examples to explain the above steps.

Example 29.1 Find the mean deviation from the mean of the following data

Sizeofitems (x_i)	5	7	9	10	12	15	16
Frequency (f_i)	2	4	5	8	4	2	3

Solution :

x_i	f_i	$f_i\,x_i$	$x_i - \overline{x}$	$\lvert x_i - \overline{x} \rvert$	$f_i(x_i - \overline{x})$
5	2	10	-5.3	5.3	10.6
7	4	28	-3.3	3.3	13.2
9	5	45	-1.3	1.3	6.5
10	8	80	0.7	0.7	5.6
12	4	48	1.7	1.7	6.8
15	2	30	4.7	4.7	9.4
16	3	48	5.7	5.7	17.1
	28	289			69.2

$$\overline{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{289}{28} = 10.7$$

Mean deviation from mean $= \dfrac{\Sigma[f_i \lvert x_i - \overline{x} \rvert]}{N}$

$$= \frac{62.7}{28} = 2.471$$

Example 29.2 Calculate the mean deviation from mean of the following distribution

Marks	0-10	10-20	20-30	30-40	40-50
No. of Students	6	7	17	16	4

Solution :

Marks	Class Marks x_i	f_i	$f_i\, x_i$	$x_i - \overline{x}$	$\lvert x_i - \overline{x} \rvert$	$f_i(x_i - \overline{x})$
0-10	5	6	30	-21	21	126
10-20	15	7	105	-11	11	77
20-30	25	17	425	-1	1	17
30-40	35	16	560	9	9	144
40-50	45	4	180	19	19	76
		50	1300			440

$$\overline{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{1300}{50} = 26$$

Mean deviation from Mean $= \dfrac{\Sigma[f_i \lvert x_i - \overline{x} \rvert]}{\text{N}} = \dfrac{440}{50} = 8.8$

29.4 MEAN DEVIATION FROM MEDIAN

Median of Discrete Frequency Deistribution.

Step 1 : Arrange the data in ascending order.

Step 2 : Find the sum of the frequencies $\Sigma f_i = \text{N}$

Step 3 : Find cumulative frequencies

Step 4 : Find $\dfrac{\text{N}}{2}$

Step 5 : The observation whose cumulative frequency is equal to or just greater than $\dfrac{\text{N}}{2}$ is the median of the data.

$$\sum_{i=1}^{n} f_i \lvert x_i - \text{median} \rvert$$

Mean deviation from median = $\frac{1}{N}\sum_{i=1}^{n} f_i \mid x_i - \text{median} \mid$.

Example 29.3 Find the mean deviation from the median for the following data.

x_i :	6	9	3	12	15	13	21	22
f_i :	4	5	3	2	5	4	4	3

Solution : Keeping the observations in the ascending order, we get the following distribution.

x_i :	3	6	9	12	13	15	21	22
f_i :	3	4	5	2	4	5	4	3
c.f.	3	7	12	14	18	23	27	30

$$N = 30 \qquad \therefore \quad \frac{N}{2} = 15$$

The observation whose c.f. is just greater than 15 is 13 (whose c.f. is 18).

$$\therefore \quad \text{Median} = 13.$$

Now we compute the absolute values of the the deviations from the median i.e., $|x_i - \text{median}|$ and compute $f_i\, |x_i - \text{median}|$ as shown in the following table.

$\lvert x_i - \text{median}\rvert$	10	7	4	1	0	2	8	9
f_i :	3	4	5	2	4	5	4	3
$f_i\,\lvert x_i - \text{median}\rvert$	30	28	20	2	0	10	32	27

Now $\qquad \Sigma f_i\, |x_i - \text{median}| = 149$

Hence mean deviation from the median

$$= \frac{1}{N}\Sigma f_i \mid x_i - \text{median} \mid = \frac{149}{30} = 4.97.$$

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Example29.4 Find the mean deviation from the median for the following distribution.

Class interval	0-10	10-20	20-30	30-40	40-50	50-60
Frequency f_i	6	8	14	16	4	2

Solution:

Class interval	**Frequency f_i**	**Cumulative frequency f_i**	x_i	$\|x_i$**-med.**$\|$	$f_i\|x_i$**-med.**$\|$
0-10	6	6	5	22.86	137.16
10-20	8	14	15	12.86	102.88
20-30	14	28	25	2.86	40.04
30-40	16	44	35	7.14	114.24
40-50	4	48	45	17.14	68.56
50-60	2	50	55	27.14	54.28
	N = 50				517.16

Here N/2th observation $= \frac{50}{2} = 25$,

this observation lies in the class interval 20-30.

$$\text{Median} = L + \left[\frac{\frac{N}{2} - \text{p.c.f}}{f}\right] i = 20 + \left(\frac{25-14}{14}\right) 10 = 27.86$$

$$\text{Mean deviation from median} = \sum f_i \frac{|x_i - \text{medain}|}{N} = \frac{517.16}{50} = 10.34$$

EXERCISE 29.1

1. The ages of 10 girls are given below :

   3 5 7 8 9 10 12 14 17 18

   What is the range ?

2. The weight of 10 students (in Kg) of class XII are given below :

   45 49 55 52 40 62 47 61 58

   What is the range ?

3. Find the mean deviation from mean of the data

   45 55 63 76 67 84 75 48 62 65

   Given mean = 64.

4. Calculate the mean deviation from mean of the following distribution.

Salary : (in rupees)	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of employees	4	6	8	12	7	6	4	3

   Given mean = Rs. 57.2

5. Calculate the mean deviation for the following data of marks obtained by 40 students in a test

Marks obtained	20	30	40	50	60	70	80	90	100
No. of students	2	4	8	10	8	4	2	1	1

6. The data below presents the earnings of 50 workers of a factory

Earnings (in rupees)	1200	1300	1400	1500	1600	1800	2000
No. of workers	4	6	15	12	7	4	2

   Find mean deviation.

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7. The distribution of weight of 100 students is given below :

Weight (in Kg)	50-55	55-60	60-65	65-70	70-75	75-80
No. of students	5	13	35	25	17	5

Calculate the mean deviation.

8. The marks of 50 students in a particular test are :

Marks	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of students	4	6	9	12	8	6	4	1

Find the mean deviation for the above data.

9. Find the mean deviation about the median for the following data.

x_i	25	20	15	10	5
f_i	7	4	6	3	5

10. Find the mean deviation about the median of the following data.

x_i	3	7	9	6	13	11
f_i	3	11	8	9	6	9

29.5 VARIANCE AND STANDARD DEVIATION OF RAW DATA

If there are n observations, $x_1, x_2, x_3, x_4, \ldots. x_n$, then

$$\text{Variance } (\sigma^2) = \frac{(x_1-\overline{x})^2+(x_2-\overline{x})^2+\ldots.+(x_n-\overline{x})^2}{n}$$

or $$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\overline{x})^2}{n}; \quad \left(\because \overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}\right)$$

The standard deviation, denoted by σ, is the positive square root of σ^2. Thus

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n}}$$

The following steps are employed to calculate the variance and hence the standard deviation of raw data. The mean is assumed to have been calculated already.

Step 1 : Make a column of deviations from the mean, namely, $x_i - \overline{x}$

Step 2 (check) : Sum of deviations from mean must be zero, i.e., $\sum_{i=1}^{n}(x_i - \overline{x}) = 0$

Step 3 : Square each deviation and write in the column headed $(x_i - \overline{x})^2$

Step 4 : Find the sum of the column in step 3.

Step 5 : Divide the sum obtained in step 4 by the number of observations. We obtain σ^2.

Step 6 : Take the positive square root of σ^2. We obtain σ (Standard deviation).

Example 29.5 The daily sale of sugar in a certain grocery shop is given below:

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
75	100	12	50	70.5	140.5

The average daily sale is 78 Kg. Calculate the variance and the standard deviation of the above data.

Solution : $\overline{x}$ = 78 kg (Given)

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
75	-3	9
100	42	1764
12	-66	4356
50	-28	784
70.5	-7.5	56.25
140.5	62.5	3906.25
	0	10,875.50

$$(\sigma^2) = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n} = \frac{10875.50}{6} = 1812.58$$

and $(\sigma) = 42.57$

Example 29.6 The marks of 10 students of section A in a test in English are given below :

7 10 12 13 15 20 21 28 29 35

Determine the variance and the standard deviation.

Solution : Here $\overline{x} = \frac{\Sigma x_i}{10} = \frac{190}{10} = 19$

x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
7	-12	144
10	-9	81
12	-7	49
13	-6	36
15	-4	16
20	1	1
21	2	4
28	9	81
29	10	100
35	16	256
	0	768

Thus $\sigma^2 = \frac{768}{10} = 76.8$

and $\sigma = \sqrt{76.8} = 8.76$

EXERCISE 29.2

1. The salary of 10 employees (in rupees) in a factory (per day) is

   50 60 65 70 80 45 75 90 95 100

   Calculate the variance and standard deviation.

2. The marks of 10 students of class X in a test in English are given below :

   9 10 15 16 18 20 25 30 32 35

   Determine the variance and the standard deviation.

3. The data on relative humidity (in %) for the first ten days of a month in a city are given below:

   90 97 92 95 93 95 85 83 85 75

   Calculate the variance and standard deviation for the above data.

4. Find the standard deviation for the data

   4 6 8 10 12 14 16

5. Find the variance and the standard deviation for the data

   4 7 9 10 11 13 16

6. Find the standard deviation for the data.

   40 40 40 60 65 65 70 70 75

   75 75 80 85 90 90 100

29.6 STANDARD DEVIATION AND VARIANCE OF RAW DATA AN ALTERNATE METHOD

If $\overline{x}$ is in decimals, taking deviations from $\overline{x}$ and squaring each deviation involves even more decimals and the computation becomes tedious. We give below an alternative formula for computing σ^2. In this formula, we by pass the calculation of $\overline{x}$.

We know
$$\sigma^2 = \sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{n}$$

$$= \sum_{i=1}^{n} \frac{x_i^2 - 2x_i\overline{x} + \overline{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n} x_i^2}{n} - \frac{2\overline{x}\sum_{i=1}^{n} x_i}{n} + \overline{x}^2$$

$$= \sum_{i=1}^{n} x_i^2 - \overline{x}^2 \quad \left(\because\ \overline{x} = \frac{\Sigma x_i}{n}\right)$$

i.e., $$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} \frac{x_i}{n}\right)^2}{n}$$

And $$\sigma = +\sqrt{\sigma^2}$$

The steps to be employed in calculation of σ^2 and σ, hence by this method are as follows :

Step 1 : Make a column of squares of observations i.e. x_i^2.

Step 2 : Find the sum of the column in step 1. We obtain $\sum_{i=1}^{n} x_i^2$.

Step 3 : Substitute the values of $\sum_{i=1}^{n} x_i^2$, n and $\sum_{i=1}^{n} x_i$ in the above formula.

We obtain σ^2.

Step 4 : Take the positive sauare root of σ^2. We obtain σ.

Example 29.7 We refer to Example 29.5 of this lesson and re-calculate the variance and standard deviation by this method.

Solution :

x_i	x_i^2
7	49
10	100
12	144
13	169
15	225
20	400
21	441
28	784
29	841
35	1225
190	4378

$$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n}$$

$$= \frac{4378 - \frac{(190)^2}{10}}{10}$$

$$= \frac{4378 - 3610}{10}$$

$$= \frac{768}{10}$$

$$= 76.8.$$

and $\sigma = \sqrt{\sigma^2} = +\sqrt{76.8} = 8.76$ (approx)

We are given we get the same value of σ^2 and σ by either methods.

29.7 STANDARD DEVIATION AND VARIANCE OF GROUPED DATA : METHOD - I

We are given k classes and their corresponding frequencies. We will denote the variance and the standard deviation of grouped data by σ_g^2 and σ_g respectively. The formulae are given below :

$$\sigma_g^2 = \frac{\sum_{i=1}^{k} [f_i (x_i - \overline{x})^2]}{N}, \quad N = \sum_{i=1}^{k} f_i$$

and $\sigma_g = +\sqrt{\sigma_g^2}$

The following steps are employed to calculate σ_g^2 and, hence σ_g : (The mean is assumed to have been calculated already).

Step 1 : Make a column of class marks of the given classes, namely x_i.

Step 2 : Make a column of deviations of class marks from the mean, namely, $x_i - \overline{x}$. Of course the sum of these deviations need not be zero, since x_i's are no more theoriginal observations.

Step 3 : Make a column of squares of deviations obtained in step 2, i.e., $(x_i - \overline{x})^2$ and write in the column headed by $(x_i - \overline{x})^2$.

Step 4 : Multiply each entry in step 3 by the corresponding frequency. We obtain $f_i(x_i - \overline{x})^2$.

Step 5 : Find the sum of the column in step 4. We obtain $\sum_{i=1}^{k}[f_i(x_i - \overline{x})^2]$

Step 6 : Divide the sum obtained in step 5 by N (total no. of frequencies). We obtain σ_g^2.

Step 7 : $\sigma_g = +\sqrt{\sigma_g^2}$

Example 29.8 In a study to test the effectiveness of a new variety of wheat, an experiment was performed with 50 experimental fields and the following results were obtained :

Yield per Hectare (in quintals)	Number of Fields
31-35	2
36-40	3
41-45	8
46-50	12
51-55	16
56-60	5
61-65	2
66-70	2

The mean yield per hectare is 50 quintals. Determine the variance and the standard deviation of the above distribution.

Solution : Given $\overline{x} = 50$

Yield per Hectare (in quintal)	No. of Fields• f_i	Class Marks x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$	$f_i(x_i - \overline{x})^2$
31-35	2	33	-17	289	578
36-40	3	38	-12	144	432
41-45	8	43	-7	49	392
46-50	12	48	-2	4	48
51-55	16	53	3	9	144
56-60	5	58	8	64	320
61-65	2	63	13	169	338
66-70	2	68	18	324	648
		50			2900

Thus $$\sigma_g^2 = \frac{\sum_{i=1}^{n}[f_i(x_i - \overline{x})^2]}{N} = \frac{2900}{50} = 58$$

and $$\sigma_g = +\sqrt{\sigma_g^2} = +\sqrt{58} = 7.61 \text{ (approx)}$$

29.8 STANDARD DEVIATION AND VARIANCE OF GROUPED DATA : METHOD - II

If $\overline{x}$ is not given or if $\overline{x}$ is in decimals in which case the calculations become rather tedious, we employ the alternative formula for the calculation of σ_g^2 as given below:

$$\sigma_g^2 = \frac{\sum_{i=1}^{n}[f_i x_i]^2 - \frac{\left(\sum_{i=1}^{k}(f_i\ x_i)\right)^2}{N}}{N}, \quad N = \sum_{i=1}^{k} f_i$$

and $$\sigma_g = +\sqrt{\sigma_g^2}$$

The following steps are employed in calculating σ_g^2 and, hence σ_g by this method:

Step 1 : Make a column of class marks of the given classes, namely, x_i.

Step 2 : Find the product of each class mark with the corresponding frequency. Write the product in the column $x_i f_i$.

Step 3 : Sum the entries obtained in step 2. We obtain $\sum_{i=1}^{n}(f_i x_i)$

Step 4 : Make a column of squares of the class marks of the given classes, namely, x_i^2.

Step 5 : Find the product of each entry in step 4 with the corresponding frequency. We obtain $f_i x_i^2$.

