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MATHEMATICS

(Algebra, Co-ordinate Geometry, Dimensional Geometry Vectors)

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Foreword

Providing education to children is a fundamental right, and it's essential for the overall development of society. The Government of Telangana plays a crucial role in ensuring that education is accessible to all, and they of ten establish institutions like the Telangana Open School Society (TOSS) to cater to children who may be unable to access formal education due to various reasons.

To provide quality education to learners studying Intermediate Educationin Telangana Open School Society starting from the 2023 academic year, the text books have been revised to align with the changing social situations and incorporate the fundamental principles of the National Education Policy 2020. The guidelines set forth in the policy aim to enhance theoverall learning experience and cater to the diverse needs of the learners. Earlier Textbooks were just guides with questions and answers. TOSS has designed the textbook with a student-centric approach, considering the different learning styles and needs of learners. This approach encourages active engagement and participation in the learning process. The textbooks include supplementary teaching materials and resources to support educators in delivering effective and engaging lessons.

This textbook of Mathematics is broadly divided into six modules : Algebra, Coordinate Geometry, Three - dimensional Geometry, Trigonometry, Calculus, and Statistics. **Book 1** contains three modules. In the module Algebra students will learn about Complex numbers, Quadratic equations, Matrices, and their applications. In the module on coordinate geometry, students will be introduced to coordinates, various forms of straight lines, circles, and conics. The module, three-dimensional geometry contains planes and vectors. Understanding all these chapters is essential for a comprehensive grasp of the subject.

We are indeed very grateful to the Government of Telangana and the Telangana State Board of Intermediate Education. Special thanks to the editor, co-coordinator, teachers, lecturers, and DTP operators who participated and contributed their services tirelessly to write this textbook.

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In a Word with you

Dear Learner

Welcome to the senior secondary course. It gives me great pleasure to know that you have opted for Mathematics as one of your subjects of study. Have you ever thought as to why we study Mathematics? Can you think of a day when you have not counted something or used mathematics? Probably not.

Mathematics is the base of human civilization. From cutting vegetables to arranging books on the shelves, from tailoring clothe to motion of planets – mathematic applies everywhere. In fact, everything we do in our daily is gerned byMathematics. Mathematics can be broadly defined as the scientific study of quantities, including their relationships, operations and their measurements, expressed by numbers and symbols. The Mathematicians claim that the learning of Mathematics can be real fun. It only requires complete concentration and love for Mathematics.

The present curriculum has six modules, namely algebra, coordinate geometry, Three dimensional geometry, functions, calculus and statistics. There will be two books to cover the six modules.

Volume 1 contains the three modules. In the module on algebra, you will be introduced to mathematical induction, complex numbers, DeMoivre's theorem, quadratic equations, theory of equations,binomial theorem and various applications.This module also explains how to solve a system of linear equations with the help of matrices and determinants.

The second module on coordinate geometry will introduce you to various forms of straightlines, circles and conic sections.

The third module is on three dimensional geometry will introduce you to planes and vectors.

We would suggest to you that you go through all the solved examples given in the learning material and then try to solve independently all questions included in exercise and practice exercise given at the end of each lesson.

If you face any difficulty, please do write to us. Your suggestions are also welcome.

Yours,
Yours Academic Officer
(Mathematics)

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MATHEMATICS

Volume - I

MODULE - I

ALGEBRA

1. Mathematical Induction
2. Complex Numbers and De Moivre's Theorem
3. Quadratic Equations and Theory of Equations
4. Matrices
5. Determinants and their Applications
6. Inverse of a Matrix and Its Applications
7. Permutations and Combinations
8. Binomial Theorem

MODULE - II

CO-ORDINATE GEOMETRY

9. Cartesian System of Coordinates
10. Straight Lines
11. Circles
12. Conic Sections

MODULE - III

THREE - DIMENSIONAL GEOMETRY AND VECTORS

13. Introduction To Three- Dimensional Geometry
14. The Planes
15. Vectors

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MATHEMATICS

Volume - II

MODULE - IV

FUNCTIONS AND TRIGONOMETRIC FUNCTIONS

MODULE - V

CACULUS

MODULE - VI

STATISTICS AND PROBABILITY

CONTENTS

Chapter 1

MATHEMATICAL INDUCTION

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- State the principle of (finite) Mathematical induction.
- Verify truth or otherwise of the statement $p(n)$ for $n = 1$.
- Verify if $p(k+1)$ is true, assuming that $p(k)$ is true;
- Use principle of mathematical induction to establish the truth or otherwise of mathematical statements.

PREREQUISITES

- Number System
- Four fundamental operations on numbers and expressions.
- Algebraic expressions and their simplifications.

INTRODUCTION

In your daily life you must be using various kinds of reasoning depending on the situation you are faced with. for instance, if you are told that that your friend just had a child, you would know that it is either a girl or a boy. In this case, you would be applying general principles to a particular case. This form of reasoning is an example of deductive logic.

Now let us consider another situation when you look around. You find students who study regularly, do well in examinations. You may formulate the general rule (rightly or wrongly) that "any one who studies regularly will do well in examinations." In this case, you would be formulating a general principle (or rule) based on several particular instances. Such reasoning is inductive, a process of reasoning by which general rules are discovered by the observation and consideration of several individual cases. Such reasoning is used in all the sciences, as well as in Mathematics.

Mathematical induction is a more precise form of this process. This precision is required because a statement is accepted to be true mathematically only if it can be shown to be true for each and every case that it refers to.

1.1 WHAT IS A STATEMENT?

In your daily interations, you must have made several assertions in the form of sentenses of these assertions, the ones that are either true or false are called statement or propositions. For intance, "I am 20 years old" and If $x = 3$, then $x^2 = 9$ are statements, but when will you leave? And 'How wonderful!' are not statements.

Notice that a statement has to be definite assertion which can be true of false, but not both. For example, $x - 5 = 7$ is not a statement, because we don't know what x, is If $x = 12$, it is true, but if $x = 5$, it is not true. Therefore '$x - 5 = 7$' is not accepted by mathematicians as a statement.

But both '$x - 5 = 7 \Rightarrow x = 12$ and $x - 5 = 7$ for any real number x' are statements, the first one true and second one false.

Example 1.1 : Which of the following sentences is statetent?

(i) India has had a woman president.

(ii) 5 is an even number

(iii) $x^n > 1$

(iv) $(a + b)^2 = a^2 + 2ab + b^2$

Solution : (i) and (ii) are statements, (i) being true and (ii) being false. (iii) is not a statement, since we can not determine whether it is true or false, unless we know the range of values that x can take.

Now look at (iv), At first glance, you may say that it is not a statement, for the very same reasons that (iii) is not. But look at (iv) carefully. It is true for any value of a and b. It is an identify. Therefore, in this case, even though we have not specified the range of values for a and b (iv) is a statement.

Some statements like the one given below are about natural numbers in general. Let us look at the statement given below:

$$1+2+\ldots+n=\frac{n(n+1)}{2}$$

This involves a general natural number n. Let us call this statement $p(n)$ [P stands for proposition].

Then p(1) would be $1=\dfrac{1(1+1)}{2}$

similary, $p(2)$ would be the statement

$$1+2=\frac{2(2+1)}{2}$$ and so on.

Let us look at some examples to help you get used to this notation.

Example 1.2 : If $p(n)$ denotes $2n > n - 1$, write $p(1)$, $p(k)$ and $p(k+1)$, where $k \in \mathrm{N}$.

Solution : Replacing n by 1, k and $k+1$, respectively in $p(n)$, we get

$p(1)$: $2^1 > 2 - 1$, i.e., $2 > 1$

$p(k)$: $2^k > k - 1$

$p(k+1)$: $2^{k+1} > (k+1) - 1$, i.e., $2^{k+1} > k$

Example 1.3 : If $p(n)$ is the statement $1+4+7+\ldots+(3n-2)=\dfrac{n(3n-1)}{2}$

write $p(1)$, $p(k)$ and $p(k+1)$.

Solution : To write $p(1)$, the terms on the left hand side (LHS) of $p(n)$ continue till $3\times 1-2$ i.e., 1 So, $p(1)$ will have only one term in its LHS, i.e., the first term.

Also, the right hand side (RHS) of $p(1)=\dfrac{1\times(3\times 1-1)}{2}=1$

Therefore, p(1) is $1 = 1$

Replacing n by 2, we get

$$p(2) : 1+4 = \frac{2\times(3\times 2-1)}{2}, \quad \text{i.e., } 5 = 5$$

Replacing n by k and k + 1, respectively, we get

$$p(k) : 1 + 4 + 7 + ... + (3k - 2) = \frac{k(3k-1)}{2}$$

$$p(k+1) : 1 + 4 + 7 + ... + (3k - 2) + [3(k+1) - 2]$$

$$= \frac{(k+1)[(3(k+1)-1]}{2}$$

$$\text{i.e., } 1 + 4 + 7 + ... + (3k + 1) = \frac{(k+1)(3k+2)}{2}.$$

Exercise 1.1

1. Determine which of the following are statements:

   (a) $1 + 2 + 4 + ... + 2^n > 20$ (b) $1 + 2 + 3 + ... + 10 = 99$

   (c) Chennai is much nicer than Mumbai.

   (d) Where is Delhi ?

   (e) $\frac{1}{1\times 2} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for $n = 5$

   (f) $\operatorname{cosec}\theta < 1$

2. Given that $p(n)$: 6 is a factor of $n^3 + 5n$, write $p(1)$, $p(2)$, $p(k)$ and $p(k+1)$ where k is a natural number.

3. Write $p(1)$, $p(k)$ and $p(k+1)$, if $p(n)$ is

   (a) $2^n \geq n + 1$ (b) $(1 + x)^n \geq 1 + nx$

   (c) n(n + 1) (n + 2) is divisible by 6

   (d) $x^n - y^n$ is divisible by $(x - y)$

   (e) $(ab)^n = a^n b^n$ (f) $\left(\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}\right)$ is a natural number.

4. Write $p(1)$, $p(2)$, $p(k)$ and $p(k+1)$, if $p(n)$ is

(a) $\frac{1}{1\times 2}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}$

(b) $1 + 3 + 5 + ... + (2n - 1) = n^2$

(c) $(1\times 2) + (2 \times 3) + ... + n(n + 1) < n(n + 1)^2$

(d) $\frac{1}{1\times 3}+\frac{1}{3\times 5}+...+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

1.2 THE PRINCIPLE OF MATHEMATICAL INDUCTION

Let $p(n)$ be a statement involving a natural number n. If

(i) it is true for $n = 1$, i.e., $p(1)$ is true ; and

(ii) assuming $k \geq 1$ and $p(k)$ to be true, it can be proved that $p(k + 1)$ is true; then $p(n)$ must be true for every natural number n.

Note that condition (ii) above does not say that $p(k)$ is true.

It says that whever $p(k)$ is true, then $p(k + 1)$ is true.

Let us see, for example, how the principle of mathematical induction allows to conclude that $p(n)$ is true for $n \in N$. By (i) $p(1)$ is true. As $p(1)$ is true. we can put $k = 1$ in (ii), so $p(1+1)$ i.e., $p(2)$ is true, we can put $k = 2$ in (ii) and conclude that $p(2 + 1)$, i.e., $p(3)$ is true. Now put $k = 3$ in (ii), so we get that $p(4)$ is true. It is clear that if we continue like this, we shall get that $p(11)$ is true.

It is also clear that in the above argument. 11 does not play any special role. We can prove that $p(137)$ is true in the same way. Indeed, it is clear that $p(n)$ is true for all $n > 1$.

Let us now see, through example, how we can apply the priciple of mathematical induction to prove various types of mathematical statements.

Example 1.4 : Prove that $1+2+3+...+n = \frac{n(n+1)}{2}$, where n is a natural number.

Solution : We have

$$p(n): \quad 1+2+3+...+n = \frac{n(n+1)}{2}$$

Therefore, p(1) is $'1 = \frac{1}{2}(1+1)'$ which is true.

Therefore, $p(1)$ is true.

Let us now see, if $p(k+1)$ is true whenever $p(k)$ is true.

Let us, therefore, assume that $p(k)$ is true, i.e.,

$$1+2+3+...+k = \frac{k}{2}(k+1) \qquad ...(i)$$

Now, $p(k+1)$ is $1+2+3+...+k+(k+1) = \frac{(k+1)(k+2)}{2}$

It will be true, if we can show that LHS = RHS

The LHS of $p(k+1) = (1+2+3+...k)+(k+1)$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= (k+1)\left(\frac{k}{2}+1\right)$$

$$= \frac{(k+1)(k+2)}{2}$$

= RHS of $p(k+1)$

So, $p(k+1)$ is true, if we assume that $p(k)$ is true.

Since $p(1)$ is also true, both the conditions of the principle of mathematical induction are fulfilled. We conclude that the given statement is true for every natural number n.

As you can see, we have proved the result in three steps the basic step [i.e., checking (i)], the Induction step [i.e., checking (ii)] and hence arriving at the end result.

Example 1.5 : Prove that $1.2 + 2.2^2 + 3.2^3 + 4.2^4 + ... + n.2^n = (n-1) . 2^{n+1} + 2$, where n is a natural number.

Solution : Here $p(n)$ is $1.2^1 + 2.2^2 + 3.2^3 + ... + n.2^n = (n-1) . 2^{n+1} + 2$

Therefore, p(1) is $1.2^1 = (1-1)\ 2^{1+1} + 2$ i.e., $2 = 2$

So, p(1) is true

We assume that p(k) is true, i.e.,

$$1.2^1 + 2.2^2 + 3.2^3 + ... + k.2^k + (k-1) . 2^{k+1} + 2 \qquad ...(i)$$

Now will prove that p(k + 1) is true.

Now $p(k+1)$ is

$$1.2^1 + 2.2^2 + 3.2^3 + ... + k.2^k + (k+1).2^{k+1} = [(k+1)-1]2^{(k+1+1)}+2$$

$$= k\ .2^{k+2} + 2.$$

LHS of $p(k+1) = 1.2^1 + 2.2^2 + 3.2^3 + ... + k.2^k + (k+1) . 2^{k+1}$

$$= (k-1)\ 2^{k+1} + 2 + (k+1) . 2^{k+1}$$

$$= 2^{k+1}\ [(k-1) + (k+1)] + 2$$

$$= 2^{k+1}\ (2k) + 2$$

$$= k . 2^{k+2} + 2$$

$$= \text{RHS of } p(k+1)$$

Therefore, $p(k+1)$ is true.

Hence, by the priciple of mathematical induction, the given statement is true for every natural number n.

Example 1.6 : Prove that $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$, where n is a natural number.

Solution : We have $p(n)$

$$1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$$

$\therefore \quad p(1)$ is $1^2 = \frac{1}{3}1(2-1)(2+1) = 1$, which is true.

Therefore $p(1)$ is true.

Assume that $p(k)$ is true i.e.,

$$1^2 + 3^2 + 5^2 + ... + (2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$$

Now, $p(k+1) = 1^2 + 3^2 + 5^2 + ... + (2k-1)^2 + [2(k+1)-1]^2$

$$= \frac{1}{3}(k+1)[2(k+1)-1]\,[2(k+1)+1]$$

$$= \frac{1}{3}(k+1)(2k+1)(2k+3)$$

LHS of $p(k+1) = 1^2 + 3^2 + 5^2 + ... + (2k-1)^2 + (2k+1)^2$

$$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$$

$$= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)]$$

$$= \frac{1}{3}(2k+1)(2k^2+5k+3)$$

$$= \frac{1}{3}(k+1)(2k+1)(2k+3)$$

$=$ RHS of $p(k+1)$

Therefore, p(k + 1) is true.

Hence, by the priciple of mathematical induction, the given statement is true for every natural number n.

Example 1.7 : Prove that $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots$ upto n terms $= \frac{n}{3n+1}$.

Solution : 1, 4, 7, ... are in Arithmetic progression whose n^{th} term is $3n-2$.
4, 7, 10 .... are is Arithmetic progression whose n^{th} term is $3n+1$.

$\therefore$ The n^{th} term is the given series is $\dfrac{1}{(3n-2)(3n+1)}$

Let $p(n)$ be the statement :

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3n-2)(3n+1)}=\frac{n}{3n+1}$$

$\therefore$ $p(1)$ is $\dfrac{1}{1.4}=\dfrac{1}{3+1}$ which is true.

$\therefore$ $p(1)$ is true.

we assume that $p(k)$ is true.

i.e., $$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$$

Now, we will prove that $p(k + 1)$ is true.

Now $p(k + 1)$ is

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}=\frac{k+1}{3k+4}$$

LHS of $p(k+1)$ $$=\frac{1}{1.4}+\frac{1}{4.7}+...+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k(3k+4)+1}{(3k+1)(3k+4)}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$=\frac{k+1}{3k+1}$$

$=$ RHS of $p(k + 1)$

Therefore $p(k + 1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Example 1.8 : Using mathmatical Induction, prove that $x^n - y^n$ is divisible by $x - y$, for every natural number n.

Solution : Let $p(n)$ be the statement : $x^n - y^n$ is divisible by $x - y$ since $x' - y' = x - y$ is divisible by $x - y$, the statement is true for $n = 1$.

Assume that the statement $p(k)$ is true.

i.e., $x^k - y^k$ is divisible by $x - y$.

Then $x^k - y^k = (x - y)p$ ...(i) where p is the quotient when $x^k - y^k$ is divided by $x - y$.

Now, we will prove that $p(k + 1)$ is true.

i.e., we prove that $x^{k+1} - y^{k+1}$ is divisible by $x - y$

From (i) we have $x^k - y^k = (x - y)p$

$\therefore \quad x^k = (x - y)p + y^k$

$\therefore \quad x^{k+1} = (x - y)px + y^k . x$

$\therefore \quad x^{k+1} - y^{k+1} = (x - y)px + y^k x - y^{k+1}$

$= (x - y)px + y^k(x - y)$

$= (x - y)(px + y^k)$

$\therefore$ $x^{k+1} - y^{k+1}$ is divisible by $x - y$.

$\therefore$ $p(k + 1)$ is true.

Hence by the principle of mathematical induction, the given statement is true for every natural number n.

Example 1.9 : Show $49^n + 16n - 1$ is divisible by 64, for every $n \in$ N.

Solution : Let $p(n)$ be the statement :

$49^n + 16n - 1$ is divisible by 64

since $49^1 + 16.1 - 1 = 64$ is divisible by 64,

$\therefore$ $p(1)$ is true.

Assume that the statement $p(k)$ is true.

i.e., $49^k + 16k - 1$ is divisible by 64.

Then $49^k + 16k - 1 = 64p$ ...(i) for some $p \in N$.

Now, we will prove that $p(k + 1)$ is true.

i.e., we show that $49^{k+1} + 16(k + 1) - 1$ is divisible by 64

From(i) we have $49^k + 16k - 1 = 64p$

$$\therefore\ 49^k = 64p - 16k + 1$$

$$\therefore\ 49^k.49 = (64p - 16k + 1).49$$

$$49^{k+1} + 16(k+1) - 1 = (64p - 16k + 1)49 + 16(k+1) - 1$$

$$= 64(49p - 12k + 1)$$

Here $49p - 12k + 1$ is an integer

$\therefore$ $49^{k+1} + 16(k + 1) - 1$ is divisible by 64

$\therefore$ $p(k + 1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Example 1.10 : Prove that $2^n > n$ for every natural number n.

Solution : We have p(n) : $2^n > n$

Therefore $p(1) : 2^1 > 1$, i.e., $2 > 1$, which is true we assume $p(k)$ to be true that is $2^k > k$ ...(i)

Now, we will prove that $p(k + 1)$ is true. i.e., $2^{k+1} > k + 1$

Now, multiplying both sides of (i) by 2, we get

$$2^{k+1} > 2k$$

$$\Rightarrow \quad 2^{k+1} > k + 1, \text{ since } k > 1$$

Therefore, $p(k + 1)$ is true.

Hence by the principle of mathematical induction, the given statement is true for every natural number n.

Some times, we need to prove a statement for all natural numbers greater than a particular natural number say a (as in example 1.11 below). In such a situation, we replace $p(1)$ by $p(a+1)$ in the statement of the principle.

Example 1.11: Prove that $n^2 > 2(n+1)$ for all $n \geq 3$, where n is a natural number.

Solution : For $n \geq 3$, let us call the following statement

$$p(n) : n^2 > 2(n+1)$$

since we have to prove the given statement for $n \geq 3$, the first relevant statement is $p(3)$. We therefore, see whether $p(3)$ is true.

$p(3) : 3^2 > 2 \times 4 \qquad$ i.e., $9 > 8$

So, $p(3)$ is true.

Let us assume that p(k) is true, where $k \geq 3$, that is

$$k^2 > 2(k+1) \qquad ....(i)$$

we wish to prove that $p(k+1)$ is true

$$p(k+1) : \ (k+1)^2 > 2(k+2)$$

$$\begin{aligned}
\text{LHS of } \ p(k+1) &= (k+1)^2 \\
&= k^2 + 2k + 1 \\
&> 2(k+1) + 2k + 1 \qquad \text{... [By (i)]} \\
&> 3 + 2k + 1, \text{ since } 2(k+1) > 3 \\
&= 2(k+2)
\end{aligned}$$

Thus, $(k+1)^2 > 2(k+2)$

Therefore, $p(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number $n \geq 3$.

Exercise 1.2

1. Using the principle of mathematical induction, prove that the following statements hold for any natural number n.

   (a) $1^2 + 2^2 + 3^2 + \text{........} + n^2 = \frac{n}{6}(n+1)(2n+1)$

   (b) $1^3 + 2^3 + 3^3 + \text{........} + n^3 = \frac{n^2(n+1)^2}{4}$

   (c) $1 + 3 + 5 + \text{........} + 2(n - 1) = n^2$.

   (d) $1 + 4 + 7 + \text{........} + (3n - 2) = \frac{n}{2}(3n - 1)$

2. Using principle of mathematical induction, prove the following equalities for any natural number n:

   (a) $\frac{1}{1.2} + \frac{1}{2.3} + \text{....} + \frac{1}{n(n+1)} = \frac{n}{n+1}$

   (b) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \text{....} + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

   (c) $1.2 + 2.3 + \text{....} + n(n+1) = \frac{n(n+1)(n+2)}{3}$

3. For every natural number n, prove that

   (a) $n^3 + 5n$ is divisible by 6

   (b) $(x^n - 1)$ is divisible by $(x - 1)$

   (c) $(n^3 + 2n)$ is divisible by 3

   (d) 4 divides $(n^4 + 2n^3 + n^2)$

KEY WORDS

1. Principle of finite mathematical induction. Let S be a subset of N. Such that (i) $1 \in S$ (ii) For any $K \in N$, $K \in S \Rightarrow K+1 \in S$ then S= N.
2. Principle of complete mathematical induction : for each $n \in N$ Let P(n) be a statement, suppose that P(1) is true. for any $K \in N$, if P(1), P(2) ... P(k) are true, then P(K + 1).
3. If x, y are natural numbers, $x \neq y$ then $x^n - y^n$ is divisible by $x - y \quad \forall \; n \in N$.

SUPPORTED WEBSITES

- http://www.wikipedia.org
- http://mathworld.wolfram.com

PRACTICE PROBLEMS

Using mathematical induction, prove each of following statements, for all $n \in N$.

1. 2.3 + 3.4 + 4.5 + ... up to n terms $= \dfrac{n(n^2+6n+11)}{3}$
2. $a + (a + d) + (a + 2d) + ...$ (up to n terms) $= \dfrac{n}{2}[2a+(n-1)d]$
3. $4^n - 3^n - 1$ is divisible by 9.
4. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + ...$ (up to n terms) $= \dfrac{n(n+1)^2(n+2)}{12}$
5. $\dfrac{1}{1.4}+\dfrac{1}{4.7}+\dfrac{1}{7.10}+.....$ (up to n terms) $= \dfrac{n}{3n+1}$

ANSWERS

EXERCISE 1.1

1. (b), (e) and (f) are statements; (a) is not, since we have not given the range of values of n, and therefore we are not in a position to decide,

if it is true or not. (c) is subjective and hence not a mathematical statement. (d) is a question, not a statement.