Step 6 : Find the sum of the entries obtained in step 5. We obtain $\sum_{i=1}^{n}(f_i x_i^2)$

Step 7 : Substitute the values of $\sum_{i=1}^{n}(f_i x_i^2)$, N and $\left[\sum_{i=1}^{n}(f_i x_i)\right]^2$ in the formula and obtain σ_g^2.

Step 8 : $\sigma_g = +\sqrt{\sigma_g^2}$

Example 29.9 Determine the variance and standard deviation for the data given in Example 29.7 by this method.

Solution :

Yields per Hectare (in quintals)•	f_i	x_i	$f_i\ x_i$	x_i^2	$f_i x_i^2$
31-35	2	33	66	1089	2178
36-40	3	38	144	1444	4332
41-45	8	43	344	1849	14792
46-50	12	48	576	2304	27648
51-55	16	53	848	2809	44944
56-60	5	58	290	3364	16820
61-65	2	63	126	3969	7938
66-70	2	68	136	4624	9248
Total	50		2500		127900

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Substituting the values of $\sum_{i=1}^{n}(f_i x_i^2)$, N and $\left[\sum_{i=1}^{n}(f_i x_i)\right]^2$

$$\sigma_g^2 = \frac{127900 - \frac{(2500)^2}{50}}{50}$$

$$\sigma_g^2 = \frac{2900}{50} = 58$$

and $\sigma_g = +\sqrt{\sigma_g^2} = +\sqrt{58} = 7.61$ (approx)

Again, we observe that we get the same value of σ_g^2 by either of the methods.

EXERCISE 29.3

1. In a study on effectiveness of a medicine over a group of patients, the following results were obtained :

Percentage of relief	0-20	20-40	40-60	60-80	80-100
No. of patients	10	10	25	15	40

Find the variance and standard deviation.

2. In a study on ages of mothers at the first child birth in a village, the following data were available :

Age(in years) at first child birth	18-20	20-22	22-24	25-26	26-28	28-30	30-32
No. of mothers	130	110	80	74	50	40	16

Find the variance and the standard deviation.

3. The daily salaries of 30 workers are given below:

Daily salary (In Rs.)	0-50	50-100	100-150	150-200	200-250	250-300
No. of workers	3	4	5	7	8	3

Find variance and standard deviation for the above data.

29.9 STANDARD DEVIATION AND VARIANCE : STEP DEVIATION METHOD

In Example 29.9, we have seen that the calculations were very complicated. In order to simplify the calculations, we use another method called the step deviation method. In most of the frequency distributions, we shall be concerned with the equal classes. Let us denote, the class size by h. Now we not only take the deviation of each class mark from the arbitrary chosen 'a' but also divide each deviation by h. Let

$$u_i = \frac{x_i - a}{h} \qquad .... (1)$$

Then $\quad x_i = hu_i + a \qquad .... (2)$

We know that $\quad \bar{x} = h\bar{u} + a \qquad .... (3)$

Subtracting (3) from (2), we get

$$x_i - \bar{x} = h(u_i - \bar{u}) \qquad .... (4)$$

In (4), squaring both sides and multiplying by f_i and summing over k, we get

$$\sum_{i=1}^{k}[f_i(x_i - \bar{x})^2] = h^2\sum_{i=1}^{k}[f_i(u_i - \bar{u})^2] \qquad .... (5)$$

Dividing both sides of (5) by N_1 we get

$$\sum_{i=1}^{k}\left[\frac{f_i(x_i - \bar{x})^2}{N}\right] = \frac{h^2}{N}\sum_{i=1}^{k}[f_i(u_i - \bar{u})^2]$$

$$\sigma_x^2 = h^2.\sigma_u^2 \qquad .... (6)$$

where σ_x^2 is the variance of the original data and σ_u^2 is the variance of the coded data or coded variance. σ_u^2 can be calculated by using the formula which involves the mean, namely,

$$\sigma_u^2 = \frac{1}{N}\sum_{i=1}^{k}[f_i(u_i - \overline{u})^2]; \quad \because N = \sum_{i=1}^{k} f_i \qquad .... (7)$$

or by using the formula which does not involve the mean, namely,

$$\sigma_u^2 = \frac{\sum_{i=1}^{k}[f_i u_i^2] - \frac{\left[\sum_{i=1}^{k}(f_i u_i)\right]^2}{N}}{N}; \; N = \sum_{i=1}^{k} f_i$$

Example 29.10 We refer to the Example 29.9 again and find the variance and standard deviation using the coded variance.

Solution : Here $h = 5$ and let $a = 48$.

Yield per Hectare (in quintal)	Number of fields f_i	Class marks x_i	$u_i = \frac{x_i - a}{h}$	$f_i u_i$	u_i^2	$f_i u_i^2$
31-35	2	33	-3	-6	9	18
36-40	3	38	-2	-6	4	12
41-45	8	43	-1	-8	1	08
46-50	12	48	0	0	0	0
51-55	16	53	1	16	1	16
56-60	5	58	2	10	4	20
61-65	2	63	3	6	9	18
66-70	2	68	4	8	16	32
	50				20	124

Thus $$\sigma_x^2 = \frac{\sum_{i=1}^{k}[f_i u_i'^2] - \frac{\left(\sum_{i=1}^{k} f_i u_i\right)^2}{N}}{N}$$

$$= \frac{124 - \frac{(20)^2}{50}}{50}$$

$$= \frac{124 - 8}{50} = \frac{58}{25}$$

Variance of the original data will be

$$\sigma_x^2 = h^2 \sigma_u^2 = 25 \times \frac{58}{25} = 58$$

and $$\sigma_x = +\sqrt{\sigma_x^2} = +\sqrt{58}$$

$$= 7.61 \text{ (approx)}$$

We, of course, get the same variance, and hence, standard deviation as before.

Example 29.11 Find the standard deviation for the following distribution giving wages of 230 persons.

Wages (in Rs.)	**No. of persons**
70-80	12
80-90	18
90-100	35
100-110	42
110-120	50
120-130	45
130-140	20
140-150	08

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Solution :

Wages (in Rs.)	Number of persons f_i	Class marks x_i	$u_i = \frac{x_i - 105}{10}$	u_i^2	$f_i u_i$	$f_i u_i^2$
70-80	12	75	-3	9	-36	108
80-90	18	85	-2	4	-36	72
90-100	35	95	-1	1	-35	35
100-110	42	105	0	0	0	0
110-120	50	115	1	1	50	50
120-130	45	125	2	4	90	180
130-140	20	135	3	9	60	180
140-150	08	145	4	16	32	128
	230				125	753

$$\sigma^2 = h^2\left[\frac{1}{N}\Sigma(f_i u_i^2) - \left(\frac{1}{N}\Sigma(f_i u_i)\right)^2\right]$$

$$= \left[\frac{753}{230} - \left(\frac{125}{230}\right)^2\right]$$

$$= 100(3.27 - 0.29) = 298$$

$$\sigma = +\sqrt{\sigma^2} = +\sqrt{298} = 17.3 \text{(approx)}$$

EXERCISE 29.4

1. The data written below gives the daily earnings of 400 workers of a flour mill.

Weekly earning (in Rs.)	No. of Workers
80-100	16
100-120	20
120-140	25
140-160	40
160-180	80
180-200	65
200-220	60
220-240	35
240-260	30
260-280	20
280-300	09

Calculate the variance and standard deviation using step deviation method.

2. The data on ages of teachers working in a school of a city are given below:

Age (in years)	20-25	25-30	30-35	35-40	40-45	45-50	50-55	55-60
No. of teachers	25	110	75	120	100	90	50	30

Calculate the variance and standard deviation using step deviation method.

3. Calculate the variance and standard deviation using step deviation method of the following data :

Age (in : years)	25-30	30-35	35-40	40-45	45-50	50-55
No. of persons	70	51	47	31	29	22

KEY WORDS

- Range : The difference between the largest and the smallest value of the given data.

- Mean deviation from mean (MD) $= \dfrac{\sum_{i=1}^{n}[f_i\,|x_i - \overline{x}\,|]}{\text{N}}$

  where $\text{N} = \sum_{i=1}^{n} f_i$, $\quad \overline{x} = \dfrac{1}{\text{N}}\sum_{i=1}^{n}(f_i x_i)$

- Variance $\sigma^2 = \sum_{i=1}^{n}\dfrac{(x_i - \overline{x})^2}{n}$ [for raw data]

- Standard derivation $(\sigma) = +\sqrt{\sigma^2} = +\sqrt{\sum_{i=1}^{n}\dfrac{(x_i - \overline{x})^2}{n}}$

- Variance for grouped data

$$\sigma_g^2 = \frac{\sum_{i=1}^{n}[f_i(x_i - \overline{x})]^2}{\text{N}},$$ x_i is the mid value of the class.

Also $\sigma_x^2 = h^2\,\sigma_u^2$ and $\sigma_u^2 = \dfrac{1}{\text{N}}\sum_{i=1}^{k}[f_i(u_i - \overline{u})^2]$

$$\text{N} = \sum_{i=1}^{k} f_i$$

or $$\sigma_u^2 = \frac{\sum_{i=1}^{k}[f_i u_i^2] - \frac{\left[\sum_{i=1}^{k}(f_i u_i)\right]}{N}}{N} \quad \text{where } N = \sum_{i=1}^{k} f_i$$

- Standard deviation for grouped data $\sigma_g = +\sqrt{\sigma_g^2}$

SUPPORTIVE WEB SITES

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com

PRACTICE EXERCISE

1. Find the mean deviation for the following data of marks obtained (out of 100) by 10 students in a test

   55 45 63 76 67 84 75 48 62 65

2. The data below presents the earnings of 50 labourers of a factory

Earnings (in Rs.)	1200	1300	1400	1500	1600	1800
No. of Labourers	4	7	15	12	7	5

   Calculate mean deviation.

3. The salary per day of 50 employees of a factory is given by the following data.

Salary (in Rs.)	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
No. of employees	4	6	8	12	7	6	4	3

   Calculate mean deviation.

4. Find the batting average and mean deviation for the following data of scores of 50 innings of a cricket player:

Run Scored	0-20	20-40	40-60	60-80	80-100	100-120
No. of Innings	6	10	12	18	3	1

5. The marks of 10 students in test of Mathematics are given below:

6 10 12 13 15 20 24 28 30 32

Find the variance and standard deviation of the above data.

6. The following table gives the masses in grams to the nearest gram, of a sample of 10 eggs.

46 51 48 62 54 56 60 71 75

Calculate the standard deviation of the masses of this sample.

7. The weekly income (in rupees) of 50 workers of a factory are given below:

Income	400	425	450	500	550	600	650
No. of workers	5	7	9	12	7	6	4

Find the variance and standard deviation of the above data.

8. Find the variance and standard deviation for the following data:

Class	0-20	20-40	40-60	60-80	80-100
Frequency	7	8	25	15	45

9. Find the standard deviation of the distribution in which the values of x are 1, 2,......, N. The frequency of each being one.

ANSWERS

EXERCISE 29.1

1. 15	2. 22	3. 9.4	4. 15.44
5. 13.7	6. 136	7. 5.01	8. 14.4
9. 6.2	10. 2.36		

EXERCISE 29.1

1. Variance = 311, Standard deviation = 17.63
2. Variance = 72.9, Standard deviation = 8.5
3. Variance = 42.6, Standard deviation = 6.53
4. Standard deviation = 4
5. Variance = 13.14, Standard deviation = 3.62
6. Standard deviation = 17.6

EXERCISE 29.3

1. Variance = 734.96, Standard deviation = 27.1
2. Variance = 12.16, Standard deviation = 3.49
3. Variance = 5489, Standard deviation = 74.09

EXERCISE 29.4

1. Variance = 2194, Standard deviation = 46.84
2. Variance = 86.5, Standard deviation = 9.3
3. Variance = 67.08, Standard deviation = 8.19

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PRACTICE EXERCISE

1. 9.4
2. 124.48
3. 15.44
4. 5219.8
5. Variance = 74.8, Standard deviation =8.6
6. 8.8
7. Variance = 5581.25, Standard deviation = 74.7
8. Variance = 840, Standard deviation = 28.9
9. Standard deviation $= \sqrt{\dfrac{N^2 - 1}{1 - 2}}$

PROBABILITY

Chapter **30**

LEARNING OUTCOMES

After studying this chapter, student will be able to:

- Define random experiment and sample space corresponding to an experiment.
- Calculate possible outcomes.
- Differentiate between various types of events such as equally likely, mutually exclusive, exhaustive, independent and dependent events.
- Explain the concept of probability.
- Calculate the probability of events using addition and multiplication theorem.
- Solve problems on probability using Baye's theorem.

PREREQUISITES

Set theory, permutations and combinations.

INTRODUCTION

'Probability' or 'Chance' is a word we often encounter in our day- to-day life. We often say that it is very probable that it will rain tonight meaning thereby we very much expect to have downpour this night. One may also say it in more likely to have a good yield of paddy in district A than in district B, meaning one expects better yield from A than from B. This expectation, of course, comes from our general knowledge about the condition of weather in the month of the season. In general, the expectation is based on one's present knowledge and belief about the event in question. Even though, there statements of expectations are by previous experience, present knowledge and analytical thinking, we need a quantitative measure to quantify the expectations. For this the theory of probability took birth in 17th century in France.

In short, the branch of Mathematics which studies the influence of chance" is the theory of probability. Hence, probability is a concept which numerically measures the degree of certainty or uncertainty of occurrence or non-occurrence of events. In this Chapter, we shall discuss various experiments and their outcomes. We shall define probability and conditional probability. We shall state and prove the addition theorem, the multiplication theorem, the Bayes and illustrate their applications through some examples.

30.1 RANDOM EXPERIMENT

Let us consider the following activities :

(i) Toss a coin and note the outcomes. There are two possible outcomes, either a head (H) or a tail (T).

(ii) In throwing a fair die, there are six possible outcomes, that is, any one of the six faces 1,2,...... 6.... may come on top.

(iii) Toss two coins simultaneously and note down the possible outcomes. There are four possible outcomes, HH,HT,TH,TT.

(iv) Throw two dice and there are 36 possible outcomes.

outcomes are (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1,6)

(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)

⋮ ⋮

(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

Each of the above mentioned activities fulfil the following two conditions.

(a) The activity can be repeated number of times under identical conditions.

(b) Outcome of an activity is not predictable beforehand, since the chance play a role and each outcome has the same chance of being selection.

Definition:

An experiment that can be repeated any number of times under identical conditions in which

(i) All possible outcomes of the experiments are known in advance

(ii) The actual outcome in a particular case is not known in advance, is called a random experiment.

Example 30.1 : Is drawing a card from well shuffled deck of cards, a random experiment ?

Solution :

(a) The experiment can be repeated, as the deck of cards can be shuffled every time before drawing a card.

(b) Any of the 52 cards can be drawn and hence the outcome is not predictable beforehand.

Hence, this is a random experiment.

Example 30.2 : Selecting a student from a class of 50 students without preference is a random experiment. Justify.

Solution:

(a) The experiment can be repeated under identical conditions.

(b) As the selection of the student is without preference, every student has equal chances of selection.

Hence, the outcome is not predictable beforehand. Thus, it is a random experiment.

Can you think of any other activities which are not random in nature.

Let us consider some activities which are not random experiments.

(i) Birth of Manish : Obviously this activity, that is, the birth of an individual is not repeatable and hence is not a random experiment.