Note that (f)is universally false.

2. P(1) : 6 is a factor $1^3 + 5.1$

   P(2) : 6 is a factor $2^3 + 5.2$

   P(k) : 6 is a factor $k^3 + 5k$

   P(k + 1) : 6 is a factor $(k + 1)^3 + 5(k + 1)$

3. (a) P(1) : $2 \geq 2$

   P(k) : $2^k \geq k + 1$

   P(k + 1) : $2^{k+1} \geq k + 2$

   (b) P(1) : $1 + x \geq 1 + x$

   P(k) : $(1 + x)^k \geq 1 + kx$

   P(k + 1) : $(1 + x)^{k+1} \geq 1 + (k + 1)x$

   (c) P(1) : 6 is divisible by 6.

   P(k) : $k(k + 1)(k + 2)$ is divisible by 6.

   P(k + 1): $(k + 1)(k + 2)(k + 3)$ is divisible by 6.

   (d) P(1) : $(x - y)$ is divisible by $(x - y)$.

   P(k) : $(x^k - y^k)$ is divisible by $(x - y)$

   P(k + 1): $(x^{k+1} - y^{k+1})$ is divisible by $(x - y)$

   (e) P(1) : $ab = ab$

   $P(k)$: $(ab)^k = a^k b^k$

   P(k + 1) : $(ab)^{k+1} = a^{k+1} . b^{k+1}$

   (f) P(1) : $\frac{1}{5}+\frac{1}{3}+\frac{7}{15}$ is a natural number.

P(k) : $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a natural number.

P(k + 1): $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$ is a natural number.

4. (a) P(1) : $\frac{1}{1\times 2}=\frac{1}{2}$

P(2) : $\frac{1}{1\times 2}+\frac{1}{2\times 3}=\frac{2}{3}$

P(k) : $\frac{1}{1\times 2}+\frac{1}{2\times 3}+....+\frac{1}{k(k+1)}=\frac{k}{k+1}$

P(k+1) : $\frac{1}{1\times 2}+....+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$

(b) P(1) : $1 = 1^2$

P(2) : $1 + 3 = 2^2$

P(k) : $1 + 3 + 5 + ... + (2k - 1) = k^2$

P(k + 1) : $1 + 3 + 5 +.... + (2k - 1) + [2(k + 1)-1] = (k + 1)^2$

(c) P(1) : $1 \times 2 < 1(2)^2$

P(2) : $(1\times 2) + (2\times 3) < 2(3)^2$

P(k) : $(1\times 2) + (2\times 3) + ... + k(k + 1) < k(k + 1)^2$

P(k + 1) : $(1\times 2) + (2\times 3) +....+(k + 1)(k + 2) < (k + 1)(k + 2)^2$

(d) P(1) : $\frac{1}{1\times 3}=\frac{1}{3}$

P(2) : $\frac{1}{1\times 3}+\frac{1}{3\times 5}=\frac{2}{5}$

P(k) : $\frac{1}{1\times 3}+\frac{1}{3\times 5}+.....+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$

P(k + 1) : $\frac{1}{1\times 3}+\frac{1}{3\times 5}+.....+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$

Chapter

2

COMPLEX NUMBERS

LEARNING OUTCOMES

After studying this lesson, you will be able to:

- describe the need for extending the set of real numbers to the set of complex numbers;
- define a complex number and cite examples;
- identify the real and imaginary parts of a complex number;
- state the condition for equality of two complex numbers;
- recognise that there is a unique complex number $x + iy$ associated with the point P(x, y) in the Argand Plane and vice-versa;
- define and find the conjugate of a complex number;
- define and find the modulus and argument of a complex number;
- represent a complex number in the polar form;
- perform algebraic operations (addition, subtraction, multiplication and division) on complex numbers;
- state and use the properties of algebraic operations (closure, commutativity, associativity, identity, inverse and distributivity) of complex numbers; and
- State and use the following properties of complex numbers in solving problems.

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(i) $|z| = 0 \quad \Leftrightarrow z = 0$ and $z_1 = z_2 \Rightarrow |z_1| = |z_2|$

(ii) $|z| = |-z| = |\bar{z}|$

(iii) $|z_1 + z_2| \le |z_1| + |z_2|$

(iv) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

(v) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, (z_2 \neq 0).$

PREREQUISITES

- Properties of real numbers.
- Solution of linear and quadratic equations
- Representation of a real number on the number line
- Representation of point in a plane.

INTRODUCTION

In the earlier classes we have learnt the properties of real numbers and studied certain operations on real numbers like addition, substraction. Multiplication and division. We have also learnt solving linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation $x^2 + 1 = 0$ has no real solution since the square of every real number is non-negative.

This suggests that we need to extend the real number system to a larger system, so that we can account for the solutions of the equation $x^2 = -1$. If this is done, it would help solving the equation $ax^2 + bx + c = 0$ for the case $b^2 - 4ac < 0$, which is not possible in the real number system.

2.1 COMPLEX NUMBERS

Consider the equation $x^2 + 1 = 0$ ....(A)

This can be written as $x^2 = -1$

or $x = \pm\sqrt{-1}$

But there is no real numbers which satisfy $x^2 = -1$. In other words, we can say that there is no real numbers whose square is -1. In order to solve such equations, let us imagine that there exist a number *'i'* which equal to $i = \sqrt{-1}$.

In 1748 a great mathematician, L. Euler named a number 'i' as *Iota* whose square is -1. This *Iota or 'i'* is defined as imaginary unit. With the introduction of the new symbol *'i'*, we can interpret the square root of a negative number as a product of a real number with *i*.

Therefore, we can denote the solution of (A) as $x = \pm i$

Thus, $-4 = 4(-1)$

$$\therefore \sqrt{-4} = \sqrt{(-1)(4)} = \sqrt{i^2 \cdot 2^2} = 2i$$

Conventionally written as $2i$.

So, we have $\sqrt{-4} = 2i$, $\sqrt{-7} = \sqrt{7}i$

$\sqrt{-4}$, $\sqrt{-7}$ are all examples of complex numbers.

Consider another quadratic equation:

$$x^2 - 6x + 13 = 0$$

This can be solved as under:

$$(x-3)^2 + 4 = 0$$

or, $(x-3)^2 = -4$

or, $x - 3 = \pm 2i$

or, $x = 3 \pm 2i$

We get numbers of the form $x + iy$ where x and y are real numbers and $i = \sqrt{-1}$.

Any number which can be expressed in the form $a + bi$ where a, b are real numbers and $i = \sqrt{-1}$.

A complex number is, generally, denoted by the letter z.

i.e., $z = a + bi$, 'a' is called the real part of z and is written as Re $(a + bi)$ and 'b' is called the imaginary part of z and is written as Imag $(a + bi)$.

If $a = 0$ and $b \neq 0$, then the complex number becomes bi which is a purely imaginary complex number.

$-7i$, $\frac{1}{2}i$, $\sqrt{3}i$ and πi are all examples of purely imaginary numbers.

If $a \neq 0$ and $b = 0$ then the complex number becomes 'a' which is a real number.

5, 2.5 and $\sqrt{7}$ are all examples of real numbers.

If $a = 0$ and $b = 0$ then the complex number becomes 0 (zero). Hence the real numbers are particular cases of complex numbers.

Example 2.1 Simplify each of the following using 'i'.

(i) $\sqrt{-36}$ (ii) $\sqrt{25}\,.\,\sqrt{-4}$

Solution: (i) $\sqrt{-36} = \sqrt{36(-1)} = 6i$

(ii) $\sqrt{25}\,.\,\sqrt{-4} = 5 \times 2i = 10i$

2.2 POSITIVE INTEGRAL POWERS OF i

We know that

$i^2 = -1$

$i^3 = i^2 \,.\, i = -1 \,.\, i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1.$

$i^5 = (i^2)^2 \,.\, i = (-1)^2 \,.\, i = (+1)\, i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1$

$i^7 = (i^2)^3\, (i) = (-1)\,(i) = -i$

$i^8 = (i^2)^4 = (-1)^4 = +1.$

Thus, we find that any higher powers of 'i' can be expressed in terms of one of four values i, -1, $-i$, $+1$

If n is a positive integer such that $n > 4$, then to find in , we first divide n by 4. Let m be the quotient and r be the remainder

Then $n = 4m + r$, where $0 \leq r < 4$.

Thus,

$$i^n = i^{4m+r} = i^{4m} . \; i^r$$
$$= (i^4)^m . \; i^r$$
$$= i^r \; (\because \; i^4 = 1)$$

Note: For any two real numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when atleast one of a and b is either 0 or positive.

If fact $\sqrt{-a} \times \sqrt{-b}$

$$= i . \sqrt{a} \times i\sqrt{b} = i^2 \sqrt{ab}$$

$= -\sqrt{ab}$, a, b where a and b are positive real numbers.

Example 2.2: Find the value of $1 + i^{10} + i^{20} + i^{30}$.

Solution: $1 + i^{10} + i^{20} + i^{30}$

$$= 1 + (i^2)^5 + (i^2)^{10} + (i^2)^{15}$$
$$= 1 + (-1)^5 + (-1)^{10} + (-1)^{15}$$
$$= 1 + (-1) + (1) + (-1)$$
$$= 1 - 1 + 1 - 1$$
$$= 0$$

Thus, $1 + i^{10} + i^{20} + i^{30} = 0$

Example 2.3: Express $8i^3 + 6i^{16} - 12i^{11}$ in the form of $a + bi$.

Solution: $8i^3 + 6i^{16} - 12i^{11}$ can be written as $8(i^2) . i + 6(i^2)^8 - 12(i^2)^5 . i$

$$= 8(-1) . i + 6(-1)^8 - 12(-1)^5 . i$$
$$= -8i + 6 - 12(-1) \; i$$
$$= -8i + 6 + 12i$$
$$= 6 + 4i$$

which is of the form of $a + bi$ where 'a' is 6 and 'b' is 4.

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EXERCISE 2.1

1. Simplify each of the following using 'i'.
   (a) $\sqrt{-27}$ (b) $-\sqrt{-9}$ (c) $\sqrt{-13}$
2. Express each of the following in the form of $a + bi$.
   (a) 5 (b) $-3i$ (c) 0
3. Simplify $10i^3 + 6i^{13} - 12\,i^{10}$.
4. Show that $i^m + i^{m+1} + i^{m+2} + i^{m+3} = 0$ for all $m \in N$.

2.3 CONJUGATE OF A COMPLEX NUMBER

Consider the equation:

$x^2 - 6x + 25 = 0$ ..... (i)

or $(x-3)^2 + 16 = 0$

or, $(x-3)^2 = -16$

or, $(x-3) = \pm\sqrt{-16} = \pm\sqrt{16(-1)}$

or, $x = 3 \pm 4i$

The roots of the above equation (i) are $3 + 4i$ and $3 - 4i$.

Consider another equation:

$x^2 + 2x + 2 = 0$ ..... (ii)

or, $(x+1)^2 + 1 = 0$

or, $(x+1)^2 = -1$

or, $(x+1) = \pm\sqrt{-1} = \pm i$

or, $x = -1 \pm i$

The roots of the equation (ii) are $-1 + i$ and $-1 - i$.

Do you find any similarity in the roots of (i) and (ii)?

The equations (i) and (ii) have roots of the type $a + bi$ and $a - bi$. Such roots are known as conjugate roots and read as $a + bi$ is conjugate to $a - bi$ and vice-versa.

The complex conjugate (or simply conjugate) of a complex number $z = a + bi$ is defined as the complex number $a - bi$ and is denoted by $\overline{z}$.

Thus if $z = a + bi$ then $\overline{z} = a - bi$.

> **Note:** ***The conjugate of a complex number is obtained by changing the sing of the imaginary part.***

Following are some examples of complex conjugates:

(i) $z = 2 + 3i$ then $\overline{z} = 2 - 3i$.

(ii) $z = 1 - i$ then $\overline{z} = 1 + i$.

(iii) $z = -2 + 10i$ then $\overline{z} = -2 - 10i$.

2.3.1 PROPERTIES OF COMPLEX CONJUGATES

(i) If z is a real number then $z = \overline{z}$ i.e., the conjugate of a real number is the number itself.

For example, let $z = 5$

This can be written as

$$z = 5 + 0i$$
$$\therefore \quad \overline{z} = 5 - 0i = 5$$
$$\therefore \quad z = 5 = \overline{z}$$

(ii) If z is a purely imaginary number then $\overline{z} = -z$

For example, if $z = 3i$

This can be written as

$$z = 0 + 3i$$
$$\therefore \quad \overline{z} = 0 - 3i = -3i$$
$$= -z$$
$$\therefore \quad \overline{z} = -z.$$

(iii) Conjugate of the conjugate of a complex number is the number itself.

i.e., $\overline{(\overline{z})} = z$

For example, if $z = a + bi$ then

$$\overline{z} = a - bi$$

Again, $\overline{(\overline{z})} = \overline{(a-bi)} = a+bi = z$

$\therefore \overline{(\overline{z})} = z$.

Example 2.4 : Find the conjugate of each of the following complex number.

(i) $3-4i$ (ii) $2i$ (iii) $(2+i)^2$ (iv) $\frac{i+1}{2}$

Solution: (i) Let $z = 3 - 4i$

then $\overline{z} = \overline{3-4i} = 3+4i$

Hence, $3 + 4i$ is the conjugate of $3 - 4i$.

(ii) *Let* $z = 2i$ or $0 + 2i$

then $\overline{z} = \overline{0+2i} = 0-2i = -2i$

Hence, $-2i$ is the conjugate of $2i$.

(iii) Let $z = (2+i)^2$

i.e., $z = (2)^2 + (i)^2 + 2(2)(i)$

$= 4 - 1 + 4i$

$= 3 + 4i$

Then $\overline{z} = \overline{3+4i} = 3 - 4i$.

Hence, $3 - 4i$ is the conjugate of $(2+i)^2$.

(iv) Let $z = \frac{i+1}{2} = \frac{1}{2}+\frac{1}{2}i$

then $\overline{z} = \overline{\left(\frac{1}{2}+\frac{1}{2}i\right)} = \frac{1}{2}-\frac{1}{2}i$

Hence, $\frac{1}{2}-\frac{i}{2}$ or $\frac{-i+1}{2}$ is the conjugate of $\frac{i+1}{2}$.

2.4 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

Let $z = a + bi$ be a complex number. Let two mutually perpendicular lines XOX' and YOY' be taken as x-axis and y-axis respectively, O being the origin.

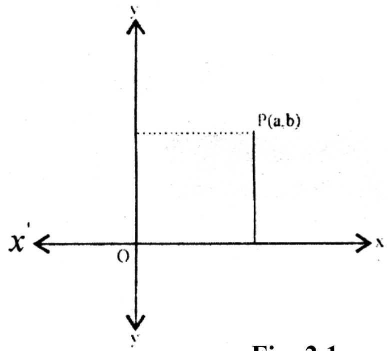

Fig. 2.1

Let P be anypoint whose coordinates are (a, b). We say that the . P(a, b) complex $z = a + bi$ is represented by the point P(a, b) as shown in Fig. 1.1

If $b = 0$, then z is real and the point representing complex number $z = a + 0i$ is denoted by $(a, 0)$. This point $(a, 0)$ lies on the x-axis.

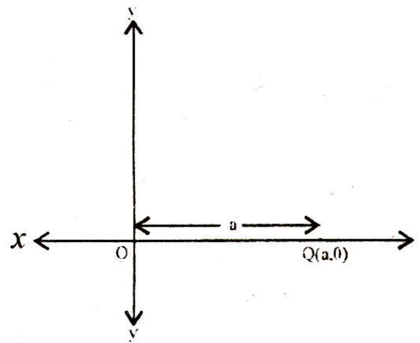

Fig. 2.2

So, xox' is called the real axis. In the Fig. 1.2 the point Q $(a, 0)$ represent the complex number $z = a + 0$i.

If $a = 0$, then z is purely imaginary and the point representing complex number $z = 0 + bi$ is denoted by $(0, b)$. The point $(0, b)$ lies on the y-axis.

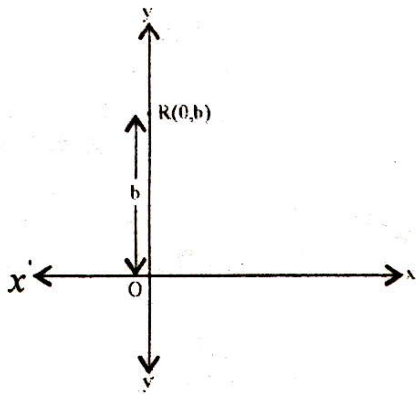

Fig. 2.3

So, YOY' is called the imaginary axis. In Fig.1.3, the point R $(0, b)$ represents the complex number $z = 0 + bi$.

The plane of two axis representing complex numbers as points is called the complex plane or Argand Plane.

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Example 2.5: Represent complex numbers $2 + 3i$ and $3 + 2i$ in the same Argand Plane.

Solution:

1. $2 + 3i$ is represented by the point A (2,3)

2. $3 + 2i$ is represented by the point B (3, 2) Clearly, the points A and B are different.

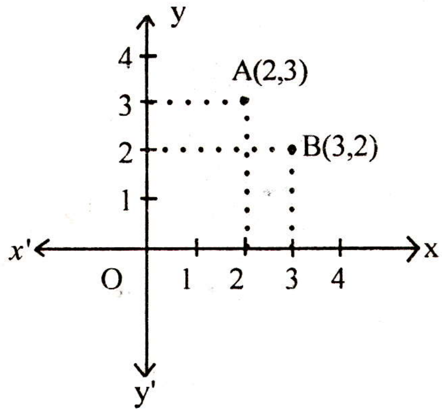

Fig. 2.4

Example 2.6: Represent complex numbers $2 + 3i$ = and $-2 - 3i$ in the same Argand Plane.

Solution:

1. $2 + 3i$ is represented by the point P(2, 3)
2. $-2 - 3i$ is represented by the point Q(−2, −3).

Points P and Q are different and lie in the I quadrant and III quadrant respectively.

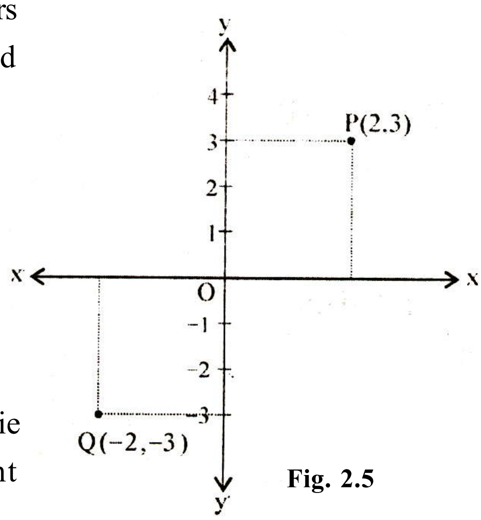

Fig. 2.5

Example 2.7: Represent complex numbers $2 + 3i$ and $2 - 3i$ in the same Argand Plan

Solution:

1. $2 + 3i$ is represented b the point R(2, 3)
2. $2 - 3i$ is represtned by the point S(2, −3)

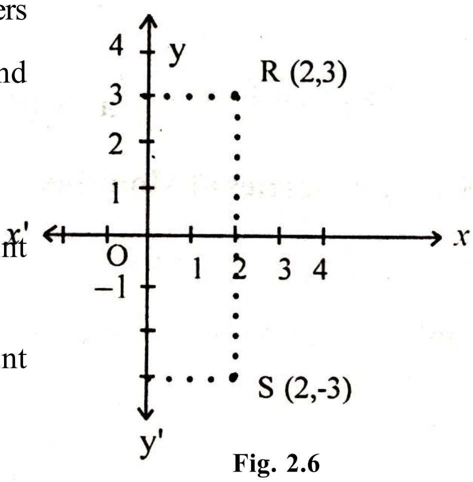

Fig. 2.6

Example 2.8: Represent complex r bers

2 + 3i, –2 – 3i in the same Argand P

Solution:

1. 2 + 3i is represented by the point P(2, 3).
2. –2 – 3i is represented by the point Q(–2, –3)
3. 2 – 3i is represented by the point R(2, –3)

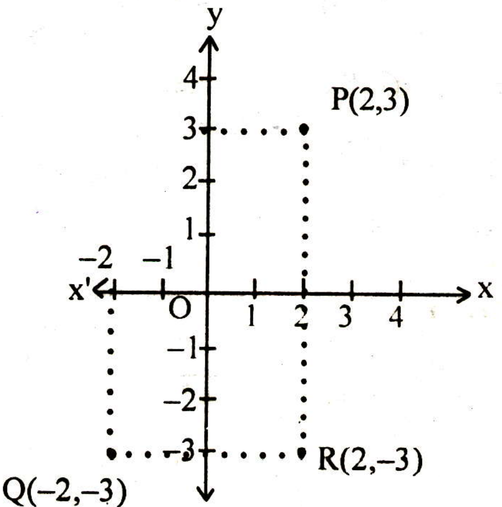

Fig. 2.7

2.5 MODULUS OF A COMPLEX NUMBER

We have learnt that any complex number $z = a + bi$ can be represented by a point in the Argand Plane. How can we find the distance of the point from the origin? Let P(a, b) be a point in the plane representing $a + bi$. Draw perpendiculars PM and PL on x -axis and y-axis respectively.

Let OM = a and MP = b. We have to find the distance of P from the origin.

$$\therefore\ \text{OP} = \sqrt{\text{OM}^2 + \text{MP}^2}$$

$$= \sqrt{a^2 + b^2}$$

OP is called the modulus or absolute value of the complex number $a + bi$.

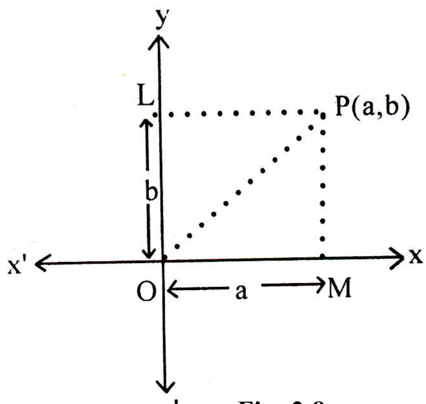

Fig. 2.8

Modulus of any complex number z such that $z = a + bi$

$\therefore z = a + bi \quad a \in \mathbf{R}, b \in \mathbf{R}$ is denoted by

$|z|$ and is given by $\sqrt{a^2 + b^2}$

$$\therefore\ |z| = |a + bi| = \sqrt{a^2 + b^2}\ .$$

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2.5.1 Properties of Modulus

(a) $|z| = 0 \Leftrightarrow z = 0$

Proof: Let $z = a + bi$, $a \in R$, $b \in R$.

then $|z| = \sqrt{a^2 + b^2}$

$|z| = 0 \Leftrightarrow a^2 + b^2 = 0$

$\Leftrightarrow a = 0$ and $b = 0$ (since a^2 and b^2 both are positive)

$\Leftrightarrow z = 0$.

(b) $|z| = |\overline{z}|$.

Proof: Let $z = a + bi$

then $|z| = \sqrt{a^2 + b^2}$

Now, $\overline{z} = a - bi$

$\therefore |\overline{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$

Thus $|z| = \sqrt{a^2 + b^2} = |\overline{z}|$ ... (i)

(c) $|z| = |-z|$

Proof: Let $z = a + bi$ then, $|z| = \sqrt{a^2 + b^2}$

$-z = -a - bi$, then $|-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2}$

Thus, $|z| = \sqrt{a^2 + b^2} = |-z|$ ...(ii)

By (i) and (ii) it can be proved that

$$|z| = |-z| = |\overline{z}|$$

Now, we consider the following examples:

Example 2.9: Find the modulus of z and $\overline{z}$ if $z = -4 + 3i$.