(ii) Multiplying 4 and 8 on a calculator.

Although this activity can be repeated under identical conditions, the outcome is always 32. Hence, the activity is not a random experiment.

30.2 SAMPLE SPACE

We throw a die once, what are possible outcomes ? Clearly, a die can fall with any of its faces at the top. The number on each of the faces is, therefore, a possible outcome. We write the set S of all possible outcomes as

$$S = \{1, 2, 3, 4, 5, 6\}$$

Again, if we toss a coin, the possible outcomes for this experiment are either a head or a tail. We write the set S of all possible outcomes as

$$S = \{H, T\}$$

The set S associated with an experiment satisfying the following properties:

(i) each element of S denotes a possible outcome of the experiment.

(ii) any trial results in an outcome that corresponds to one and only one element of the set S is called the sample space of the experiment and the elements are called sample points. Sample space is generally denoted by S.

Example 30.3 Write the sample space in two tosses of a coin.

Solution: Let H denote a head and T denote a tail in the experiment of tossing of a coin.

Sample Space S = {HH, HT, TH, TT}

Example 30.4 : Write the sample space for each of the following experiments:

(i) A coin is tossed three times and the result at each toss is noted.

(ii) From five players A, B, C, D and E , two players are selected for a match.

(iii) Six seeds are sown and the number of seeds germinating is noted.

Solution: (i) S = {TTT, TTH, THT, HTT, HHT, HTH, THH, HHH}

number of elements in the sample space is $2 \times 2 \times 2 = 8$

(iii) S = {AB, AC, AD, AE, BC, BD, BE, CD, CE, DE}

Here $n(S) = 10$

(iii) S = {0, 1, 2, 3, 4, 5, 6} Here $n(S) = 7$

30.3 DEFINITION OF VARIOUS TERMS

Event : An event (E) is a subset of the sample space (S) i.e. E is a subset of S.

Let us consider the example of tossing a coin. In this experiment, we may be interested in 'getting a head'. Then the outcome 'head' is an event.

In an experiment of throwing a die, our interest may be in, 'getting an even number'. Then the outcomes 2, 4 or 6 constitute the event.

We often use the capital letters A, B, C etc. to represent the events.

Example 30.5 Let E denote the experiment of tossing three coins at a time. List all possible outcomes and the events that

(i) the number of heads exceeds the number of tails.

(ii) getting two heads.

Solution:

The sample space S is

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

If E_1 is the event that the number of heads exceeds the number of tails, and E_2 the event getting two heads. Then

E_1 = {HHH, HHT, HTH, THH}

and E_2 = {HHT,, HTH, THH}

30.3.1 Equally Likely Events

Outcomes of a trial are said to be equally likely if taking into consideration all the relevant evidences there is no reason to expect one in preference to the other.

Example:

(i) In tossing an unbiased coin, getting head or tail are equally likely events.

(ii) In throwing a fair die, all the six faces are equally likely to come.

(iii) In drawing a card from a well shuffled deck of 52 cards, all the 52 cards are equally likely to come.

(iv) In the experiment of throwing a die. the following events

$A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$ are equally likely events.

30.3.2 Mutually Exclusive Events

Definition: Two or more events are said to be mutually exclusive if the occurrence of one of the events prevents the occurrence of any of the remaining events. Thus events $E_1, E_2 \ldots E_K$ are said to be mutually exclusive if $E_i \cap E_j = \phi$ for $i \neq j,\ 1 \le i,\ j \le k$.

Examples :

(i) In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive. If any one of these faces comes at the top, the possibility of others, in the same trial is ruled out.

(ii) When two coins are tossed, the event that both should come up tails and the event that there must be at least one head are mutually exclusive.

Mathematically events are said to be mutually exclusive if their intersection is a null set (i.e., empty)

30.3.3 Exhaustive Events

Two are more events are said to be exhaustive if the performance of the experiment always results in the occurence of atleast one of them. Thus events $E_1, E_2 .... E_K$ are said to be exhaustive if $E_1 \cup E_2 \cup E_3 .... \cup E_k = S$.

For example, when a die is rolled, the event of getting an even number and the event of getting an odd number are exhaustive events. Or when two coins are tossed the event that at least one head will come up and the event that at least one tail will come up are exhaustive events.

Mathematically a collection of events is said to be exhaustive if the union of these events is the complete sample space.

Examples

(i) In a throwing a die, the events 1, 2, 3, 4, 5, 6 are exhaustive.

(ii) In tossing an unbiased coin, getting head or tail are exhaustive events.

30.3.4 Independent and Dependent Events

A set of events is said to be independent if the happening of any one of the events does not affect the happening of others. If, on the other hand, the happening of any one of the events influence the happening of the other, the events are said to be dependent.

Examples :

(i) In tossing an unbiased coin the event of getting a head in the first toss is independent of getting a head in the second, third and subsequent throws.

(ii) If we draw a card from a pack of well shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But, however, if the first card drawn is not replaced then the second card is dependent on the first draw (in the sense that it cannot be the card drawn the first time).

Excercise 30.1

1. Selecting a student from a school without preference is a random experiment. Justify.
2. Adding two numbers on a calculator is not a random experiment. Justify.
3. Write the sample space of tossing three coins at a time.
4. Write the sample space of tossing a coin and a die.
5. Two dice are thrown simultaneously, and we are interested to get six on top of each of the die. Are the two events mutually exclusive or not ?
6. Two dice are thrown simultaneously. The events A, B, C, D are as below:

   A : Getting an even number on the first die.

   B : Getting an odd number on the first die.

   C : Getting the sum of the number on the dice < 7.

   D : Getting the sum of the number on the dice > 7.

   State whether the following statements are True or False.

   (i) A and B are mutually exclusive.

   (ii) A and B are mutually exclusive and exhaustive.

   (iii) A and C are mutually exclusive.

   (iv) C and D are mutually exclusive and exhaustive.
7. A ball is drawn at random from a box containing 6 red balls, 4 white balls and 5 blue balls. There will be how many sample points, in its sample space?
8. In a single rolling with two dice, write the sample space and its elements.
9. Suppose we take all the different families with exactly 2 children. The experiment consists in asking them the sex of the first and second child.

   Write down the sample space.

30.4 Probability

If an experiment with 'n' exhaustive, mutually exclusive and equally likely outcomes, m outcomes are favourable to the happening of an event A, the probability 'p' of happening of A is given by

$$P = P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total Number of possible outcomes}} = \frac{m}{n} \qquad \text{...(i)}$$

Since the number of cases favourable to the non-happening of the event A are $n - m$, the probability 'q' that 'A' will not happen is given by

$$q = \frac{n-m}{n} = 1 - \frac{m}{n}$$

$$= 1 - p \quad \text{[using (i)]}$$

$$\therefore \qquad p + q = 1$$

Obviously, p as well as q are non-negative and cannot exceed unity.

ie., $\qquad 0 \le p \le 1, \quad 0 \le q \le 1$

Thus, the probability of occurrence of an event lies between 0 and 1[including 0 and 1].

Remarks:

1. Probability 'p' of the happening of an event is known as the probability of success and the probability 'q' of the non-happening of the event as the probability of failure.
2. Probability of an impossible event is 0 and that of a sure event is 1

   if P(A) = 1, the event A is certainly going to happen and

   if P(A) = 0, the event is certainly not going to happen.
3. The number (m) of favourable outcomes to an event cannot be greater than the total number of outcomes (n).

Let us consider some examples

Example 30.6: A die is rolled once. Find the probability of getting 5.

Solution: There are six possible ways in which a die can fall, out of these only one is favourable to the event.

$$\therefore \; P(5) = \frac{1}{6}$$

Example 30.7: A coin is tossed once. What is the probability of the coin coming up with head ?

Solution: The coin can come up either 'head' (H) or a tail (T). Thus, the total possible outcomes are two and one is favourable to the event.

So, $$P(H) = \frac{1}{2}$$

Example 30.8: A die is rolled once. What is the probability of getting an odd number ?

Solution: There are six possible outcomes in a single throw of a die. Out of these; 1, 3 and 5 are the favourable cases.

$$\therefore \quad P\,(\text{Odd Number}) = \frac{3}{6} = \frac{1}{2}$$

Example 30.9: A die is rolled once. What is the probability of the number '7' coming up ? What is the probability of a number 'less than 7' coming up ?

Solution: There are six possible outcomes in a single throw of a die and there is no face of the die with mark 7.

$$\therefore \quad P(\text{number } 7) = \frac{0}{6} = 0$$

[**Note:** That the probability of impossible event is zero]

As every face of a die is marked with a number less than 7,

$$\therefore \quad P(< 7) = \frac{6}{6} = 1$$

[**Note:** That the probability of an event that is certain to happen is 1]

Example 30.10: In a simultaneous toss of two coins, find the probability of (i) getting 2 heads (ii)exactly 1 head

Solution: Here, the possible outcomes are

HH, HT, TH, TT

i.e., Total number of possible outcomes = 4.

(i) Number of outcomes favourable to the event (2 heads) = 1 (i.e., HH).

$$P(2 \text{ heads}) = \frac{1}{4}$$

(ii) Now the event consisting of exactly one head has two favourable cases, namely HT and TH .

$$P(\text{exactly one head}) = \frac{2}{4} = \frac{1}{2}$$

Example 30.11: Three coins are tossed simultaneously. Find the probability of getting two tails and one head.

Solution: Heren the possible outcomes are

HHH, HHT, HTH, THH, HTT, THT, TTH, TTT

i.e., total number of possible outcomes = 8

Number of outcomes favourable to the event

(two tails and one head) = 3 (i.e., HTT, THT, TTH)

$\therefore$ P(Two tails and one head)

$$= \frac{3}{8}.$$

Example 30.12: A page is opened at random from a book containing 200 pages. What is the probability that the number on the page is a perfect squars.

Solution: The possible outcomes are

1, 2, 3, ... 200

i.e., Total number of possible outcomes = 200

Number of outcomes favourable to the event (Perfect square)

$= \{1, 4, 9, \ldots 196\} = 14$

$\therefore \quad P(\text{perfect square}) = \dfrac{14}{200} = \dfrac{7}{100}$

Example 30.13: In a single throw of two dice, what is the probability that the sum is 9?

Solution: The number of possible outcomes is $6 \times 6 = 36$. We write them as given below :

(1, 1),(1, 2), (1, 3) (1, 4), (1, 5), (1, 6)
(2, 1),(2, 2), (2, 3) (2, 4), (2, 5), (2, 6)
(3, 1),(3, 2), (3, 3) (3, 4), (3, 5), (3, 6)
(4, 1),(4, 2), (4, 3) (4, 4), (4, 5), (4, 6)
(5, 1),(5, 2), (5, 3) (5, 4), (5, 5), (5, 6)
(6, 1),(6, 2), (6, 3) (6, 4), (6, 5), (6, 6)

The outcomes (3, 6), (4, 5), (5, 4) and (6, 3) are favourable to the said event, i.e., the number of favourable outcomes is 4.

Hence $P(\text{a total of } 9) = \dfrac{4}{36} = \dfrac{1}{9}$.

Example 30.14: From a bag containing 10 red, 4 blue and 6 black balls, a ball is drawn at random. What is the probability of drawing

(i) a red ball ? (ii) a blue ball ? (iii) not a black ball ?

Solution: There are 20 balls in all. So, the total number of possible outcomes is 20. (Random drawing of balls ensure equally likely outcomes)

(i) Number of red balls = 10

$\therefore \quad P(\text{a red ball}) = \dfrac{10}{20} = \dfrac{1}{2}$

(ii) Number of blue balls = 4

$P(\text{a blue ball}) = \dfrac{4}{20} = \dfrac{1}{5}$

(iii) Number of balls which are not black $= 10 + 4$

$= 14$

$$P\text{ (Not a black ball)} = \frac{14}{20} = \frac{7}{10}$$

Example 30.15: A card is drawn at random from a well shuffled deck of 52 cards. If A is the event of getting a Queen and B is the event of getting a card bearing a number greater than 4 but less than 10, find P(A) and P (B).

Solution: Well shuffled pack of cards ensures equally likely outcomes.

$\therefore$ the total number of possible outcomes is 52.

(i) There are 4 Queens in a pack of cards.

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

(ii) The cards bearing a number greater than 4 but less than 10 are 5, 6, 7, 8 and 9.

Each card bearing any of the above number is of 4 suits diamond, spade, club or heart.

Thus, the number of favourable outcomes $5 \times 4 = 20$

$$= \frac{20}{52} = \frac{5}{13}.$$

Example 30.16: If a car is selected from a well shuffled deck of 52 cards, what is the probability of drawing.

(i) a soade ? (ii) a king ? (iii) a king of spade?

Solution: Well shuffled pack of cards ensures equally likely outcomes.

$\therefore$ the total number of possible outcomes is 52.

(i) Let A be the event of getting a spade.

There are 13 spades in a pack of cards

$$\therefore P(A) = \frac{13}{52} = \frac{1}{4}.$$

(ii) Let B be the event of getting a king.

There are 4 kings in a pack of cards.

$$\therefore \ P(B) = \frac{4}{52} = \frac{1}{13}.$$

(iii) Let C be the event of getting a king of spade.

Thus is 1 king of spade in a pack of cards

$$\therefore \ P(C) = \frac{1}{52}.$$

Example 30.17: What is the chance that a leap year, selected at random, will contain 53 Sundays?

Solution: A leap year consists of 366 days consisting of 52 weeks and 2 extra days. These two extra days can occur in the following possible ways.

(i) Sunday and Monday

(ii) Monday and Tuesday

(iii) Tuesday and Wednesday

(iv) Wednesday and Thursday

(v) Thursday and Friday

(vi) Friday and Saturday

(vii) Saturday and Sunday

Out of the above seven possibilities, two outcomes,

e.g., (i) and (vii), are favourable to the event

$$\therefore \ P(53 \text{ Sundays}) = \frac{2}{7}$$

Exercise 30.2

1. A die is rolled once. Find the probability of getting 3.
2. A coin is tossed once. What is the probability of getting the tail ?
3. What is the probability of the die coming up with a number greater than 3 ?

4. In a simultaneous toss of two coins, find the probability of getting ' at least' one tail.
5. From a bag containing 15 red and 10 blue balls, a ball is drawn 'at random'. What is the probability of drawing (i) a red ball ? (ii) a blue ball ?
6. If two dice are thrown, what is the probability that the sum is (i) 6 ? (ii) 8? (iii) 10? (iv) 12?
7. If two dice are thrown, what is the probability that the sum of the numbers on the two faces is divisible by 3 or by 4 ?
8. If two dice are thrown, what is the probability that the sum of the numbers on the two faces is greater than 10 ?
9. What is the probability of getting a red card from a well shuffled deck of 52 cards ?
10. If a card is selected from a well shuffled deck of 52 cards, what is the probability of drawing (i) a spade ? (ii) a king ? (iii) a king of spade ?
11. A pair of dice are thrown. Find the probability of getting
    (i) a sum as a prime number
    (ii) a doublet, i.e., the same number on both dice
    (iii) a multiple of 2 on one die and a multiple of 3 on the other.
12. Three coins are tossed simultaneously. Find the probability of getting (i) no head (ii) at least one head (iii) all heads

30.5 CALCULATION OF PROBABILITY USING COMBINATORICS (PERMUTATIONS AND COMBINATIONS)

In the preceding section, we calculated the probability of an event by listing down all the possible outcomes and the outcomes favourable to the event. This is possible when the number of outcomes is small, otherwise it becomes difficult and time consuming process. In general, we do not require the actual listing of the outcomes, but require only the total number of possible outcomes and the number of outcomes favourable to the event. In many cases, these can be found by applying the knowledge of permutations and combinations, which you have already studied.