Solution: $z = -4 + 3i$, then $|z| = \sqrt{(-4)^2 + (3)^2}$

$= \sqrt{16 + 9} = \sqrt{25} = 5$.

and $\overline{z} = -4 - 3i$

then, $|\overline{z}| = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$.

Thus, $|z| = 5 = |\overline{z}|$.

Example 2.10 : Find the modulus of z and $-z$ if $z = 5 + 2i$. And also show that $|z| = |-z|$

Solution: $z = 5 + 2i$, then $-z = -5 - 2i$

$$|z| = \sqrt{5^2 + 2^2} = \sqrt{29} \text{ and}$$

$$|-z| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

Thus $|z| = \sqrt{29} = |-z|$.

Example 2.11: Find the modulus of $z, -z$ and $\overline{z}$ where $z = 1 + 2i$.

Solution: $z = 1 + 2i$ then $-z = -1 - 2i$ and $\overline{z} = 1 - 2i$

$$|z| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$|-z| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$

and $$|\overline{z}| = \sqrt{(1)^2 + (-2)^2} = \sqrt{5}$$

Thus, $|z| = |-z| = |\overline{z}|$.

Example 2.12: Find the modulus of:

(i) $1 + i$ (ii) 2π (iii) 0 (iv) $-\frac{1}{2}i$

Solution: (i) Let $z = 1 + i$

then $$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Thus, $|1 + i| = \sqrt{2}$

(ii) Let $z = 2\pi$ or $2\pi + 0i$

Then $$|z| = \sqrt{(2\pi)^2 + (0)^2} = 2\pi$$

Thus, $|2\pi| = 2\pi$.

If z is real then $|z| = z$

(iii) $z = 0$ or $0 + 0i$

then $$|z| = \sqrt{0^2 + 0^2} = 0$$

Thus, $|z| = 0$.

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(iv) Let $z = -\frac{1}{2}i$ or $0 - \frac{1}{2}i$

then $|z| = \sqrt{0^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{2}$

Thus, $\left|-\frac{1}{2}i\right| = \frac{1}{2}$.

If z is purely imaginary number, then $z \neq |z|$.

Example 2.13 : Find the absolute value of the conjugate of the complex number $z = -2 + 3i$.

Solution: Let $z = -2 + 3i$ then $\overline{z} = -2 - 3i$

Absolute value of $\overline{z} = |\overline{z}| = |-2 - 3i| = \sqrt{(-2)^2 + (-3)^2}$

$= \sqrt{4+9} = \sqrt{13}$

Example 2.14: Find the modulus of the complex numbers shown in an Argand Plane (Fig. 1.9)

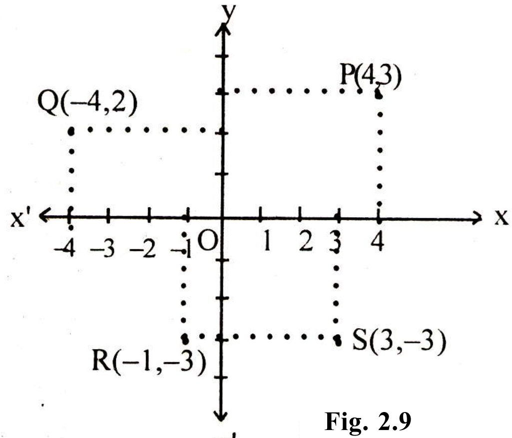

Fig. 2.9

Solution: (i) P(4, 3) represents the complex number $z = 4 + 3i$.

$\therefore\ |z| = \sqrt{4^2 + 3^2} = \sqrt{25}$

or $|z| = 5$.

(ii) Q(–4, 2) represents the complex number $z = -4 + 2i$

$|z| = \sqrt{(-4)^2 + (2)^2} = \sqrt{16+4} = \sqrt{20}$

$\therefore\ |z| = 2\sqrt{5}$.

(iii) R(–1, –3) represents the complex number $z = -1 - 3i$.

$\therefore\ |z| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{1+9}$

or $|z| = \sqrt{10}$

(iv) S(3, −3) represents the complex number $z = 3 - 3i$.

$$\therefore\ |z| = \sqrt{(3)^2 + (-3)^2} = \sqrt{9+9}$$

or $|z| = \sqrt{18} = 3\sqrt{2}.$

EXERCISE 2.2

1. Find the conjugate of each of the following:

   (a) $-2i$ (b) $-5 - 3i$ (c) $-\sqrt{2}$ (d) $(-2 + i)^2$

2. Represent the following complex numbers on Argand Plane:

   (a) (i) $2 + 0i$ (ii) $-3 + 0i$ (iii) $0 - 0i$ (iv) $3 - 0i$

   (b) (i) $0 + 2i$ (ii) $0 - 3i$ (iii) $4i$ (iv) $-5i$

   (c) (i) $2 + 5i$ and $5 + 2i$ (ii) $3 - 4i$ and $-4 + 3i$

   (iii) $-7 + 2i$ and $2 - 7i$ (iv) $-2 - 9i$ and $-9 - 2i$

   (d) (i) $1 + i$ and $-1 - i$ (ii) $6 + 5i$ and $-6 - 5i$

   (iii) $-3 + 4i$ and $3 - 4i$ (iv) $4 - i$ and $-4 + i$

   (e) (i) $1 + i$ and $1 - i$ (ii) $-3 + 4i$ and $-3 - 4i$

   (iii) $6 - 7i$ and $6 + 7i$ (iv) $-5 - i$ and $-5 + i$

3. (a) Find the modulus of following complex numbers:

   (i) 3 (ii) $(i + 1)(2 - i)$ (iii) $2 - 3i$

   (iv) $4 + \sqrt{5}\, i$

   (b) For the following complex numbers, verify that $|z| = |\bar{z}|$.

   (i) $-6 + 8i$ (ii) $-3 - 7i$

   (c) For the following complex numbers, verify that $|z| = |-z|$.

   (i) $14 + i$ (ii) $11 - 2i$

   (d) For the following complex numbers, verify that $|z| = |-z| = |-\bar{z}|$.

   (i) $2 - 3i$ (ii) $-6 - i$ (iii) $7 - 2i$

2.6 EQUALITY OF TWO COMPLEX NUMBERS

Let us consider two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ such that $z_1 = z_2$.

we have $a + bi = c + di$

or $(a - c) + (b - d)i = 0 = 0 + 0i$

Comparing real and imaginary parts on both sides, we have

$a - c = 0$ or $a = c$

$\Rightarrow$ real part of z_1 = real part of z_2

and $b - d = 0$ or $b = d$

$\Rightarrow$ imaginary partof z_1 = imaginary part of z_2

Therefore, we can conclude that two complex numbers are equal if and only if their real parts and imaginary parts are respectively equal.

In general $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Properties: $z_1 = z_2 \Rightarrow |z_1| = |z_2|$

Let $z_1 = a + bi,\ z_2 = c + di$

$z_1 = z_2$ gives $a = c$ and $b = d$

Now $|z_1| = \sqrt{a^2 + b^2}$ and $|z_2| = \sqrt{c^2 + d^2}$

$= \sqrt{a^2 + b^2}$ (since $a = c$ and $b = d$)

$\Rightarrow |z_1| = |z_2|$.

Example 2.15: For what value of x and y, $5x + 6yi$ and $10 + 18i$ are equal?

Solution: It is given that $5x + 6y\,i = 10 + 18\,i$

Comparing real and imaginary parts, we have

$5x = 10$ or $x = 2$

and $6y = 18$ or $y = 3$

For $x = 2,\ y = 3$, the given complex numbers are equal.

2.7 ADDITION OF COMPLEX NUMBERS

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers then their sum $z_1 + z_2$ is defined by

$z_1 + z_2 = (a + c) + (b + d)i$

For example, if $z_1 = 2 + 3i$ and $z_2 = -4 + 5i$

then $z_1 + z_2 = [2 + (-4)] + [3 + 5]i$

$= -2 + 8i$

Example 2.16: Simplify

(i) $(3 + 2i) + (4 - 3i)$

(ii) $(2 + 5i) + (-3 - 7i) + (1 - i)$

Solution: (i) $(3 + 2i) + (4 - 3i) = (3 + 4) + (2 - 3)i = 7 - i$

(ii) $(2 + 5i) + (-3 - 7i) + (1 - i) = (2 - 3 + 1) + (5 - 7 - 1)i$

$= 0 - 3i$

or $(2 + 5i) + (-3 - 7i) + (1 - i) = - 3i$

2.7.1 Geometrical Represention of Addition of Two Complex Numbers

Let two complex numbers z_1 and z_2 be represented by the points P(a, b) and Q(c, d).

Their sum, $z_1 + z_2$ is represented by the point R ($a + c$, $b + d$) in the same Argand Plane.

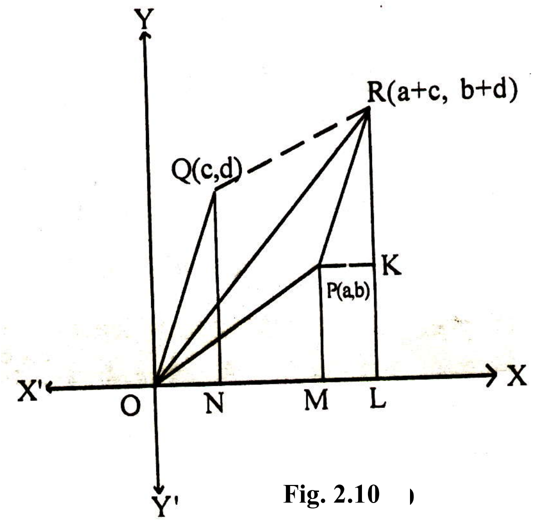

Fig. 2.10

Join OP, OQ, OR, PR and QR.

Draw perpendiculars PM, QN, RL from P, Q, R respectively on X-axis.

Draw perpendicular PK to RL

In ΔQON

ON = c

and QN = d

In Δ ROL In Δ POM

RL $= b + d$ PM $= b$

and OL $= a + c$ OM $= a$

Also PK = ML

= OL − OM

$= a + c - a$

$= c =$ ON.

RK = RL − KL

= RL − PM

$= b + d - b$

$= d =$ QN

Δ QON and Δ RPK,

ON = PK, QN = RK and $\angle$QNO = $\angle$RKP = 90^0.

$\therefore$ ΔQON $\cong$ ΔRPK

$\therefore$ OQ = PR and OQ // PR.

$\Rightarrow$ OPRQ is a parallelogram and OR its diagonal.

Therefore, we can say that the sum of two complex numbers is represented by the diagonal of a parallelogram.

Example 2.17 : Prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.

Solution : We have proved that the sum of two complex numbers z_1 and z_2 represented by the diagonal of a parallelogram OPRQ (see fig. 1.11).

In Δ OPR

OR $\leq$ OP + PR

or OR $\leq$ OP + OQ ($\because$ OQ = PR)

or $|z_1 + z_2| \leq |z_1| + |z_2|$.

Example 2.18: If $z_1 = 2 + 3i$ and $z_2 = 1 + i$

verify that $|z_1 + z_2| \leq |.z_1| + |z_2|$

Solution: $z_1 = 2 + 3i$ and $z_2 = 1 + i$ represented by the points (2, 3) and (1, 1) respectively. Their sum ($z_1 + z_2$) will be represented by the point (2 + 1,3+ 1) i.e. (3, 4).

Verification:

$$|z_1| = \sqrt{2^2+3^2} = \sqrt{13} = 3.6 \text{ approx.}$$

$$|z_2| = \sqrt{1^2+1^2} = \sqrt{2} = 1.41 \text{ approx.}$$

$$|z_1+z_2| = \sqrt{3^2+4^2} = \sqrt{25} = 5$$

$$|z_1|+|z_2| = 3.6+1.41 = 5.01$$

$$\therefore \; |z_1 + z_2| \leq |z_1| + |z_2|$$

2.7.2 Subtraction of the Complex Numbers

Let two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ be represented by the points (a, b) and (c, d) respectively.

$$\therefore \; z_1 - z_2 = (a + bi) - (c + di)$$

$$= (a - c) + (b - d)i$$

which represents a point $(a - c, b - d)$.