Let us consider the following examples :

Example 30.18: A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn are white and blue ?

Solution : Total number of balls = 3 + 6 + 7 = 16

Now, out of 16 balls, 2 can be drawn in $^{16}C_2$

$$\therefore \text{ Exhaustive number of cases} = {}^{16}C_2 = \frac{16\times 15}{2} = 120$$

Out of 6 white balls, 1 ball can be drawn in $^{6}C_1$ ways and out of 7 blue balls, one can be drawn is $^{7}C_1$ ways. Since each of the former case is associated with each of the later case, therefore total number of favourable cases are

$$^{6}C_1 \times {}^{7}C_1 = 6 \times 7 = 42$$

$$\therefore \text{ Required probability} = \frac{42}{120} = \frac{7}{20}.$$

Remarks :

When two or more balls are drawn from a bag containing several balls, there are two ways in which these balls can be drawn.

(i) **Without replacement :** The ball first drawn is not put back in the bag, when the second ball is drawn. The third ball is also drawn without putting back the balls drawn earlier and so on. Obviously, the case of drawing the balls without replacement is the same as drawing them together.

(ii) **With replacement :** In this case, the ball drawn is put back in the bag before drawing the next ball. Here the number of balls in the bag remains the same, every time a ball is drawn.

In these types of problems, unless stated otherwise, we consider the problem of without replacement.

Example 30.19: Find the probability of getting both red balls, when from a bag containing 5 red and 4 black balls, two balls are drawn,

(i) with replacement.

(ii) without replacement.

Solution: (i)Total number of balls in the bag in both the draws = 5 + 4 = 9

Hence, by fundamental principle of counting, the total number of possible outcomes = $9 \times 9 = 81$

Similarly, the number of favourable cases = $5 \times 5 = 25$.

Hence, probability (both red balls) = $\frac{25}{81}$.

(ii) Total number of possible outcomes is equal to the number of ways of selecting 2 balls out of 9 balls = ${}^9C_2$

Number of favourable cases is equal to the number of ways of selecting 2 balls out of 5 red balls = ${}^5C_2$

Hence, P (both red balls) = $\frac{{}^5C_2}{{}^9C_2} = \frac{5\times4}{9\times8} = \frac{5}{18}$.

Example 30.20: Three cards are drawn from a well-shuffled pack of 52 cards. Find the probability that they are a king, a queen and a jack.

Solution: From a pack of 52 cards, 3 cards can be drawn in 52 C3 ways, all being equally likely.

$\therefore$ Exhaustive number of cases = ${}^{52}C_3$

A pack of cards contains 4 kings, 4 queens and 4 jacks .A king, a queen and a Jack can each be drawn in ${}^4C_1$ ways and since each way of drawing a king can be associated with each of the ways of drawing a queen and a jack, the total number of favourable cases = ${}^4C_1 \times {}^4C_1 \times {}^4C_1$

$\therefore$ Required probability $= \frac{{}^4C_1 \times {}^4C_1 \times {}^4C_1}{{}^{52}C_3}$

$$= \frac{4\times 4\times 4}{\frac{52\times 51\times 50}{1\times 2\times 3}}$$

$$= \frac{16}{5525}$$

Example 30.21: From 25 tickets, marked with the first 25 numerals, one is drawn at random. Find the probability that it is a multiple of 5.

Solution: Numbers (out of the first 25 numerals) which are multiples of 5 are 5, 10, 15, 20 and 25, i.e., 5 in all. Hence, required favourable cases are=5.

$\therefore$ Required probability $= \frac{5}{25} = \frac{1}{5}$

Example 30.22: A and B are among 20 persons who sit at random along a round table. Find the probability that there are any six persons between A and B.

Solution: Let A occupy any seat at the round table.

Then there are 19 seats left for B. But it six persons are to be seated between A and B, then B has only two ways to sit. Thus the required probability is $\frac{2}{19}$.

Exercise 30.2

1. A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn at random are both white?
2. A bag contains 5 red and 8 blue balls. What is the probability that two balls drawn are red and blue ?
3. A bag contains 20 white and 30 black balls. Find the probability of getting 2 white balls, when two balls are drawn at random

   (a) with replacement (b) without replacement

4. Three cards are drawn from a well-shuffled pack of 52 cards. Find the probability that all the three cards are jacks.

5. Two cards are drawn from a well-shuffled pack of 52 cards. Show that the chances of drawing both aces is $\frac{1}{221}$.

6. In a group of 10 outstanding students in a school, there are 6 boys and 4 girls. Three students are to be selected out of these at random for a debate competition. Find the probability that

   (i) one is boy and two are girls.

   (ii) all are boys.

   (iii) all are girls.

7. Out of 21 tickets marked with numbers from 1 to 21, three are drawn at random. Find the probability that the numbers on them are in A.P.

8. Two cards are drawn at random from 8 cards numbered 1 to 8. What is the probability that the sum of the numbers is odd, if the cards are drawn together ?

9. A team of 5 players is to be selected from a group of 6 boys and 8 girls. If the selection is made randomly, find the probability that there are 2 boys and 3 girls in the team.

10. An integer is chosen at random from the first 200 positive integers.Find the probability that the integer is divisible by 6 or 8.

30.6 Axioms of Probability

Axomatic definition of Probability

Probability is a number that is assigned to each number of a collection of events from a random experiment that satisfies the following properties.

If S is the sample space and E is any event in a random exeriment.

1. $0 \leq P(E) \leq 1$ for each event E in S

2. $P(S) = 1$

3. If E_1 and E_2 are any mutually exclusive $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

30.7 SOME RESULTS ON PROBABILITY OF EVENTS

Result 1 : Probability of impossible event is zero.

Solution : Impossible event contains no sample points. Therefore, the certain event S and the impossible event ϕ are mutually exclusive.

Hence, $S \cup \phi = S$

$$\Rightarrow \quad P(S \cup \phi) = P(S)$$

$$P(S) + P(\phi) = P(S)$$

$$P(\phi) = 0$$

Result 2 : Probability of the complementary event $\bar{A}$ of A is given by

$$P(\bar{A}) = 1 - P(A)$$

Solution : A and $\bar{A}$ •are disjoint events. Also,

$$A \cup \bar{A} = S \Rightarrow P(A \cup \bar{A}) = P(S)$$

Using additive laws (ii) and (iii) , we get

$$P(A) + P(\bar{A}) = 1$$

$$\Rightarrow \quad P(\bar{A}) = 1 - P(A).$$

Result 3 : Prove that $0 \le P(A) \le 1$, for any A in S.

Solution : We know that

$$A \subset S \quad \Rightarrow \quad P(A) \le P(S)$$

$$\Rightarrow \quad P(A) \le 1$$

We know that $\quad P(A) \ge 0$

Hence, $\quad 0 \le P(A) \le 1.$

Result 4 : If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Solution:

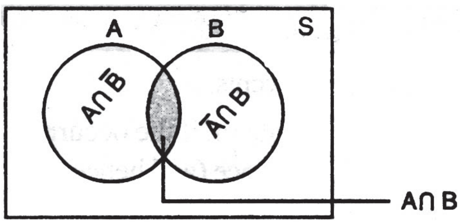

Fig. 30.1

From the above figure, we can write

$$A \cup B = A \cup (\bar{A} \cap B)$$

$$\Rightarrow \quad P(A \cup B) = P[A \cup (\bar{A} \cap B)] \quad ...(1)$$

Since the events A and $(\bar{A} \cap B)$ are disjoint, therefore law (iii) gives

$$P[A \cup (\bar{A} \cap B)] = P(A) + P(\bar{A} \cap B)$$

Substituting this value in (1) , we get

$$P(A \cup B) = P(A) + P(\bar{A} \cap B)$$

or $P(A \cup B) = P(A) + [P(\bar{A} \cap B) + P(A \cap B)] - P(A \cap B)$ ...(2)

From Fig. 30.1, we see that

$$(\bar{A} \cap B) \cup (A \cap B) = B$$

$$P[(\bar{A} \cap B) \cup (A \cap B)] = P(B).$$

Further, the events $(\bar{A} \cap B)$ and $(A \cap B)$ are disjoint, so from additive law, we get

$$P(\bar{A} \cap B) + P(A \cap B) = P(B)$$

Substituting this value in (2), we get

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Result 5 : If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

Solution: From additive law, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad ...(1)$$

Since A and B are mutually exclusive events,

Therefore $A \cap B = \phi$

$\Rightarrow \quad P(A \cap B) = P(\phi) = 0$

Substituting this value in equation (1), we get

$$P(A \cup B) = P(A) + P(B),$$

which is additive law for mutually exclusive events.

Example 30.23: A card is drawn from a well-shuffled deck of 52 cards. What is the probability that it is either a spade or a king ?

Solution : If a card is drawn at random from a well-shuffled deck of cards, the likelyhood of any of the 52 cards being drawn is the same. Obviously, the sample space consists of 52 sample points.

If A and B denote the events of drawing a 'spade card' and a 'king' respectively, then the event A consists of 13 sample points, whereas the event B consists of 4 sample points. Therefore,

$$P(A) = \frac{13}{52}, \qquad P(B) = \frac{4}{52}$$

The compound event $(A \cap B)$, consists of only one sample point, viz.; king of spade. So,

$$P(A \cap B) = \frac{1}{52}$$

Hence, the probability that the card drawn is either a spade or a king is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$$

$$= \frac{16}{52} = \frac{4}{13}.$$

Example 30.24: In an experiment with throwing 2 fair dice, consider the events

A : The sum of numbers on the faces is 8

B : Doubles are thrown.

What is the probability of getting A or B ?

Solution : In a throw of two dice, the sample space consists of $6 \times 6 = 36$.

The favourable outcomes to the event A (the sum of the numbers on the faces is 8) are

$$A = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$$

The favourable outcomes to the event B (Double means both dice have the same number) are

$$B = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

$$A \cap B = \{(4, 4)\}.$$

Now $\quad P(A) = \frac{5}{36}, \quad P(B) = \frac{6}{36}, \quad P(A \cap B) = \frac{1}{36}$

Thus, the probability of A or B is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{36} + \frac{6}{36} - \frac{1}{36} = \frac{10}{36} = \frac{5}{18}.$$

Example 30.25: The probabilities that a student will receive an A, B , C or D grade are 0.30, 0.35, 0.20 and 0.15 respectively. What is the probability that a student will receive at least a B grade ?

Solution : The event at least a 'B' grade means that the student gets either a B grade or an A grade.

$$\therefore \quad P(\text{ at least B grade }) = P(\text{ B grade }) + P(\text{ A grade })$$

$$= 0.35 + 0.30$$

$$= 0.65$$

MODULE - VI
Statistics and Probability

Notes

Example 30.26: Find the probability of the event getting at least 1 tail, if four coins are tossed once.

Solution: In tossing of 4 coins once, the sample space has 16 samples points.

P (at least one tail) = P(1 or 2 or 3 or 4 tails)

= 1 – P (0 tail) (By law of complimentation)

= 1 – P (H H H H)

The outcome favourable to the event four heads is 1.

$$\therefore \qquad P(HHHH) = \frac{1}{16}.$$

Substituting this value in the above equation,

we get

$$\therefore \qquad P(\text{at least one tail}) = 1 - \frac{1}{16} = \frac{15}{16}$$

In many instances, the probability of an event may be expressed as odds - either odds in favour of an event or odds against an event.

If A is an event :

The odds in favour of A = $\frac{P(A)}{P(\bar{A})}$ or P (A) to P ($\bar{A}$) ,

where P (A) is the probability of the event A and P ($\bar{A}$) is the probability of the event 'not A'.

Similarly, the odds against A are

$$\frac{P(\bar{A})}{P(A)} \quad \text{or } P(A) \text{ to } P(\bar{A})$$

Example 30.27: The probability of the event that it will rain is 0.3. Find the odds in favour of rain and odds against rain.

Solution : Let A be the event that it will rain.

$$\therefore \qquad P(A) = .3$$

By law of complementation,

$$P(\bar{A}) = 1 - 0.3 = 0.7$$

Now, the odds in favour of rain are $\frac{0.3}{0.7}$ or 3 to 7 (or 3 : 7).

The odds against rain are $\frac{0.7}{0.3}$ or 7 to 3.

When either the odds in favour of A or the odds against A are given, we can obtain the probability of that event by using the following formulae

If the odds in favour of A are a to b, then

$$P(A) = \frac{a}{a+b}$$

If the odds against A are a to b, then

$$P(A) = \frac{b}{a+b}$$

This can be proved very easily.

Suppose the odds in favour of A are a to b. Then, by the definition of odds,

$$\frac{P(A)}{P(\bar{A})} = \frac{a}{b}$$

From the law of complimentation,

$$P(\bar{A}) = 1 - P(A)$$

Therefore, $\frac{P(A)}{1-P(A)} = \frac{a}{b}$ or $b\,P(A) = a - a\,P(A)$

or $(a + b)\,P(A) = a$ or $P(A) = \frac{a}{a+b}$.

Similarly, we can prove that

$$P(A) = \frac{b}{a+b}$$

when the odds against A are b to a.

Example 30.28: Determine the probability of A for the given odds

(a) 3 to 1 in favour of A (b) 7 to 5 against A.

Solution: (a) $P(A) = \frac{3}{3+1} = \frac{3}{4}$

(b) $P(A) = \frac{5}{7+5} = \frac{5}{12}$

Example 30.29: If two dice are thrown, what is the probability that the sum is

(a) greater than 8 ? (b) neither 7 nor 11 ?

Solution: If S denotes the sum on two dice, then we want P(S !Ü 8) . The required event can happen in the following mutually exclusive ways :

(i) S = 9, (ii) S = 10, (iii) S = 11 and (iv) S = 12 .

Hence, by addition probability theorem for mutually exclusive events, we get

$P(S > 8) = P(S = 9) + P(S = 10) + P(S = 11) + P(S = 12)$ ...(1)

In a throw of two dice, the sample space contains 6 × 6 = 36 points. The number of favourable cases can be enumerated as shown below :

S = 9 : (3, 6), (4, 5), (5, 4), (6, 3) i.e., 4 sample points.

$\therefore \quad P(S = 9) = \frac{4}{36}$

S = 10 : (4, 6), (5, 5), (6, 4), i.e., 3 sample points.

$\therefore \quad P(S = 10) = \frac{3}{36}$

S = 11 : (5, 6), (6, 5) i.e., 2 sample points

$\therefore \quad P(S = 11) = \frac{2}{36}$

S = 12 : (6, 6) i.e., 1 sample point.

$P(S = 12) = \frac{1}{36}$

Substituting these values in equation (1), we get

$P(S > 8) = \frac{4}{36}+\frac{3}{36}+\frac{2}{36}+\frac{1}{36} \qquad = \frac{10}{36} = \frac{5}{18}.$

(b) Let A and B denote the events of getting the sum 7 and 11 respectively on a pair of dice.

S = 7 : (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)i.e., 6 sample points.

$\therefore \quad P(S = 7) = \frac{6}{36} \quad \text{or} \quad P(A) = \frac{6}{36}$

S = 11 : (5, 6), (6, 5) i.e., 2 sample points.

$\therefore \quad P(S = 11) = \frac{2}{36} \quad \text{or} \quad P(B) = \frac{2}{36}.$

Since A and B are disjoint events, therefore

$$P(\text{either} A \text{ or } B) = P(A) + P(B)$$

$$= \frac{6}{36} + \frac{2}{36}$$

$$= \frac{8}{36}$$

Hence, by law of complementation,

$$P(\text{neither 7 nor 11}) = 1 - P(\text{either 7 or 11})$$

$$= 1 - \frac{8}{36}$$

$$= \frac{28}{36}$$

$$= \frac{7}{9}$$

Example 30.30: Are the following probability assignments consistent ? Justify your answer.

(a) $P(A) = P(B) = 0.6, \quad P(A \text{ and } B) = 0.05$

(b) $P(A) = 0.5, P(B) = 0.4, \quad P(A \text{ and } B) = 0.1$

(c) $P(A) = 0.2, \quad P(B) = 0.7, \quad P(A \text{ and } B) = 0.4$

Solution: (a) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$$= 0.6 + 0.6 - 0.05$$

$$= 1.15$$

P(A or B) > 1 is not possible, hence the given probabilities are not consistent.

(b) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$= 0.5 + 0.4 - 0.1$

$= 0.8$

which is less than 1.

As the number of outcomes favourable to event 'A and B' should always be less than or equal to those favourable to the event A,

Therefore, $P(A \text{ or } B) \leq P(A)$

and similarly $P(A \text{ and } B) \leq P(B)$

In this case, P (A and B) = 0.1, which is less than both P (A) = 0.5 and P (B) = 0.4. Hence, the assigned probabilities are consistent.

(c) In this case, P (A and B) = 0.4, which is more than P (A) = 0.2.

$[\because P(A \text{ and } B) \leq P(A)]$

Hence, the assigned probabilities are not consistent.

Example 30.31: An urn contains 8 white balls and 2 green balls. A sample of three balls is selected at random. What is the probability that the sample contains at least one green ball ?

Solution : Urn contains 8 white balls and 2 green balls.

$\therefore$ Total number of balls in the urn = 10

Three balls can be drawn in ${}^{10}C_3$ ways = 120 ways.

Let A be the event "at least one green ball is selected".

Let us determine the number of different outcomes in A. These outcomes contain either one green ball or two green balls.

There are ${}^{2}C_1$ ways to select a green ball from 2 green balls and for this remaining two white balls can be selected in ${}^{8}C_2$ ways.

Hence, the number of outcomes favourable to one green ball

$= {}^2C_1 \times {}^8C_2$

$= 2 \times 28 = 56$

Similarly, the number of outcomes favourable to two green balls

$= {}^2C_2 \times {}^8C_1 = 1 \times 8 = 8$

Hence, the probability of at least one green ball is

P (at least one green ball)

$= P(\text{ one green ball }) + P(\text{ two green balls })$

$$= \frac{56}{120} + \frac{8}{120}$$

$$= \frac{64}{120} = \frac{8}{15}.$$

Example 30.32: Two balls are drawn at random with replacement from a bag containing 5 blue and 10 red balls. Find the probability that both the balls are either blue or red.

Solution : Let the event A consists of getting both blue balls and the event B is getting both red balls. Evidently A and B are mutually exclusive events.

By fundamental principle of counting, the number of outcomes favourable to A = $5 \times 5 = 25$

Similarly, the number of outcomes favourable to B = $10 \times 10 = 100$.

Total number of possible outcomes = $15 \times 15 = 225$.

$$P(A) = \frac{15}{225} = \frac{1}{9} \text{ and } P(B) = \frac{100}{225} = \frac{4}{9}$$

Since the events A and B are mutually exclusive, therefore

$P(A \cup B) = P(A) + P(B)$

$$= \frac{1}{9} + \frac{4}{9}$$

$$= \frac{5}{9}.$$

Thus, P (both blue or both red balls) $= \frac{5}{9}$.

Exercise 30.4

1. A card is drawn from a well-shuffled pack of cards. Find the probability that it is a queen or a card of heart.
2. In a single throw of two dice, find the probability of a total of 7 or 12.
3. The odds in favour of winning of Indian cricket team in 2010 world cup are 9 to 7. What is the probability that Indian team wins ?
4. The odds against the team A winning the league match are 5 to 7. What is the probability that the team A wins the league match.
5. Two dice are thrown. Getting two numbers whose sum is divisible by 4 or 5 is considered a success. Find the probability of success.
6. Two cards are drawn at random from a well-shuffled deck of 52 cards with replacement.

   What is the probability that both the cards are either black or red ?
7. A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability that the card is an ace or a black card.
8. Two dice are thrown once. Find the probability of getting a multiple of 3 on the first die or a total of 8.
9. (a) In a single throw of two dice, find the probability of a total of 5 or 7.

   (b) A and B are two mutually exclusive events such that P (A) = 0.3 and P (B) = 0.4. Calculate P (A or B).
10. A box contains 12 light bulbs of which 5 are defective. All the bulbs look alike and have equal probability of being chosen. Three bulbs are picked up at random. What is the probability that at least 2 are defective?
11. Two dice are rolled once. Find the probability

    (a) that the numbers on the two dice are different,

    (b) that the total is at least 3.
12. A couple have three children. What is the probability that among the children, there will be at least one boy or at least one girl ?

13. Find the odds in favour and against each event for the given probability

    (a) P(A) = 0.7 (b) $P(A) = \frac{4}{5}$

14. Determine the probability of A for the given odds

    (a) 7 to 2 in favour of A (b) 10 to 7 against A.

15. If two dice are thrown, what is the probability that the sum is

    (a) greater than 4 and less than 9 ?

    (b) neither 5 nor 8 ?

16. Which of the following probability assignments are inconsistent ? Give reasons.

    (a) P (A) = 0.5 , P (B) = 0.3 , P (A and B) = 0.4

    (b) P (A) = P (B) = 0.4, P (A and B) = 0.2

    (c) P (A) = 0.85, P (B) = 0.8, P (A and B) = 0.61

17. Two balls are drawn at random from a bag containing 5 white and 10 green balls. Find the probability that the sample contains at least one white ball.

18. Two cards are drawn at random from a well-shuffled deck of 52 cards with replacement.
    What is the probability that both cards are of the same suit ?

30.8 MULTIPLICATION LAW OF PROBABILITY FOR INDEPENDENT EVENTS

Let us recall the definition of independent events.

Two events A and B are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence (and hence non-occurrence) of the other.

Can you think of some examples of independent events ?

The event of getting 'H' on first coin and the event of getting 'T' on the second coin in a simultaneous toss of two coins are independent events.

What about the event of getting 'H' on the first toss and event of getting 'T' on the second toss in two successive tosses of a coin ? They are also independent events.

Let us consider the event of 'drawing an ace' and the event of 'drawing a king' in two successive draws of a card from a well-shuffled deck of cards without replacement.

Are these independent events ?

No, these are not independent events, because we draw an ace in the first draw with probability $\frac{4}{52}$, Now, we do not replace the card and draw a king from the remaining 51 cards and this affect the probability of getting a king in the second draw, i.e., the probability of getting a king in the second draw without replacement will be $\frac{4}{51}$.

Note : If the cards are drawn with replacement, then the two events become independent. Is there any rule by which we can say that the events are independent?

How to find the probability of simultaneous occurrence of two independent events?

If A and B are independent events, then

$$P\ (A \text{ and } B) = P(A)\ .\ P(B)$$

or

$$P(A \cap B) = P(A)\ .\ P(B)$$

Thus, the probability of simultaneous occurrence of two independent events is the product of their separate probabilities.

Note : The above law can be extended to more than two independent events, i.e.,

$$P(A \cap B \cap C) = P(A)\ .\ P(B)\ .\ P(C)\ .....$$

On the other hand, if the probability of the event 'A' and 'B' is equal to the product of the probabilities of the events A and B, then we say that the events A and B are independent.

Example 30.33 : If A and B are independent events of a random experiment, show that $\overline{A}$ and $\overline{B}$ are also independent.

Solution: If A and B are independent then

$$P(A \cap B) = P(A)\ P(B)$$

$$P(\overline{A} \cap \overline{B}) = P\left[\overline{(A \cup B)}\right]$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [P(A) + P(B) - P(A)\ P(B)]$$

$$= 1 - [P(A) + P(B)[1 - P(A)]$$

$$= (1 - P(A)) - P(B)\ (1 - P(A))$$

$$= [1 - P(A)]\ \ [(1 - P(B)]$$

$$= P(\overline{A}) - P(\overline{B}).$$

$\therefore$ $\overline{A}$ and $\overline{B}$ are independent.

Example 30.34: A die is tossed twice. Find the probability of a number greater than 4 on each throw.

Solution: Let us denote by A, the event 'a number greater than 4' on first throw. B be the event 'a number greater than 4' in the second throw. Clearly A and B are independent events. In the first throw, there are two outcomes, namely, 5 and 6 favourable to the event A.

$\therefore$ $$P(A) = \frac{2}{6} = \frac{1}{3}$$

Similarly, $$P(B) = \frac{1}{3}$$

Hence $$P(A\ andB) = P(A)\ .\ P(B)$$

$$= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}.$$

Example 30.35: Arun and Tarun appear for an interview for two vacancies. The probability of Arun's selection is $\frac{1}{3}$ and that of Tarun's selection is $\frac{1}{5}$. Find the probability that

(a) both of them will be selected.

(b) none of them is selected.

(c) at least one of them is selected

(d) only one of them is selected.

Solution: Probability of Arun's selection $P(A) = \frac{1}{3}$

Probability of Tarun's selection $P(T) = \frac{1}{5}$

(a) P((both of them will be selected) = P(A) P(T)

$$= \frac{1}{3} \times \frac{1}{5}$$

$$= \frac{1}{15}$$

(b) P (none of them is selected)

$$= P(\overline{A})\ P(\overline{T})$$

$$= \left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

$$= \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}.$$

(c) P (at least one of them is selected)

$$= 1 - P\text{ (None of them is selected)}$$

$$= 1 - P(\overline{A})\ P(\overline{T})$$

$$= 1-\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

$$= 1-\left(\frac{2}{3}\times\frac{4}{5}\right)$$

$$= 1 - \frac{8}{15} = \frac{7}{15}$$

(d) P (only one of of them is selected)

$$= P(A)\ \ P(\overline{T}) + P(\overline{A})\ P(T)$$

$$= \frac{1}{3} \times \frac{4}{5} + \frac{2}{3} \times \frac{1}{5}$$

$$= \frac{6}{15} = \frac{2}{15}.$$

Example 30.36: A problem in statistics is given to three students, whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ .respectively. What is the probability that problem will be solved ?

Solution: Let p_1, p_2 and p_3 be the probabilities of three persons of solving the problem.

Here, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$ and $p_3 = \frac{1}{4}$.

The problem will be solved, if at least one of them solves the problem.

$\therefore$ P (at least one of them solves the problem)

= 1 − P(None of them solves the problem)

Now, the probability that none of them solves the problem will be

P (none of them solves the problem) = $(1 - p_1)(1 - p_2)(1 - p_3)$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

Putting this value in (1), we get

P (at least one of them solves the problem) $= 1 - \frac{1}{4}$

$$= \frac{3}{4}$$

Hence, the probability that the problem will be solved is $\frac{3}{4}$.

Example 30.37: Two balls are drawn at random with replacement from a box containing 15 red and 10 white balls. Calculate the probability that

(a) both balls are red.

(b) first ball is red and the second is white.

(c) one of them is white and the other is red.

Solution :

(a) Let A be the event that first drawn ball is red and B be the event that the second ball drawn is red. Then as the balls drawn are with replacement,

Therefore $P(A) = \frac{15}{25} = \frac{3}{5}, \; P(B) = \frac{3}{5}$

As A and B are independent events

therefore P (both red)= P (A and B)

$$= P(A) \times P(B)$$

$$= \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}.$$

(b) Let A : First ball drawn is red.

B : Second ball drawn is white.

$$\therefore \quad P(A \text{ and } B) = P(A) \times P(B)$$

$$= \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}.$$

(c) If WR denotes the event of getting a white ball in the first draw and a red ball in the second draw and the event RW of getting a red ball in the first draw and a white ball in the second draw.

Then as ‘RW’ and WR’ are mutually exclusive events, therefore

P (a white and a red ball)

$$= P(WR \text{ or } RW)$$

$$= P(WR) + P(RW)$$

$$= P(W)\ P(R) + P(R)\ P(W)$$

$$= \frac{2}{5}\cdot\frac{3}{5} + \frac{3}{5}\cdot\frac{2}{5}$$

$$= \frac{6}{25} + \frac{6}{25} = \frac{12}{25}.$$

Example 30.38: The odds against Manager X settling the wage dispute with the workers are 8 : 6 and odds in favour of manager Y settling the same dispute are 14 : 16.

(i) What is the chance that neither settles the dispute, if they both try independently of each other ?

(ii) What is the probability that the dispute will be settled ?

Solution : Let A be the event that the manager X will settle the dispute and B be the event that the manager Y will settle the dispute. Then, clearly

(i) $P(A) = \frac{6}{14} = \frac{3}{7}$, $P(\bar{A}) = 1 - P(A) = 1 - \frac{3}{7} = \frac{4}{7}$

$P(B) = \frac{14}{30} = \frac{7}{15}$, $P(\bar{B}) = 1 - \frac{14}{30} = \frac{16}{30} = \frac{8}{15}$

The required probability that neither settles the dispute is given by

$$P(\bar{A} \cap \bar{B}) = P(\bar{A})\ p(\bar{B})$$

$$= \frac{4}{7}\cdot\frac{8}{15} = \frac{32}{105}$$

(Since A, B are independent, therefore, $\bar{A}$, $\bar{B}$ also independent)

(ii) The dispute will be settled, if at least one of the managers X and Y settles the dispute. Hence, the required probability is given by

$P(A \cup B) =$ P[At least one of X and Y settles the dispute]

$= 1 - P$ [None settles the dispute]

$= 1 - P(\bar{A} \cap \bar{B})$

$$= 1 - \frac{32}{105} = \frac{73}{105}.$$

Example 30.39: A and B are events with P(A) = 0.5 P(B) = 0.4 and $P(A \cap B) = 0.3$. Find the probability that

(i) P(A does not occur) $= P(\bar{A})$

$= 1 - P(A)$

$= 1 - 0.5$

$= 0.5$

(ii) P(neither A nor B occur) $= P(\bar{A} \cap \bar{B})$

$= P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - [0.5 + 0.4 - 0.3]$

$= 0.4.$

Example 30.40: If A, B, C are three events show that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Solution: Write $B \cup C = D$

$$P(A \cup B \cup C) = P(A \cup D)$$

$$= P(A) + P(D) - P(A \cap D)$$

$$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C))$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P[(A \cap B) \cup (A \cap C)]$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Example 30.41: A dice is thrown 3 times. Getting a number '5 or 6' is a success. Find the probability of getting

(a) 3 successes (b) exactly 2 successes (c) at most 2 successes (d) at least 2 successes.

Solution : Let S denote the success in a trial and F denote the ' not success' i.e. failure.