$\therefore$ The difference i.e., $z_1 - z_2$ is represented by the point $(a - c, b - d)$.

Thus, to subtract a complex number from another, we subtract corresponding real and imaginary parts separately.

Example 2.19 : Find $z_1 - z_2$ in each of following if:

(a) $z_1 = 3 - 4i$, $z_2 = -3 + 7i$

(b) $z_1 = -4 + 7i$, $z_2 = -4 - 5i$

Solution : (a) $z_1 - z_2 = (3 - 4i) - (-3 + 7i)$

$$= (3 - 4i) + (3 - 7i)$$

$$= (3 + 3) + (-4 - 7)i$$

$$= 6 + (-11)\, i = 6 - 11i$$

(b) $z_1 - z_2 = (-4 + 7i) - (-4 - 5i)$

$= (-4 + 7i) + (4 + 5i)$

$= (-4 + 4) + (7 + 5)i$

$= 0 + 12i = 12i$

Example 2.20 : What should be added to i to obtain $(5 + 4i)$?

Solution: Let $z = a + bi$ be added to i to obtain $5 + 4i$

$\therefore \quad i + (a + bi) = 5 + 4i$

or $\quad a + (b + 1)i = 5 + 4i$

Equating real and imaginary parts, we have

$a = 5$ and $b + 1 = 4$ or $b = 3$

$\therefore$ $z = 5 + 3i$ is to be added to i to obtain $5 + 4i$.

2.8 PROPERTIES: WITH RESPECT TO ADDITION OF COMPLEX NUMBERS

1. **Closure :** The sum of two complex numbers will always be a complex number. $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, $a_1, b_1, a_2, b_2 \in R$.

   Now $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$ which is again a complex number.

   This proves the closure property of complex numbers.

   Thus, $(1 + i) + (2 + 3i) = (1 + 2) + (1 + 3)i = 3 + 4i$ which is again a complex number.

   Similarly, the difference of two complex numbers will always be a complex number. For example, $(2 + 4i) - (1 - 4i) = (2 - 1) + \{4 - (-4)\}i = 1 + 8i$, which is again a complex number.

2. **Commutative:** If z_1 and z_2 are two complex numbers then

   $$z_1 + z_2 = z_2 + z_1.$$

   Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$

   Now $z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i)$

   $= (a_1 + a_2) + (b_1 + b_2)i$

$$= (a_2 + a_1) + (b_2 + b_1)i$$

[commutative property of real numbers]

$$= (a_2 + b_2\, i) + (a_1 + b_1 i)$$

$$= z_2 + z_1$$

i.e., $z_1 + z_2 = z_2 + z_2$.

Hence, addition of complex numbers is commutative.

For example, $z_1 = 8 + 7i$, $z_2 = 9 - 3i$ then

$z_1 + z_2 = (8 + 7i) + (9 - 3i)$ and $z_1 + z_2 = (9 - 3i) + (8 + 7i)$

$= (8 + 9) + (7 - 3)i$ and $= (9 + 8) + (-3 + 7)i$

$z_2 + z_1 = 17 + 14i$ and $z_1 + z_2 = 17 + 14i$

We get, $z_1 + z_2 = z_2 + z_1$

Now, $z_1 - z_2 = (a_1 + b_1 i) - (a_2 + b_2 i)$

$= (a_1 - a_2) + (b_1 - b_2)i$

and $z_2 - z_1 = (a_2 + b_2 i) - (a_1 + b_1 i)$

$= (a_2 - a_1) + (b_2 - b_1)i$

$= -(a_1 - a_2) - (b_1 - b_2)i$

$= -(a_1 + b_1 i) + (a_2 + b_2 i)$

$\therefore z_1 - z_2 \neq z_2 - z_1$

Hence, subtraction of complex numbers is not commutative.

For example, if $z_1 = 8 + 7i$ and $z_2 = 9 - 3i$ then

$z_1 - z_2 = (8 + 7i) - (9 - 3i)$ and $z_2 - z_1 = (9 - 3i) - (8 + 7i)$

$= (8 - 9) + (7 + 3)i = (9 - 8) + (-3 - 7)i$

or $z_1 - z_2 = -1 + 10i$ and $z_2 - z_1 = 1 - 10i$

$\therefore z_1 - z_2 \neq z_2 - z_1$.

3. **Associative :** If $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ and $z_3 = a_3 + b_3 i$ are three complex numbers, then

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

Now $z_1 + (z_2 + z_3)$

$$= (a_1 + b_1 i) + \{(a_2 + b_2 i) + (a_3 + b_3 i)\}$$
$$= (a_1 + b_1 i) + \{(a_2 + a_3) + (b_2 + b_3)i\}$$
$$= \{(a_1 + (a_2 + a_3)\} + \{b_1 + (b_2 + b_3)\}i$$
$$= \{(a_1 + a_2) + (b_1 + b_2)i\} + (a_3 + b_3 i)$$
$$= \{(a_1 + b_1 i) + (a_2 + b_2 i)\} + (a_3 + b_3 i)$$
$$= (z_1 + z_2) + z_3.$$

Hence, the associativity property holds good in the case of addition of complex numbers.

For example, if $z_1 = 2 + 3i$, $z_2 = 3i$ and $z_3 = 1 - 2i$, then

$$z_1 + (z_2 + z_3) = (2 + 3i) + \{(3i) + (1 - 2i)\}.$$
$$= (2 + 3i) + (1 + i)$$
$$= (3 + 4i)$$
$$(z_1 + z_2) + z_3 = \{(2 + 3i) + (3i)\} + (1 - 2i)$$
$$= (2 + 6i) + (1 - 2i)$$
$$= (3 + 4i)$$

and $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

The equality of two sums is the consequence ofthe associative property of addition of complex numbers.

Like commutativity, it can be shown that associativity also does not hold good in the case of subtraction.

4. Existence of Additive Identitiy

If $x + yi$ be a complex number, then there exists a complex number $(0 + 0i)$

Such that $(x + yi) + (0 + 0i) = x + yi.$

Let $z_2 = x + yi$ be the additive identity of $z_1 = 2 + 3i$ then

$z_1 + z_2 = z_1$

i.e., $(2 + 3i) + (x + yi) = 2 + 3i$

or $(2 + x) + (3 + y)i = 2 + 3i$

or $(2 + x) = 2;\ 3 + y = 3$

or $x = 0$ and $y = 0$

i.e., $z_2 = x + yi = 0 + 0i$ is the additive identity.

i.e., if $z = a + bi$ is any complex number, then

$(a + bi) + (0 + 0i) = a + bi$

i.e., $(0 + 0i)$ is the additive identity.

$$z_1 - z_2 = (2 + 3i) - (0 + 0i)$$
$$= (2 - 0) + (3 - 0)i$$
$$= 2 + 3i$$
$$= z_1$$

$\therefore\ z_2 = 0 + 0i$, is the identity w.r.t. subtraction also.

as $(a + bi) - (0 + 0i) = a + bi$

5. Existence of Additive Inverse

For every complex number $a + bi$ there exists a unique complex number $-a - bi$ such that $(a + bi) + (-a - bi) = 0 + 0i$.

Example: Let $z_1 = 4 + 5i$ and $z_2 = x + yi$ be the additive inverse of z_1

Then, $z_1 + z_2 = 0$

or $(4 + 5i) + (x + yi) = 0 + 0i$

or $(4 + x) = 0$ and $5 + y = 0$

or $x = -4$ and $y = -5$

Thus, $z_1 = -4 - 5i$ is the additive invese of $z_1 = 4 + 5i$

In general, additive inverse of a complex number is obtained by changing the signs ofreal and imaginaryparts.

Consider $z_1 - z_2 = 0$

or $(4 + 5i) - (x + yi) = 0 + 0i$

or $(4 - x) + (5 - y)i = 0 + 0i$

or $4 - x = 0$ and $5 - y = 0$

or $x = 4$ and $y = 5$

i.e., $z_1 - z_2 = 0$ gives $z_2 = 4 + 5i$

Thus, in subtraction, the number itselfis the inverse.

i.e., $(a + bi) - (a + bi) = 0 + 0i$ or 0.

EXERCISE 2.3

1. Simplify:

   (a) $(\sqrt{2}+\sqrt{5}i)+(\sqrt{5}-\sqrt{2}i)$ (b) $\frac{2+i}{3}+\frac{2-i}{6}$

   (c) $(1+i)-(1-6i)$ (d) $(\sqrt{2}-\sqrt{3}i)-(-2-7i)$

2. If $z_1 = (5 + i)$ and $z_2 = (6 + 2i)$ then:

   (a) find $z_1 + z_2$ (b) find $z_2 + z_1$

   (c) Is $z_1 + z_2 = z_2 + z_1$?

   (d) find $z_1 - z_2$ (e) find $z_2 - z_1$

   (f) Is $z_1 - z_2 = z_2 - z_1$?

3. If $z_1 = (1 + i)$, $z_2 = (1 - i)$ and $z_3 = (2 + 3i)$, then

   (a) find $z_1 + (z_2 + z_3)$ (b) find $(z_1 + z_2) + z_3$

   (c) Is $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$? (d) find $z_1 - (z_2 - z_3)$

   (e) find $(z_1 - z_2) - z_3$

   (f) Is $z_1 - (z_2 - z_3) = (z_1 - z_2) - z_3$?

4. Find the additive inverse of the following:

   (a) $12 - 7i$ (b) $4 - 3i$

5. What shoud be added to $(-15 + 4i)$ to obtain $(3 - 2i)$?

6. Show that $\left\{\overline{(3+7i)-(5+2i)}\right\} = \overline{(3+7i)}-\overline{(5+2i)}$

2.9 ARGUMENT OF A COMPLEX NUMBER

Let P(a, b) represent the complex number $z = a + bi, a \in R, b \in R$ and OP makes an angle θ with the positive direction of x -axis.

Draw PM $\perp$ OX

Let OP = r

In right ΔOMP

OM = a

MP = b

$\therefore$ $r \cos \theta$ = a.

$r \sin \theta = b$.

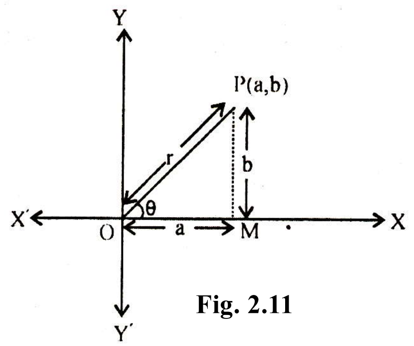

Fig. 2.11

Then $z = a + bi$ can be written as $z = r(\cos \theta + i \sin \theta)$ ...(i)

where $r = \sqrt{a^2 + b^2}$ and $\tan \theta = \dfrac{b}{a}$

or $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

This is known as the polar form of the complex number z and r are respectively called the modulus and argument of the complex number.

2.10 MULTIPLICATION OF TWO COMPLEX NUMBERS

Two complex numbers can be multiplied by the usual laws of addition and multiplication as is done in the case of numbers.

Let $z_1 = (a + bi)$ and $z_2 = (c + di)$ then,

$$z_1 \cdot z_2 = (a + bi)(c + di)$$

$$= a(c + di) + bi(c + di)$$

or $= ac + adi + bci + bdi^2$

or $= (ac - bd) + (ad + bc)i$ (since $i^2 = -1$)

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If $(a + bi)$ and $(c + di)$ are two complex numbers, their product is defined as the complex number $(a - bd) + (ad + bc)i$.

Example 2.21: Evaluate:

(i) $(1+2i)(1-3i)$ (ii) $(\sqrt{3}+i)(\sqrt{3}-i)$ (iii) $(3-2i)^2$

Solution: (i) $(1 + 2i)(1 - 3i) = \{(1 - (-6)\} + (-3 + 2)i$

$= 7 - i$

(ii) $(\sqrt{3}+i)(\sqrt{3}-i) = \{3 - (-1)\} + (-\sqrt{3}+\sqrt{3})i$

$= 4 + 0i$

(iii) $(3 - 2i)^2 = (3 - 2i)(3 - 2i)$

$= (9 - 4) + (-6 - 6)i$

$= 5 - 12i$

2.10.1 Properties of Multiplication

$|z_1 . z_2| = |z_1| . |z_2|$.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore |z_1| = r_1 . \sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1$

Similarly, $|z_2| = r_2$.