Therefore,

$$P(S)=\frac{2}{6}=\frac{1}{3}$$

$$P(F)=\frac{4}{6}=\frac{2}{3}$$

(a) As the trials are independent, by multiplication theorem for independent events,

$$P(SSS) = P(S)\ P(S)\ P(S)$$

$$= \frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}=\frac{1}{27}$$

$$P(SSS)= P(S)\ P(S)\ P(F)$$

$$=\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}=\frac{2}{27}$$

Since the two successes can occur in ${}^3C_2$ ways

$\therefore$ P (exactly two successes) $= {}^3C_2 \times \frac{2}{27} = 3\times\frac{2}{27}=\frac{2}{9}$

(c) P (at most two successes) 1 – P(3successes)

$$= 1-\frac{1}{27}=\frac{26}{27}$$

(d) P (at least two successes) = P (exactly 2 successes) + P (3 successes)

$$=\frac{2}{9}+\frac{1}{27}=\frac{7}{27}.$$

Example 30.42: A card is drawn from a pack of 52 cards so that each card is equally likely to be selected. Which of the following events are independent?

(i) A : the card drawn is a spade
B : the card drawn is an ace

(ii) A : the card drawn is black
B : the card drawn is a king

(iii) A : the card drawn is a king or a queen
B : the card drawn is a queen or a jack

Solution : (i) There are 13 cards of spade in a pack.

$$P(A) = \frac{13}{52} = \frac{1}{4}$$

There are four aces in the pack.

$$\therefore \quad P(B) = \frac{4}{52} = \frac{1}{13}$$

$$A \cap B = \{\text{an ace of spade}\}$$

$$\therefore \quad P(A \cap B) = \frac{1}{52}$$

Now $\quad P(A)\ P(B) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$

Since $\quad P(A \cap B) = P(A)\ .\ P(B)$

Hence, the events A and B are independent.

(ii) There are 26 black cards in a pack.

$$\therefore \quad P(A) = \frac{26}{52} = \frac{1}{2}$$

There are four kings in the pack.

$$\therefore \quad P(B) = \frac{4}{52} = \frac{1}{13}$$

$$A \cap B = \{2 \text{ black kings}\}$$

$$\therefore \quad P(A \cap B) = \frac{2}{52} = \frac{1}{26}$$

Now $\quad P(A) \times P(B) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26}$

Since $\quad P(A \cap B) = P(A)\ \ P(B)$

Hence, the events A and B are independent.

(iii) There are 4 kings and 4 queens in a pack of cards.

$\therefore$ Total number of outcomes favourable to the event A is 8.

$\therefore \qquad P(A) = \frac{8}{52} = \frac{2}{13}$

Similarly, $\qquad P(B) = \frac{2}{13}$

$A \cap B = \{4 \text{ queens}\}$

$\therefore \qquad P(A \cap B) = \frac{4}{52} = \frac{1}{13}$

$\therefore \qquad P(A) \times P(B) = \frac{2}{13} \times \frac{2}{13} = \frac{4}{169}.$

Here, $\quad P(A \cap B) \neq P(A) \,.\, P(B)$

Hence, the events A and B are not independent.

Example 30.43: Consider a group of 36 students. Suppose that A and B are two properties that each student either has or does not have. The events are

A : Student has blue eyes

B : Student is a male

Out of 36, there are 12 male and 24 female students and half of them in each has blue eyes. Are these events independent ?

Solution : With regard to the given two properties, i.e., either has or does not have, the 36 students are distributed as follows :

	Blue eyes A	Not blue eyes $(\bar{A})$	Total
Male (B)	6	6	12
Female$(\bar{B})$	12	12	24
Total	18	18	36

If we choose a student at random, the probabilities corresponding to the events A and B are

$$P(A) = \frac{18}{36} = \frac{1}{2}$$

$$P(B) = \frac{12}{36} = \frac{1}{3}$$

$$P(A \cap B) = \frac{6}{36} = \frac{1}{3}$$

Also $P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

Here, $P(A \cap B) = P(A).\ P(B)$

Hence, the events A and B are independent.

Example 30.44: Suppose that we toss a coin three times and record the sequence of heads and tails. Let A be the event ' at most one head occurs' and B the event ' both heads and tails occur'. Are these event independent ?

Solution : The sample space in tossing a coin three times will be

$$S = \{HHH,\ HHT, HTH, HTT, THH, TTH, THT, TTT\}$$

Also $A \cap B = \{TTH, THT, HTT\}$

$\therefore$ $P(A) = \frac{4}{8} = \frac{1}{2}$, $P(B) = \frac{6}{8} = \frac{3}{4}$, $P(A \cap B) = \frac{3}{8}$

Moreover, $P(A) \times P(B) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$

Which equals $P(A \cap B)$. Hence, A and B are independent.

Exercise 30.5

1. A husband and wife appear in an interview for two vacancies in the same department The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that

(a) Only one of them will be selected ?

(b) Both of them will be selected ?

(c) None of them will be selected ?

(d) At least one of them will be selected ?

2. Probabilities of solving a specific problem independently by Raju and Soma are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(a) the problem is solved.

(b) exactly one of them solves the problem.

3. A die is rolled twice. Find the probability of a number greater than 3 on each throw.

4. Sita appears in the interview for two posts A and B, selection for which are independent.

The probability of her selection for post A is $\frac{1}{5}$ and for post B is $\frac{1}{7}$.

Find the probability that she is selected for

(a) both the posts

(b) at least one of the posts.

5. The probabilities of A, B and C solving a problem are $\frac{1}{3}, \frac{2}{7}$ and $\frac{3}{8}$ respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them will solve it.

6. A draws two cards with replacement from a well-shuffled deck of cards and at the same time B throws a pair of dice. What is the probability that

(a) A gets both cards of the same suit and B gets a total of 6 ?

(b) A gets two jacks and B gets a doublet ?

7. Suppose it is 9 to 7 against a person A who is now 35 years of age living till he is 65 and 3:2 against a person B now 45 living till he is 75. Find the chance that at least one of these persons will be alive 30 years hence.

8. A bag contains 13 balls numbered from 1 to 13. Suppose an even number is considered a 'success'. Two balls are drawn with replacement, from the bag. Find the probability of getting

(a) Two successes (b) exactly one success

(c) at least one success (d) no success

9. One card is drawn from a well-shuffled deck of 52 cards so that each card is equally likely to be selected. Which of the following events are independent?

(a) A : The drawn card is red B : The drawn card is a queen

(b) A : The drawn card is a heart B: The drawn card is a face card

30.9 CONDITIONAL PROBABILITY

Suppose that a fair die is thrown and the score noted. Let A be the event, the score is 'even'. Then

$$A = \{2, 4, 6\}$$

$$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}$$

Now suppose we are told that the score is greater than 3. With this additional information what will be P (A) ?

Let B be the event, 'the score is greater than 3'. Then B is {4, 5, 6 }. When we say that B has occurred, the event 'the score is less than or equal to 3' is no longer possible. Hence the sample space has changed from 6 to 3 points only. Out of these three points 4, 5 and 6; 4 and 6 are even scores.

Thus, given that B has occurred, P (A) must be $\frac{2}{3}$.

Let us denote the probability of A given that B has already occurred by P(A|B) .

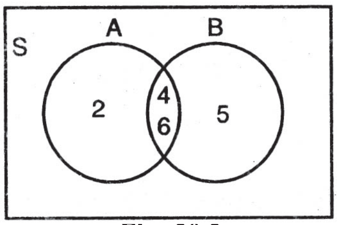

Fig. 30.2

Again, consider the experiment of drawing a single card from a deck of 52 cards. We are interested in the event A consisting of the outcome that a black ace is drawn.

Since we may assume that there are 52 equally likely possible outcomes and there are two black aces in the deck, so we have

$$P(A) = \frac{2}{52}$$

However, suppose a card is drawn and we are informed that it is a spade. How should this information be used to reappraise the likelihood of the event A ?

Clearly, since the event B "A spade has been drawn" has occurred, the event “not spade” is no longer possible. Hence, the sample space has changed from 52 playing cards to 13 spade cards. The number of black aces that can be drawn has now been reduced to 1.

Therefore, we must compute the probability of event A relative to the new sample space B.

Let us analyze the situation more carefully.

The event A is "a black ace is drawn". We have computed the probability of the event A knowing that B has occurred. This means that we are computing a probability relative to a new sample space B. That is, B is treated as the universal set. We should consider only that part of A which is included in B.

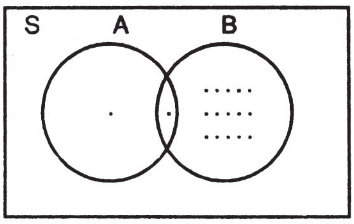

Fig. 30.3

Hence, we consider $A \cap B$ (see figure 31.3).

Thus, the probability of A given B, is the ratio of the number of entries in $A \cap B$ to the number of entries in B. Since $n(A \cap B) = 1$ and $n(B) = 13$

then $$P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{13}$$

Note that $n(A \cap B) = 1 \Rightarrow P(A \cap B) = \frac{1}{52}$

$$n(B) = 13 \Rightarrow P(B) = \frac{13}{52}$$

$$\therefore \quad P(A/B) = \frac{1}{13} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{P(A \cap B)}{P(B)}$$

This leads to the definition of conditional probability as given below :

Let A an B be two events defined on a sample space S. Let P(B) > 0, then the conditional probability of A, provided B has already occurred, is denoted by P(A|B) and mathematically written as :

$$P(A/B) = \frac{P(A \cap B)}{P(A)}, \ P(B) > 0$$

Similarly, $$P(B/A) = \frac{P(A \cap B)}{P(A)}, \ P(A) > 0$$

The symbol P(A | B) is usually read as "the probability of A given B".

Example 30.45: Consider all families “with two children (not twins). Assume that all the elements of the sample space {BB, BG, GB,GG} are equally likely. (Here, for instance, BG denotes the birth sequence “boy girls”). Let A be the event {BB} and B be the event that ‘atleast one boy’. Calculate P (A | B).

Solution: Here, $A = \{BB\}$

$$B = \{BB, BG, GB\}$$

$$A \cap B = \{BB\}$$

$$\therefore \quad P(A \cap B) = \frac{1}{4}$$

$$P(B) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Hence $$P(A / B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Example 30.46: Assume that a certain school contains equal number of female and male students. 5 % of the male population is football players. Find the probability that a randomly selected student is a football player male.

Solution: Let M = Male

F = Football player

We wish to calculate $P(M \cap F)$ From the given data,

$P(M) = \frac{1}{2}$ ($\because$ School contains equal number of male and female students)

$P(F/M) = 0.05$

But from definition of conditional probability, we have

$$P(F/M) = \frac{P(M \cap F)}{P(M)}$$

$$\Rightarrow \quad P(M \cap F) = P(M) \; P(F/M)$$

$$= \frac{1}{2} \times 0.05 = 0.025$$

Example 31.47: If A and B are two events, such that P (A) = 0.8, P(B) = 0.6, $P(A \cap B) = 0.5$ find the value of

(i) $P(A \cup B)$ (ii) P(B/A) (iii) P(A/ B)

Solution: (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.8 + 0.6 - 0.5 = 0.9$$

(ii) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.5}{0.8} = \dfrac{5}{8}$

(iii) $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$= \frac{0.5}{0.6} = \frac{5}{6}$$

Example 30.48: Find the chance of drawing 2 white balls in succession from a bag containing 5 red and 7 white balls, the balls drawn not being replaced.

Solution : Let A be the event that ball drawn is white in the first draw. B be the event that ball drawn is white in the second draw.

$\therefore \quad P(A \cap B) = P(A)\, P(B/A)$

Here $P(A) = \dfrac{7}{12}$, $P(B|A) = \dfrac{6}{11}$

$\therefore \quad P(A \cap B) = \dfrac{7}{12} \times \dfrac{6}{11} = \dfrac{7}{22}$

Example 30.49: A coin is tossed until a head appears or until it has been tossed three times. Given that head does not occur on the first toss, what is the probability that coin is tossed three times ?

Solution : Here, it is given that head does not occur on the first toss. That is, we may get the head on the second toss or on the third toss or even no head.

Let B be the event, " no heads on first toss".

Then $B = \{TH, TTH, TTT\}$

These events are mutually exclusive.

$$P(B) = P(TH) + P(TTH) + P(TTT) \qquad .... (1)$$

Now $P(TH) = \dfrac{1}{4}$ ($\because$ This event has the sample space of four outcomes)

and $P(TTH) = P(TTT) = \dfrac{1}{8}$

($\because$ This event has the sample space of eight outcomes)

Putting these values in (1), we get

$$P(B) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Let A be the event "coin is tossed three times".

Then $A = \{TTH, TTT\}$

$\therefore$ We have to find $P(A \mid B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Here $A \cap B = A$

$$\therefore \quad P(A|B) = \frac{P(A)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Exercise 30.6

1. A sequence of two cards is drawn at random (without replacement) from a well-shuffled deck of 52 cards. What is the probability that the first card is red and the second card is black ?

2. Consider a three child family for which the sample space is

   { BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG }

   Let A be the event " the family has exactly 2 boys " and B be the event " the first child is a boy". What is the probability that the family has 2 boys, given that first child is a boy ?

3. Two cards are drawn at random without replacement from a deck of 52 cards. What is the probability that the first card is a diamond and the second card is red ?

4. If A and B are events with $P(A) = 0.4$, $P(B) = 0.2$, $P(A \cap B) = 0.1$ find the probability of A given B. Also find $P(B|A)$.

5. From a box containing 4 white balls, 3 yellow balls and 1 green ball, two balls are drawn one at a time without replacement. Find the probability that one white and one yellow ball is drawn.

30.10 THEOREMS ON MULTIPLICATION LAW OF PROBABILITY, CONDITIONAL PROBABILITY AND TOTAL PROBABILITY

Theorem 1 : For two events A and B,

$$P(A \cap B) = P(A) \,.\, P(B|A)$$

and $$P(A \cap B) = P(B) \,.\, P(A|B)$$

where P (B|A) represents the conditional probability of occurrence of B, when the event A has already occurred and P (A|B) is the conditional probability of happening of A, given that B has already happened.

Proof : Let n (S) denote the total number of equally likely cases, n (A) denote the cases favourable to the event A, n (B) denote the cases favourable to B and n(A ∩ B) denote the cases favourable to both A and B.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)}$$

$$P(B) = \frac{n(B)}{n(S)}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} \qquad ...(1)$$

For the conditional event A|B , the favourable outcomes must be one of the sample points of B, i.e., for the event A|B, the sample space is B and out of the n (B) sample points, n (A ∩ B) pertain to the occurrence of the event A, Hence,

$$P(A/B) = \frac{n(A \cap B)}{n(B)}$$

Rewriting (1), we get $P(A \cap B) = \frac{n(B)}{n(S)} = \frac{n(A \cap B)}{n(B)}$ =P(B) . P(A|B)

Similarly, we can prove

$$P(A \cap B) = P(A).\, P(B|A) \,.$$

Note : If A and B are independent events, then

$$P(A|B) = P(A) \quad \text{and } P(B|A) = P(B)$$

$$P(A \cap B) = P(A) \cdot P(B)$$

Theorem 2: Two events A and B of the sample space S are independent, if and only if

$$P(A \cap B) = P(A) \,.\, P(B)$$

Proof : If A and B are independent events,

then $\quad P(A|B) = P(A)$

we know that $\quad P(A\,|\,B) = \dfrac{P(A \cap B)}{P(B)}$

$$\Rightarrow \quad P(A \cap B) = P(A)\,P(B)$$

Hence, if A and B are independent events, then the probability of 'A and B' is equal to the product of the probability of A and probability of B.