Now, $z_1 . z_2 = r_1(\cos\theta_1 + i\sin\theta_1) . r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 . r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)i]$

$= r_1 . r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.

$$\begin{bmatrix} \text{since } \cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \text{ and} \\ \sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \end{bmatrix}$$

$|z_1 . z_2| = r_1.r_2\sqrt{\cos^2(\theta_1+\theta_2)+\sin^2(\theta_1+\theta_2)}$

$= r_1 . r_2$.

$\therefore |z_1 . z_2| = r_1 . r_2 = |z_1| . |z_2|$.

and argument $z_1 . z_2 = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$.

Example 2.22 : Find the modulus of the complex number $(1 + i)(4 - 3i)$

Solution: $z = (1 + i)(4 - 3i)$

then $|z| = |(1 + i)(4 - 3i)|$

$= |1 + i|\,|4 - 3i|$ (since $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

But $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

$|4-3i| = \sqrt{4^2+(-3)^2} = 5$

$\therefore |z| = \sqrt{2} \cdot 5 = 5.\sqrt{2}.$

2.11 DIVISION OF TWO COMPLEX NUMBERS

Division of complex numbers involves multiplying both numerator and denominator with the conjugate of the denominator. We will explain it through an example.

Let $z_1 = a + bi$ and $z_2 = c + di$ then.

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} \quad (c+di) \neq 0$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)\,(c-di)}{(c+di)\,(c-di)}$$

(multiplying numerator and denominator with the conjugate of the denominator)

$$= \frac{(ac+bd) + (bc-ad)i}{c^2+d^2}$$

Thus, $$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\,i$$

Example 2.23 : Divide $3 + i$ by $4 - 2i$

Solution : $$\frac{3+i}{4-2i} = \frac{(3+i)(4+2i)}{(4-2i)(4+2i)}$$

MODULE - I
Algebra

Notes

Multiplying numerator and denominator by the conjugate of $(4 - 2i)$ we get

$$= \frac{10+10i}{20}$$

$$= \frac{1}{2} + \frac{1}{2}i$$

Thus, $$\frac{3+i}{4-2i} = \frac{1}{2} + \frac{1}{2}i$$

2.11.1 Properties of Division

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Proof:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$|z_1| = r_1\sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1$$

Similarly, $|z_2| = r_2$

and $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

Then, $$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)\,(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\,\frac{(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}\,[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Thus, $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}\sqrt{\cos^2(\theta_1-\theta_2)+\sin^2(\theta_1-\theta_2)} = \frac{r_1}{r_2}$

$\therefore$ Argument of $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \theta_1 - \theta_2$

Example 2.24: Find the modulus of the complex numbef

$$\frac{2+i}{3-i}$$

Solution: Let $z = \frac{2+i}{3-i}$

$$\therefore |z| = \left|\frac{2+i}{3-i}\right| = \left|\frac{2+i}{3-i}\right| \quad \left(\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right)$$

$$= \frac{\sqrt{2^2+1^2}}{\sqrt{3^2+(-1)^2}} = \frac{\sqrt{5}}{\sqrt{10}} = \frac{1}{\sqrt{2}}$$

$$\therefore |z| = \frac{1}{\sqrt{2}}$$

2.12 PROPERTIES OF MULTIPLICATION OF TWO COMPLEX NUMBERS

1. **Closure**

   If $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers then their product $z_1 . z_2$ is also a complex number.

2. **Cummutative**

   If $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers then $z_1 . z_2 = z_2 . z_1$

   For example, let $z_1 = 3 + 4i$ and $z_2 = 1 - i$

   then $z_1 z_2 = (3 + 4i)(1 - i)$

   $= 3(1 - i) + 4i(1 - i)$

$= 3 - 3i + 4i - 4i^2$

$= 3 - 3i + 4i - 4(-1)$

$= 3 + i + 4 = 7 + i$

Again, $z_2 z_1 = (1 - i)(3 + 4i)$

$= (3 + 4i) - i(3 + 4i)$

$= 3 + 4i - 3i - 4i^2$

$= 3 + i + 4 = 7 + i$

$\therefore \; z_1 z_2 = z_2 z_1 = 7 + i$

3. Associativity

If $z_1 = (a + bi)$, $z_2 = c + di$ and $z_3 = (e + fi)$ then

$z_1 (z_2 . z_3) = (z_1 . z_2) z_3$

Let us verify it with an example :

If $z_1 = (1 + i)$, $z_2 = (2 + i)$ and $z_3 = (3 + i)$ then

$z_1(z_2 . z_3) = (1 + i) \{(2 + i)(3 + i)\}$

$= (1 + i) \{(6 - 1) + (3 + 2)i\}$

$= (1 + i)(5 + 5i)$

$= (5 - 5) + (5 + 5) i$

$= 0 + 10i = 10i$

and $(z_1 . z_2) z_3 = \{(1 + i)(2 + i)\}(3 + i)$

$= \{(2 - 1) + (1 + 2)i\}(3 + i)$

$= (1 + 3i)(3 + i)$

$= (3 - 3) + (1 + 9)i$

$= 0 + 10i = 10i$

$\therefore \; z_1(z_2 . z_3) = (z_1 . z_2) z_3 = 10i$

4. Existence of Multiplicative Identity :

For every non-zero complex number $z_1 = a + bi$ there exists a unique complex number $(1 + 0i)$ such that

$(a + bi)(1 + 0i) = (1 + 0i)(a + bi) = a + bi$

Let $z_1 = x + yi$ be the multipicative identity of $z_1 = a + bi$

Then $zz_1 = z.$

i.e., $(a + bi)(x + yi) = a + bi$

or $(ax - by) + (ay + bx)i = a + bi$

or $ax - by = a, \quad ay + bx = b$

or $x = 1$ and $y = 0$

i.e., $z_1 = x + yi = 1 + 0i$ be the multipicative identity.

The complex number $1 + 0i$ is the identity for multiplication.

Let us verify it with an example:

If $z = 2 + 3i$ then

$z \cdot (1 + 0i) = (2 + 3i)(1 + 0i)$

$= (2 - 0) + (3 + 0)i$

$= 2 + 3i$

5. Existence of Multiplicative inverse:

Multiplicative inverse is a complex number that when multiplied to a given non -zero complex number yields one. In other words, for every non-zero complex number $z = a + bi$, there exists a unique complex number $(x + yi)$ such that their product is $(1 + 0i)$.

i.e., $(a + bi)(x + yi) = 1 + 0i$

or $(ax - by) + (bx + ay)i = 1 + 0i$

Equating real and imaging parts, we have

$ax - by = 1$ and $bx + ay = 0$

By cross multiplication

$$\frac{x}{a} = \frac{y}{-b} = \frac{1}{a^2+b^2}.$$

$$\Rightarrow x = \frac{a}{a^2+b^2} = \frac{\mathrm{Re}(z)}{|z|^2} \quad \text{and} \quad y = \frac{-b}{a^2+b^2} = \frac{-\mathrm{Im}(z)}{|z|^2}$$

Thus, the multiplicative inverse of a non-zero complex number $z = (a + bi)$ is

$$x + yi = \left[\frac{\mathrm{Re}(z)}{|z|^2} - \frac{-\mathrm{Im}(z)}{|z|^2} i\right] = \frac{\bar{z}}{|z|^2}.$$

Example 2. 25 : Find the multiplication inverse of $2 - 4i$.

Solution: Let $z = 2 - 4i$

we have, $\bar{z} = 2 + 4i$ and $|z|^2 = |2^2 + (-4)^2| = 20$

$\therefore$ Required multiplicative inverse is

$$\frac{\bar{z}}{|z|^2} = \frac{2+4i}{20} = \frac{1}{10} + \frac{1}{5}i$$

Verification:

If $\frac{1}{10} + \frac{1}{5}i$ be the muliplicative inverse of $2 - 4i$, their product must be equal to $1 + 0i$

$$\text{We have,} \quad (2-4i)\left(\frac{1}{10} + \frac{1}{5}i\right) = \left(\frac{2}{10} + \frac{4}{5}\right) + \left(\frac{2}{5} - \frac{4}{10}\right)i$$

$= 1 + 0i$ which is true.

6. Distributive Property of Multiplication over Addition

Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ and $z_3 = a_3 + b_3 i$

Then $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Let us verify it with an example:

If $z_1 = 3 - 2i$, $z_2 = -1 + 4i$ and $z_3 = -3 - i$ then

$$z_1(z_2 + z_3) = (3 - 2i)\ \{(-1 + 4i) + (-3 - i)\}$$
$$= (3 - 2i)\ (-1 + 4i - 3 - i)$$
$$= (3 - 2i)\ (-4 + 3i)$$
$$= (-12 + 6) + (9 + 8)i$$
$$= -6 + 17i$$

and $z_1z_2 = (3 - 2i)\ (-1 + 4i)$
$$= (-3 + 8) + (12 + 2)i$$
$$= 5 + 14i$$

and $z_1z_3 = (3 - 2i)\ (-3 - i)$
$$= (-9 - 2) + (-3 + 6)i$$
$$= -11 + 3i$$

Now $z_1z_2 + z_1z_3 = (5 + 14i) + (-11 + 3i)$
$$= -6 + 17i$$
$$\therefore \quad z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

EXERCISE 2.4

1. Simplify each of the following:

   (a) $(1+2i)(\sqrt{2}-i)$ (b) $(\sqrt{2}+i)^2$

   (c) $(3+i)(1-i)(-1+i)$ (d) $(2+3i) \div (1-2i)$

   (e) $(1+2i) \div (1+i)$ (e) $(1+0i) \div (3+7i)$

2. Compute multiplicative inverse of each of the following complex numbers:

   (a) $3 - 4i$ (b) $\sqrt{3}+7i$ (c) $\dfrac{3+5i}{2-3i}$

3. If $z_1 = 4 + 3i$, $z_2 = 3 - 2i$ and $z_3 = i + 5$ verify that $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$.

4. If $z_1 = 2 + i$, $z_2 = -2 + i$ and $z_3 = 2 - i$ then verify that $(z_1 . z_2)z_3 = z_1(z_2 . z_3)$

De Moivre's Theorem

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- Use it in studying the n^{th} roots of unity.
- Obtain a version of De Moivre's Theorem for rational indices.
- Solve an equation and find the roots in the polar form of any complex number, even it is a positive number, negative number and fraction.
- Helps us find the power and roots of complex numbers easily.

PREREQUISITES

- Properties of complex number.
- Solution of linear and quadratic equations.
- Representation of a complex number in the Argand Plane.

INTRODUCTION

In the previous chapter we learnt that $\text{cis}\theta_1 \,.\, \text{cis}\theta_2 = \text{cis}(\theta_1 + \theta_2)$ and hence $(\text{cis}\theta)^2 = \text{cis}\, 2\theta$. In this chapter we extend this result for any integer. The extension is called De Moivre's theorem for integral indices. We use it in studing the n^{th} roots of unity. We also obtain a verson of De Moivre's theorem for rational indices.

De Moiver's Theorem - Integral and Rational Indices

In this section Demoivre's theorem is proved. By using this theorem all the n^{th} roots of a complex number $Z \neq 0$ are determined. As a particular case, all the n^{th} roots of unity are determined.

De Moivre's Theorem for Integral Index

For any real number θ and any Integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof: Let θ be a given real number. We distinguish three cases.

Case(i) Let n be a positive integer. We prove the theorem using the principle of mathematical induction on n.

Let $p(n)$ be the statement $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

If $n = 1$, then LHS $= \cos\theta + i\sin\theta = \cos 1\theta + i\sin 1\theta =$ RHS.

$\therefore$ $p(1)$ is true.