Conversely, if $P(A \cap B) = P(A)\,P(B)$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{gives}$$

$$P(A\,|\,B) = \frac{P(A)\,P(B)}{P(B)} = P(A)$$

That is, A and B are independent events.

Theorem 3: (Theorem of Total Probability)

Let $E_1, E_2 \,.....\, E_n$ are mutually exclusive and exhaustive events for a sample space s with $P(E_i) > 0 \;\forall\; i = 1, 2, \ldots n$. Let A be any event associated with S, then

$$P(A) = P(E_1)\;P(A/E_1) + P(E_2)\;P(A/E_2) + \ldots. + P(E_n)\;P(A/E_n)$$

$$= \sum_{i=1}^{n} P(E_i)P(A/E_i)$$

Proof : Given $E_1, E_2 \ldots E_n$ are mutually exclusive and exhaustive events for S.

MODULE - VI
Statistics and Probability

Notes

$S = E_1 \cup E_2 .... E_n$ and $E_i \cap E_j = \phi$

We can write $A = A \cap S$ $(\because A \subset S)$

$= A \cap (E_1 \cup E_2 \cup ... \cup E_n)$

$= (A \cap E_1) \cup (A \cap E_2) \cup ... \cup (A \cap E_n)$

Since all E_i's are mutually exclusive, so $A \cap E_1, A \cap E_2$ ... will also be mutually exclusive

$$\Rightarrow P(A) = P(A \cap E_1) + P(A \cap E_2) + ... + P(A \cap E_n)$$

$$= P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + ... + P_n\,P(A/E_n)$$

(By using the multiplication rule of the probability)

$$= \sum_{i=1}^{n} P(E_i)\,P(A/E_i)$$

30.11 BAYE'S THEOREM

Suppose $E_1, E_2, ... E_n$ are n mutually exclusive and exhaustive events of a random experiments with $P(E_i) \neq 0$, for $i = 1, 2, ... n$. Then for any event A of the random experiment with $P(A) \neq 0$.

$$P(E_K / A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)}, \; i = 1, 2, ...n.$$

Proof: Given that $P(E_i) > 0$ for $i = 1, 2, ... n$.

By hupothesis, $i \neq j$, $E_i \cap E_j = \phi$ and $\bigcup_{i=1}^{n} E_i = S$, the sample space of the experiment.

Since $A \subseteq S$ for any even A, we have

$$A = A \cap S = A \cap \left(\bigcup_{i=1}^{n} E_i \right) = \bigcup_{i=1}^{n} (A \cap E_i)$$

Also, for $i \neq j$, $(A \cap E_i) \cap (A \cap E_j) = A \cap (E_i \cap E_j)$

$$= A \cap \phi$$
$$= \phi.$$

Therefore $P(A) = \sum_{i=1}^{n} P(A \cap E_i)$

$$= \sum_{i=1}^{n} P(E_i)P(A / E_i)$$

(by using multiplication theorem)

Hence $P(E_k / A) = \dfrac{P(E_k \cap A)}{P(A)}$

$$= \frac{P(E_k)\, P(A / E_k)}{\sum_{i=1}^{n} P(E_i)\, P(A / E_i)}$$

Example 30.50 : A shop keeper buys a particular type of electric bulbs from three manufactures M_1, M_2 and M_3. He buys 25% of his requirement from M, 45% from M_2 and 30% from M_3. Based on the part experience he found that 2% of type M_3 bulbs are detective, where as only 1% of type M_1 and type M_2 are detective. If a bulb chosen by him at random is found defective, let us find the probability that it was of type M_3.

Solution : If E is the event that the bulb chosen is defective, then $P(M_3/E)$ is the required probability.

Given $P(M_1) = 0.25$ $P(M_2) = 0.45$, and $P(M_3) = 0.3$

$P(E/M_1) = 0.01$, $P(E/M_2) = 0.01$ and $P(E/M_3) = 0.02$

By Baye's theorem

$$P(M_3/E) = \frac{P(E/M_3)P(M_3)}{P(E/M_1)P(M_1) + P(E/M_2)P(E/M_3)P(M_3)}$$

$$= \frac{0.02 \times 0.3}{(0.01 \times 0.25) + (0.01 \times 0.45) + (0.2 \times 0.3)}$$

$$= 0.46.$$

KEY WORDS

- The set of possible outcomes of a random experiment is called sample sapce.
- An event is a subset of the sample space.
- **Events Relation :** The complement of an event A consists of all those outcomes which are not favourable to the event A, and is denoted by 'not A' or by $\overline{A}$.
- **Event 'A or B' :** The event 'A or B' occurs if either A or B or both occur.
- **Event 'A and B' :** The event 'A and B' consists of all those outcomes which are favourable to both the events A and B.
- **Addition Law of Probability :** For any two events A and B of a sample space S

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

- **Additive Law of Probability for Mutually Exclusive Events :** If A and B are two mutually exclusive events, then

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$$

- **Odds in Favour of an Event :** If the odds for A are a to b, then

$$P(A) = \frac{a}{a+b}$$

If odds against A are a to b, then

$$P(A) = \frac{a}{a+b}$$

- Two events are mutually exclusive, if occurrence of one precludes the possibility of simultaneous occurrence of the other.
- Two events A and B are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence (and hence non-occurrence) of the other.

- If A and B are independent events, then

  P(A and B) = P(A) . P(B)

  or P(A ∩ B) = P(A) . P(B)

- For two events A and B,

  $P(A \cap B) = P(A)\, P(B|A), \quad P(A) > 0$

  or $P(A \cap B) = P(B)\, P(A|B), \quad P(B) > 0$

  where P (B|A) represents the conditional probability of occurrence of B, when the event A has already happened and P (A|B) represents the conditional probability of happening of A, given that B has already happened.

- **Baye's theorem:** If $E_1, E_2, \ldots E_n$ are mutially exclusive and exhaustive events of a random experiment with $P(E_i) > 0$ for $i = 1, 2, \ldots, n$ them.

$$P(E_K / A) = \frac{P(E_K) + P(A/E_K)}{\sum_{i=1}^{n} P(E_i)\ P(A/E_i)}; \qquad K = 1, 2, \ldots n.$$

SUPPORTIVE WEB SITES

http : //www.wikipedia.org

http:// math world . wolfram.com

PRACTICE EXERCISE

1. In a simultaneous toss of four coins, what is the probability of getting
   (a) exactly three heads ? (b) at least three heads ?
   (c) atmost three heads ?
2. Two dice are thrown once. Find the probability of getting an odd number on the first die or a sum of seven.
3. An integer is chosen at random from first two hundred integers. What is the probability that the integer chosen is divisible by 6 or 8 ?

4. A bag contains 13 balls numbered from 1 to 13. A ball is drawn at random. What is the probability that the number obtained it is divisible by either 2 or 3 ?

5. Find the probability of getting 2 or 3 heads, when a coin is tossed four times.

6. Are the following probability assignments consistent ? Justify your answer.

   (a) P(A) = 0.6, P(B) = 0.5, P(A and B) = 0.4

   (b) P(A) = 0.2, P(B) = 0.3, P(A and B) = 0.4

   (c) P(A) = P(B) = 0.7, P(A and B) = 0.2

7. A box contains 25 tickets numbered 1 to 25. Two tickets are drawn at random. What is the probability that the product of the numbers is even ?

8. A drawer contains 50 bolts and 150 nuts. Half of the bolts and half of the nuts are rusted. If one item is chosen at random, what is the probability that it is rusted or is a bolt ?

9. A lady buys a dozen eggs, of which two turn out to be bad. She chose four eggs to scramble for breakfast. Find the chances that she chooses

   (a) all good eggs
   (b) three good and one bad eggs
   (c) two good and two bad eggs
   (d) at least one bad egg.

10. Two cards are drawn at random without replacement from a well-shuffled deck of 52 cards. Find the probability that the cards are both red or both kings.

11. Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, 1 girl and 3 boys. One child is selected at random from each group. Show the chances that three selected children consist of 1 girl and 2 boys is $\frac{13}{32}$.

12. A die is thrown twice. Find the probability of a prime number on each throw.

13. Kamal and Monika appears for an interview for two vacancies. The probability of Kamal's selection is $\frac{1}{3}$ and that of Monika's rejection is $\frac{4}{5}$. Find the probability that only one of them will be selected.

14. A bag contains 7 white, 5 black and 8 red balls. Four balls are drawn without replacement. Find the probability that all the balls are black.

15. For two events A and B, it is given that

    $P(A) = 0.4, \quad P(B) = P \quad \text{and } P(A \cup B) = 0.6$

    (a) Find p so that A and B are independent events.

    (b) For what value of p, of A and B are mutually exclusive ?

16. The odds against A speaking the truth are 3 : 2 and the odds against B speaking the truth are 5 : 3. In what percentage of cases are they likely to contradict each other on a identical issue ?

17. Let A and B be the events such that

    $P(\overline{A}) = \frac{1}{2}, \ P(\overline{B}) = \frac{2}{3}, \ P(A \cup B) = \frac{1}{4}.$

    Compute P (A|B) and P (B|A).

18. Suppose that a can B_1 contains 2 white and 3 black balls and another can B_2 contains 3 white and 4 black balls. One can is selected at random and a ball is drawn from it. If the ball drawn is found black find the probability that the can chosen was B_1.

ANSWERS

Exercise 30.1

1. Both properties are satisfied
2. Outcome is predictable
3. S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
4. {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}
5. No.

6. (i) True (ii) True (iii) False (iv) True 7. 15

7. 15

8. {(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1,6)
(2, 1) (2, 2) (2, 3) (2, 4) (2,5) (2, 6)
(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)
(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)
(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6)
(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

9. {MM, MF, FM, FF}

Exercise 30.2

1. $\frac{1}{6}$ 2. $\frac{1}{2}$ 3. $\frac{1}{2}$ 4. $\frac{3}{4}$

5. (i) $\frac{3}{5}$ (ii) $\frac{2}{3}$

6. (i) $\frac{5}{36}$ (ii) $\frac{5}{36}$ (iii) $\frac{1}{12}$ (iv) $\frac{1}{36}$

7. $\frac{5}{9}$ 8 $\frac{1}{12}$ 9. $\frac{1}{2}$

10. (i) $\frac{1}{4}$ (ii) $\frac{1}{13}$ (iii) $\frac{1}{52}$

11. (i) $\frac{5}{12}$ (ii) $\frac{1}{6}$ (iii) $\frac{11}{36}$

12. (i) $\frac{1}{8}$ (ii) $\frac{7}{8}$ (iii) $\frac{1}{8}$

Exercise 30.3

1. $\frac{1}{8}$ 2. $\frac{20}{39}$ 3.(a) $\frac{4}{25}$ (b) $\frac{38}{245}$

4. $\frac{1}{5525}$

6. (i) $\frac{3}{10}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{30}$

7. $\frac{10}{133}$ 8 $\frac{4}{7}$ 9. $\frac{60}{143}$ 10. $\frac{1}{4}$

Exercise 30.4

1. $\frac{4}{13}$ 2. $\frac{7}{36}$ 3. $\frac{9}{16}$

4. $\frac{7}{12}$ 5. $\frac{4}{9}$ 6. $\frac{1}{2}$

7. $\frac{7}{13}$ 8. $\frac{5}{12}$

9. (a) $\frac{5}{18}$ (b) 0.710. $\frac{4}{11}$

11. (a) $\frac{5}{6}$ (b) $\frac{35}{36}$ 12. $\frac{3}{4}$

13. (a) The odds for A are 7 to 3 . The odds against A are 3 to 7

(b) The odds for A are 4 to 1 and The odds against A are 1 to 4

14. (a) $\frac{7}{9}$ (b) $\frac{7}{17}$ 15. (a) $\frac{5}{9}$ (b) $\frac{3}{4}$

16. (a), (c)

17. $\frac{4}{7}$ 18. $\frac{1}{4}$

Exercise 30.5

1. (a) $\frac{2}{7}$ (b) $\frac{1}{35}$ (c) $\frac{24}{35}$ (d) $\frac{11}{35}$

2. (a) $\frac{2}{3}$ (b) $\frac{1}{2}$ 3. $\frac{1}{4}$ 4. (a) $\frac{1}{35}$ (b) $\frac{11}{35}$

5. $\frac{1}{2}$ 6. (a) $\frac{5}{144}$ (b) $\frac{1}{1014}$ 7. $\frac{53}{80}$

8. (a) $\frac{36}{169}$ (b) $\frac{84}{169}$ (c) $\frac{120}{169}$ (d) $\frac{149}{169}$

9. (a) Independent (b) Independent

Exercise 30.6

1. $\frac{13}{51}$ 2. $\frac{1}{2}$ 3. $\frac{25}{204}$

4. $\frac{1}{2}, \frac{1}{4}$ 5. $\frac{3}{7}$

PRACTICE EXERCISE

1. (a) $\frac{1}{4}$ (b) $\frac{5}{16}$ (c) $\frac{15}{16}$

2. $\frac{7}{12}$ 3. $\frac{1}{4}$ 4. $\frac{8}{13}$

5. $\frac{5}{8}$ 6. Only (a) is consistent

7. $\frac{436}{625}$ 8. $\frac{5}{8}$

9. (a) $\frac{14}{33}$ (b) $\frac{16}{33}$ (c) $\frac{1}{11}$ (d) $\frac{19}{33}$

10. $\frac{55}{221}$ 12. $\frac{1}{4}$ 13. $\frac{2}{15}$

14. $\frac{1}{969}$ 15, $\frac{1}{3}$ 16. $\frac{19}{40}$

17. $\frac{3}{4}, \frac{1}{2}$ respectively. 18. $\frac{21}{41}$

Chapter 31

RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

LEARNING OUTCOMES

After studying chapter, student will be able to

- Distinguish between discrete and continuous random variables.
- Calculate mean, variance and standard deviation of a probability distribution.
- Compute probabilities of a binomial random variable and a Poisson random variable.

PREREQUISITES

Random Experiments and Events, Probability and Binomial Coefficients.

INTRODUCTION

In a random experiment, we may be interested quite often in the numerical measure of the different out comes. It is true that in some experiments, the outcomes are directly expressed in quantitative measures. For example, consider tosses of unbiased coins. There will be 2^{nd} elementary events. We may be more interested to know the number of heads (or tails) in each outcome. For this purpose, we introduce random variables whose value is determined by the outcome of a random experiment is called a random variable.

31.1 DEFINITION

Let S be the sample space associated with a random experiment A function X : S $\to$ R is called a random variable.

Note : If X is a random variable then $X^{-1}(P(R)) = P(S)$

Here P stands for the probability function and P(S) stands for the power set of S .

Example 31.1

Let S be the sample space of the experiment of rolling a fair die.

Then X : S $\to$ R given by X(n) = 0, if n is even.

= 1, If n X : S $\to$ is odd.

is a random variable.

Here S = {1, 2, 3, 4, 5, 6} and X(1) = X(3) = X(5) = 1; X(2) = X(4) = X(6) = 0.

Example 31.2

Let two coins be tossed simultaneously. Let S denote the sample space of the experiment. Then

S = {HH. HT, TH, TT}

if X denotes the number of heads obtained then X is a random variable.

Here X takes the values 0, 1, 2

X(TT) = 0, X(HT) = 1, X(TH) = 1, X(HH) = 2.

31.1.1 Definition

Suppose X : S $\to$ R is a random variable. If the range of X is either finite or countably infinite, then X is called a discrete random variable.

A random variable which can take all real values in an interval (a, b) is called a conitnuous random variable.

31.1.2 Probability distribution of a random variable

Suppose X is a discrete random variable with range E = $\{x_i / i \geq 1\}$. E may be finite or countably infinite. With each possible outcome x_i, we associate a number P(X = x_i) = p(x_i), called the probability of x_i. The number P(x_i), i = 1, 2, 3, ... must satisfy the following conditions.

(i) $P(x_i) \geq 0$ for every i (ii) $\sum_{i \geq 1} P(x_i) = 1$

The set $\{P(X = x_i) = P(x_i)\}$ is called the probability distibution of the discrete random variable X. The probability distribution of the discrete random variable X is given in the following table.

$X = x_i$	x_1	x_2	x_3	...	x_n
$P(X = x_i)$	$P(x_1)$	$P(x_2)$	$P(x_3)$	...	$P(x_n)$

Example 31.3 : For the radom experiment of tossing two coins simultaneously, the sample space S = {HH, HT, TH, TT}.

For every x define $X(x)$ as the number of heads in X. Then $X(x)$ is a random variable. Range of X = {0, 1, 2}.

Now $P(X = 0)$ = Probability of getting no heads

= P({TT}) = 1/4

$P(X = 1)$ = Probability of getting one head

= P({HT, TH}) = 2/4 = 1/2.

$P(X = 2)$ = Probability of getting two heads

= P({HH}) = 1/4

The probability distribition of the random variable X is given in the following table.

$X = x_i$	0	1	2
$P(X = x_i)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

31.1.3 Definition

The mean (μ) and variance (σ^2) of a discrete random variable X are

$$\mu = \sum x_n P(X = x_n), \quad \sigma^2 = \sum (x_n - \mu)^2 P(X = x_n)$$

$$\sigma^2 = \sum \left[x_n^2 P(X = x_n) \right] - \mu^2$$

The standard deviation σ is the square root of the variance.

MODULE - VI
Statistics and Probability

Notes

Example 31.4

Find the mean and variance of the following distribution.

$X = x_i$	-2	-1	0	1	2
$P(X = x_i)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Sol : Mean of $X = \mu = \sum_{i=-2}^{2} x_i \ P(X = x_i)$

$$= (-2).\frac{1}{8} + (-1).\frac{2}{8} + 0.\frac{3}{8} + 1.\frac{1}{8} + 2.\frac{1}{8}$$

$$= \frac{1}{8}(-2 - 2 + 0 + 1 + 2)$$

$$= -\frac{1}{8}$$

Variance of $X = \sigma^2 = \sum_{i=-2}^{2} x_i^2 \ P(X = x_i) - \mu^2$

$$= 4\left(\frac{1}{8}\right) + 1\left(\frac{2}{8}\right) + 0\left(\frac{3}{8}\right) + 1\left(\frac{1}{8}\right) + 4\left(\frac{1}{8}\right) - \left(-\frac{1}{8}\right)^2$$

$$= \frac{11}{8} - \frac{1}{64}$$

$$= \frac{87}{64}.$$

Example 31.5

A cubical die is thrown. Find the mean and variance of X, giving the number on the face that shows up.

Sol : Let S be the sample space and X be the random variable associated with S, where P(X) is given by the following table

$X = x_i$	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Mean of X $= \mu = \sum_{i=1}^{6} x_i \, P(X = x_i)$

$$= 1.\frac{1}{6} + 2.\frac{1}{6} + 3.\frac{1}{6} + 4.\frac{1}{6} + 5.\frac{1}{6} + 6.\frac{1}{6}$$

$$= \frac{1}{6}\left(\frac{6 \times 7}{2}\right) = \frac{7}{2}$$

Variance of X $= \sigma^2 = \sum_{i=1}^{6} x_i^2 \, P(X = x_i) - \mu^2$

$$= 1^2.\frac{1}{6} + 2^2.\frac{1}{6} + 3^2.\frac{1}{6} + 4^2.\frac{1}{6} + 5^2.\frac{1}{6} + 6^2.\frac{1}{6} - \left(\frac{7}{2}\right)^2$$

$$= \frac{1}{6}\left(\frac{6 \times 7 \times 13}{6}\right) - \frac{49}{4} = \frac{35}{12}$$

Example 31.6

The probability distribution of a random variable X is given below.

$X = x_i$	1	2	3	4	5
$P(X = x_i)$	K	2K	3K	4K	5K

Find the value of K and the mean and variance of X.

Solution: We have $\sum_{i=1}^{5} P(X = x_i) = 1$

$\Rightarrow$ K + 2K + 3K + 4K + 5K = 1

$\Rightarrow$ 15K = 1

$\Rightarrow$ $K = \frac{1}{15}$

The mean μ of $X = \sum_{i=1}^{5} x_i P(X = x_i)$

$$= 1(K) + 2(2K) + 3(3K) + 4(4K) + 5(5K)$$

$$= 55K.$$

$$= 55 \times \frac{1}{15}$$

$$= \frac{11}{3}$$

Variance σ^2 of $X = \sum_{i=1}^{5} x_i^2 P(X = x_i) - \mu^2$

$$= 1(K) + 4(2K) + 9(3K) + 16(4K) + 25(5K) - \left(\frac{11}{3}\right)^2$$

$$= K + 8K + 27K + 64K + 125K - \left(\frac{11}{3}\right)^2$$

$$= 225K - \left(\frac{11}{3}\right)^2$$

$$= 225\left(\frac{1}{15}\right) - \frac{121}{9}$$

$$= 15 - \frac{121}{9}$$

$$= \frac{14}{9}.$$

31.2 THEORETICAL DISCRETE DISTRIBUTIONS BINOMIAL AND POISSON DISTRIBUTIONS

Suppose that an experiment has only two possible outcomes. For instance, when a coin is tossed the possible out comes are head and tail. Each performance of an experiment with two possible outcomes are termed as success and failure. If p is taken as the probability of success and q is the probability of failure, it follows that p + q = 1. Many problems can be solved by determining the probability of x successes when an experiment consists of n independent Bernoulli trials.

We shall discuss two theoretical frequency distributions in this section. One is Binomial distribution and the other is poission distribution.

31.2.1 Binomial distribution

Definition

A discrete random variable X is said to follow a binomial distribution (or simply a binomial variable with parameters n and p) where $0 < p < 1$ if

$$P(X = x) = n_{C_x}\, p^x\, q^{n-x},\ x \in \{0, 1, 2, ..., n\}$$

If X is a binomial variate with parameters n, p; then it is also described by writing $X \sim B(n, p)$

The distribution of X is summarised in the following table.

x	0	1	2	...	r ...	n
$P(X = x)$	$nC_0\ p^0 q^n$	$nC_1\ pq^{n-1}$	$nC_2\ p^2 q^{n-2}$	...	$nC_r pq^{n-r}$ ...	$nC_n\ p^n q^0$

Theorem

If $X \sim B(n, p)$, then the mean μ and the variance σ^2 of X are equal to np and npq respectively.

n is number of trials, p be the probability of sucess and q be the probability of failure.

31.3 POISSON DISTRIBUTION

Poisson distribution is a discrete probability distribution. S.D. Poisson introduced poisson distribution as a rare distribution of rare events i.e., the events whose probability of occurence is very small but the number of trials which could lead to the occurrence of the event, are very large.

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The poisson distribution can be derived as a limiting case of the Binomial Distribution under the following conditions.

(i) p, the probability of the occurence of the event is very small.

(ii) n, the number of trials in indefinitely large i.e., $n \to \infty$

(iii) np = λ (say) is finite, where λ is a positive real number and λ is called parameter of the poisson distribution.

In otherwords, poisson distribution provides an approximation of the binomial probabilities, when the number of trials (n) is very large and the probability (p) is very small, as in rare events like the number of telephone calls received at a particular telephone exchange in some unit of time, number of defective material, number of ears passing a crossing per minute during the busy hours of a day etc.

31.3.1 Definition

If X is a discrete random variable that can assume value 0, 1, 2, 3, ... such that for some fixed $\lambda > 0$.

$p(X = x) = \dfrac{e^{-\lambda}\lambda^x}{\lfloor x}$, $x = 0, 1, 2.......$ then X is said to follow a Possion distribution with parameter λ then its mean μ, variance $\sigma^2 = \lambda$.

Example 31.7

A die is thrown 3 times. If getting an odd numbers is a success, what is the probability of

a) 3 successes

b) at least 2 successes

c) At most 1 successes

Sol : Given X: "an odd number"

Then p = P(an odd number)

$$= \frac{3}{6} = \frac{1}{2}$$

q = P (not an odd number)

$$= \frac{3}{6} = \frac{1}{2}$$

Here $n = 3$

a) P(3 successes) $= 3C_3 \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

b) P(atleast 2 successes)

= P(2 success or 3 success)

= P(2 success) + p(3 success)

$$= 3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)+3C_3\left(\frac{1}{2}\right)^3$$

$$= 3.\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

c) P(At most 1 successes) = P(0 success or 1 success)

= P (0 success) + p(1 success)

$$= 3C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 + 3C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2$$

$$= \frac{1}{8} + 3.\frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Example 31.8

A fair coin is tossed 8 times. If the number of heads turned up is denoted by the variable X. Then find the mean and variance of X.

Sol : Here $n = 8$

Prabbability of geting in a head on a coin $p = \frac{1}{2}$

$$p = \frac{1}{2} \Rightarrow q = 1 - p = \frac{1}{2},\ n = 8$$

$$\text{Mean } (\mu)\ np = 8\left(\frac{1}{2}\right) = 4$$

$$\text{Variance } (\sigma^2) = npq = 8\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 2$$

Example 31.9

6 Coins are tossed simuletaneously. Find the probability of getting atleast 5 heads.

Sol : Here $n = 6$ the probability of getting a head.

$$P = \frac{1}{2} \Rightarrow q = 1 - p = \frac{1}{2}.$$

The probability of getting r heads in a random throw of 6 coins is

$$P(X = r) = {}^6C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{6-r}$$

$$= {}^6C_r\left(\frac{1}{2}\right)^6, \quad r = 0, 1, ..., 6$$

The probability of getting atleast 5 heads is

$$\Rightarrow p(X \geq 5) = P(X = 5) + P(X = 6)$$

$$= {}^6C_5\left(\frac{1}{2}\right)^6 + {}^6C_6\left(\frac{1}{2}\right)^6$$

$$= (6+1)\left(\frac{1}{2}\right)^6 = \frac{7}{64}.$$

Example 31.10

Find the parameters of the binomial variate whose mean and variance are 4, $\frac{4}{3}$ respectively.

Sol : n, p are parameters of the binomial distribution

Mean of the binomial distribution = 4

$$\text{i.e., } npq = \frac{4}{3}$$

$$\frac{npq}{np} = \frac{\left(\frac{4}{3}\right)}{4}$$

$$\Rightarrow q = \frac{1}{3}, \qquad \because p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\therefore \quad n\,p = 4$$

$$\Rightarrow \quad n\left(\frac{1}{3}\right) = 4 \Rightarrow n = 12$$

$$\therefore \quad n = 12,\ p = \frac{1}{3}.$$

Example 31.11

If X is a binomial variate with $16\,p(\mathrm{X} = 4) = p(\mathrm{X} = 2)$ and $n = 6$ then, find the parameter p.

Sol : Given $p + q = 1$

$$\because \quad 16\ (\mathrm{X} = 4) = p(\mathrm{X} = 2)$$

$$\Rightarrow 16({}^6\mathrm{C}_4)\ q^2 \,.\, p^4 = {}^6\mathrm{C}_2\ q^4\ p^2$$

$$\Rightarrow 16 = \frac{q^4 p^2}{p^4 q^2} = \left(\frac{q}{p}\right)^2$$

$$\Rightarrow \frac{q}{p} = 4 \Rightarrow 1 - p = 4p$$

$$\therefore \quad p = \frac{1}{5}$$

$$\Rightarrow \ q = \frac{4}{5}$$

$\therefore$ Parameter $p = \dfrac{1}{5}$

Example 31.12

If X is a poisson variate such that $P(X = 0) = P(X = 1) = k$, Then show that $k = e^{-1}$.

Sol : Let $\lambda = 0$ be the parameter of a poisson variate X

Given $P(X = 0) = P(X = 1) = k$

$$\frac{e^{-\lambda} . \lambda^0}{0!} = \frac{e^{-\lambda} . \lambda^1}{1!} = k$$

$$\Rightarrow e^{-\lambda} = e^{-\lambda} . \lambda \Rightarrow \lambda = 1$$

$$\therefore \quad k = \frac{e^{-1} \lambda^0}{0!} = e^{-1}$$

$$\therefore \quad k = e^{-1} .$$

Example 31.13

If X follows a poisson distribution and $p(X = 1) = 3\,p(X = 2)$, Then find the variance of X.

Sol : Let $\lambda > 0$ be the parameter of a poission variate X

Given $P(X = 1) = 3(p(x = 2))$

$$\frac{e^{-\lambda} . \lambda^1}{1!} = 3. \frac{e^{-\lambda} . \lambda^2}{2!}$$

$$\Rightarrow \ \lambda = \frac{2}{3}$$

$$\therefore \qquad \text{Variance } \lambda = \frac{2}{3}$$

EXERCISE 31.1

1. 8 coins are tossed simultaneously. Find the probability of getting atleast 6 heads.

2. The mean and variance of a binomial distribution are 4 and 3 respectively Fix the distribution and find $p(X \geq 1)$.

3. If $X - B(n, p)$, $\mu = 20$, $\sigma^2 = 10$, then find n and p.

4. For a poisson variate X, $p(X = 2) = p(X = 3)$. Find the variance of X.

5. A poisson variable satisfies $p(X = 1) = p(X = 2)$ Find $p(X = 5)$.

KEY WORDS

1. If p is the probability of a success, q be the probability of a failure such the $p + q = 1$ and n is the number of Bernoulli trials, then the probability distribution of a discrete random variable X, called a binomial variate is given by

   $$p(X = k) = n_{C_k}\, p^k q^{n-k},\ k = 0, 1, 2, ..., n$$

   This is called the binomial distribution.

   Here n and p are called the parameters of the distribution.

   In this case X is expressed as $X \sim B(n, p)$.

2. If X is a binomial variate with parameters n and p i.e., $X \sim B(n, p)$, then the mean of the distribution $\mu = np$ and the variance of the distribution $\sigma^2 = npq$ The standard deviation of this distribution is given by $\sqrt{npq}$.

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3. The probability distribution of a discrete random variable X (called the Poisson variable) given by $p(X = k) = \frac{e^{-\lambda}\lambda^k}{k\,!}$, k = 0, 1, 2, ...and λ = 0,is called the Poisson distribution.

   Here λ is called the parameter of X .

4. If x is a Poisson variate with parameter λ then its mean μ = variance $\sigma^2 = \lambda$.

SUPPORTIVE WEBSITES

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

ANSWERS

EXERCISE 31.1

(1) $\frac{37}{256}$

(2) $1 - \left(\frac{3}{4}\right)^{16}$

(3) $n = 40,\ p = \frac{1}{2}$

(4) 3

(5) $\frac{e^{-2}\,2^5}{5\,!}$