Assume that p(k) is true for $k \in$ N.

i.e., $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Multiplying both the sides of the above equation with $(\cos\theta + i\sin\theta)$ we get

$(\cos\theta + i\sin\theta)^k (\cos k\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta) = (\cos\theta + i\sin\theta)$

$$(\cos\theta + i\sin\theta)^{k+1} = \cos k\theta.\cos\theta + i\sin k\theta\cos\theta + i\cos k\theta\sin\theta + i^2\sin k\theta\sin\theta$$

$$= (\cos k\theta.\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$

$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$

$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

$=$ RHS.

$\therefore$ $p(k+1)$ is true.

By the principle of mathematical induction. P(n) is true for all positive integers n.

i.e., $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in Z^+$.

Case(ii) Let $n = 0$ then LHS $= (\cos\theta + i\sin\theta)^0 = 1$

$$= \cos 0\theta + i\sin 0\theta$$

$$= \text{RHS}.$$

Hence $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ is this case also.

Case(iii) Let n be a negative integer and $n = -m$ where $m \in Z^+$.

$$\therefore \text{ LHS} = (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} = \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \quad \text{form case (i)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + i^2\sin^2 m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos(-m)\theta + i\sin(-m)\theta$$

$$= \cos n\theta + i\sin n\theta = \text{RHS}.$$

Example 2.1: Simplify : $(\cos 2\alpha + i\sin 2\alpha)^6 \,.\, (\cos 3\alpha - i\sin 3\alpha)^5$

Solution: GE $= (\cos 12\alpha + i\sin 12\alpha)\ (\cos 15\alpha - i\sin 15\alpha)$

$= (\text{cis } 12\alpha)\ [\text{cis }(-15\alpha)]$

$= \text{cis}(12\alpha - 15\alpha) = \text{cis }(-3\alpha) = \cos 3\alpha - i\sin 3\alpha.$

Example 2.2: If $x = \text{cis }\alpha$ and $y = \text{cis }\beta$ and find $xy + \frac{1}{xy}$.

Solution: $xy = \text{cis}\alpha \,.\, \text{cis}\beta = \text{cis}(\alpha+\beta)$, then $\frac{1}{xy} = cis[-(\alpha+\beta)]$

$$\Rightarrow xy + \frac{1}{xy} = \cos(\alpha+\beta) + i\sin(\alpha+\beta) + \cos(\alpha+\beta) - i\sin(\alpha+\beta)$$

$$= 2\cos(\alpha + \beta).$$

Example 2.3 : If n is a positive integer, show that

$$(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}} \cos\left(\frac{n\pi}{4}\right)$$

Solution : $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$(1+i)^n = \left(\sqrt{2}\right)^n \left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^n = 2^{n/2}\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right] \quad ...(i)$$

$$(1-i)^n = \left(\sqrt{2}\right)^n \left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^n = 2^{n/2}\left[\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right] \quad ...(ii)$$

By adding (1) & (2) we get

$$(1+i)^n + (1-i)^n = 2^{n/2}\left(2\cos\frac{n\pi}{4}\right) = 2^{\frac{n+2}{2}} \cos\frac{n\pi}{4}.$$

EXERCISE 2.5

I 1. Find the values of the following

(i) $(1+i\sqrt{3})^3$ (ii) $(1-i)^8$ (iii) $(1+i)^{16}$

(iv) $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$

2. If n is an integer then show that

$$(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$$

II.1. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$ then for any $n \in N$ show that $\alpha^n + \beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$.

2. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2} = \sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

3. Show that $(1+i)^{2n} + (1+i)^{2n} = 2^{n+1}\cos\left(\frac{n\pi}{2}\right)$ where n is a positive integer.

De Moivre's Theorem - Rational Index

Theorem : If $\frac{p}{q}(q>1)$ is a rational number, then $\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$ is one of the q roots of $(\cos\theta + i\sin\theta)^p$.

Proof : Let $n = \frac{p}{q}$ so that $nq = p$. Since q is a positive integer

$$(\cos n\theta + i\sin n\theta)^q = \cos nq\theta + i\sin nq\theta$$

$$\therefore \quad (\cos n\theta + i\sin n\theta)^q = \cos p\theta + i\sin p\theta = (\cos\theta + i\sin\theta)^p$$

$\therefore$ $(\cos n\theta + i\sin n\theta)$ is one of the q^{th} roots of $(\cos\theta + i\sin\theta)^p$

$\Rightarrow \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)$ is one of the q roots of $(\cos\theta + i\sin\theta)^{p/q}$

Note: $\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$.

n^{th} roots of Unity

$$\sqrt[n]{1} = (\cos 0 + i\sin 0)^{1/n} = \{\cos(2k\pi+0) + i\sin(2k\pi+0)\}^{1/n}$$

$$= \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) \quad \text{where } k = 0, 1, 2, \ldots n-1.$$

It can be seen that n^{th} roots of unity differ in argument by $\frac{2\pi}{n}$, but have the same modulus unity.

Graphically the n^{th} roots of unity are the points $A_1, A_2, \ldots A_n$ where $OA_1 = OA_2 = \ldots = OA_n = 1$ and $\angle A_1OX = \frac{2\pi}{n}$, $\angle A_2OX = \frac{4\pi}{n}$, ... $\angle A_nOX = \frac{2k\pi}{n}$.

Also the points lie on the circle with centre at the origin and with radius unity. These points divide the circle into n equal arcs.

Let $\quad cis\left(\frac{2\pi}{n}\right) = \alpha$

$$\therefore \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = \alpha^k$$

$\therefore$ n^{th} roots of unity are $\alpha^0, \alpha^1, \alpha^2 \ldots \alpha^{n-1}$

$$\therefore \quad 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha} = 0 \qquad (\because \alpha^n = 1)$$

$\therefore$ Sum of the n^{th} roots of unity is zero.

Cube roots of Unity

The cube roots of unity are obtained by solving the equation $x^3 = 1$.

$x^3 = 1 = \cos 0 + i \sin 0 = \cos(2k\pi = 0) + i\sin(2k\pi + 0)$

$\therefore$ The roots of the equation are x_k where

$$x_k = \cos\left(\frac{0+2k\pi}{3}\right) + i\sin\left(\frac{0+2k\pi}{3}\right) \qquad k = 0, 1, 2$$

$\therefore$ The cube roots of unity are

$$\cos 0 + i\sin 0; \quad \cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}, \ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

$$\Rightarrow \quad 1, \omega, \omega^2 \text{ where } w = cis\frac{2\pi}{3} \text{ or } cis\frac{4\pi}{3}$$

$$[\because w = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} \Rightarrow w^2 = \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}$$

$$\Rightarrow w^2 = \cos\left(2\pi + \frac{2\pi}{3}\right) + i\sin\left(2\pi + \frac{2\pi}{3}\right)$$

$$\Rightarrow w^2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}]$$

Note: $1 + w + w^2 = 1 + \cos 120^0 + i \sin 120^0 + \cos 240^0 + i \sin 240^0$

$$= 1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2} = 1 - 1 = 0$$

Also $w^3 = w \,.\, w^2 = 1$.

Example 2.4 : Find the values of $\left(1 + i\sqrt{3}\right)^{\frac{1}{5}}$.

Solution : Let $1 + i\sqrt{3} = r(\cos\theta + i\sin\theta) \Rightarrow r\cos\theta = 1,\ r\sin\theta = \sqrt{3}$

Then $r = \sqrt{3+1} = 2$ and $\cos\theta = \frac{1}{2},\ \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

$$\therefore \left(1 + i\sqrt{3}\right)^{1/5} = 2^{1/5}\left[\cos\left(\frac{2k\pi + (\pi/3)}{5}\right) + i\sin\left(\frac{2k\pi + \pi/3}{5}\right)\right]$$

where $k = 0, 1, 2, 3, 4$

$$= 2^{1/5}\text{cis}\left(\frac{6k+1}{15}\right)\pi,\ k = 0,1,2,3,4$$

The five values of $\left(1 + i\sqrt{3}\right)^{1/5}$ are

$$2^{1/5}cis\left(\frac{\pi}{15}\right),\ 2^{1/5}cis\left(\frac{7\pi}{15}\right),\ 2^{1/5}cis\left(\frac{13\pi}{15}\right),\ 2^{1/5}cis\left(\frac{19\pi}{15}\right), 2^{1/5}cis\left(\frac{25\pi}{15}\right).$$

Example 2.5 : Solve the equation $x^4 + 1 = 0$

Solution : Given $x^4 + 1 = 0 \;\Rightarrow x^4 = -1 \;\Rightarrow x = (-1)^{1/4}$

We have $-1 = \cos\pi + i\sin\pi = \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$

$$\Rightarrow (-1)^{1/4} = \left[\cos(2k+1)\pi + i\sin(2k+1)\pi\right]^{1/4}$$

$$= \text{c}is\left(2k+1\right)\frac{\pi}{4},\ k = 0,1,2,3$$

The four values of x are $cis\left(\frac{\pi}{4}\right),\ cis\left(\frac{3\pi}{4}\right), cis\left(\frac{5\pi}{4}\right),\ cis\left(\frac{7\pi}{4}\right)$

$$\Rightarrow \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}} \text{ and } \frac{1-i}{\sqrt{2}}$$

Example 2.6 : If $1, w, w^2$ are the cube roots of unity, prove that

(a) $1 + w^2 + w^7 = 0$

(b) $(1 - w + w^2)(1 + w - w^2) = 4$

(c) $(1 + w)^3 - (1 + w^2)^3 = 0$

(d) $(1 - w + w^2) = -8$ and $(1 + w - w^2)^3 = -8$

Sol : (a) $1 + w^2 + w^7 = 0$

$$\begin{aligned} LHS &= 1 + w^2 + (w^3)^2 . \ w \\ &= 1 + w^2 + w && (\because w^3 = 1) \\ &= 0 && (\because 1 + w + w^2 = 0) \\ &= RHS \end{aligned}$$

$\therefore$ LHS = RHS.

(b) $(1 - w + w^2)(1 + w - w^2) = 4$

$$\begin{aligned} LHS &= (1 - w + w^2)(1 + w - w^2) \\ &= (1 + w^2 - w)(1 + w - w^2) \\ &= (-w - w)(-w^2 - w^2) && (\because 1 + w^2 = -w,\ 1 + w = -w^2) \\ &= (-2w)(-2w^2) \\ &= 4w^3 && (\because w^3 = 1) \\ &= 4 \\ &= RHS \end{aligned}$$

$\therefore$ LHS = RHS

(c) $(1 + w)^3 - (1 + w^2)^3 = 0$

$$\begin{aligned} LHS &= (1 + w)^3 - (1 + w^2)^3 \\ &= (-w^2)^3 - (-w)^3 && (\because 1 + w = -w^2;\ 1 + w^2 = -w) \\ &= -w^6 + w^3 \\ &= -(w^3)^2 + w^3 \\ &= -(1)^2 + 1 \\ &= -1 + 1 \\ &= 0 \\ &= RHS \end{aligned}$$

$\therefore$ LHS = RHS.

(d) $(1 - w + w^2) = -8$

$$\begin{aligned}
\text{LHS} &= (1 - w + w^2)^3 \\
&= (1 + w^2 - w)^3 \\
&= (-w - w)^3 \\
&= (-2w)^3 \\
&= -8w^3 \qquad\qquad (\because\ w^3 = 1) \\
&= -8.
\end{aligned}$$

(ii) $\text{LHS} = (1 + w - w^2)^3$

$$\begin{aligned}
&= (-w^2 - w^2)^3 \\
&= (-2w^2)^3 \\
&= -8w^6 \\
&= -8(w^3)^2 \\
&= -8. \\
&= \text{RHS}
\end{aligned}$$

$\therefore$ LHS = RHS.

Example 2.7 : Show that one of the values of $\left[\dfrac{1+\sin\pi/8+i\cos\pi/8}{1+\sin\pi/8-i\cos\pi/8}\right]^{8/3}$ is equal to -1.

Sol : We have $\sin\left(\dfrac{\pi}{8}\right) = \cos\left(\dfrac{\pi}{2}-\dfrac{\pi}{8}\right) = \cos\dfrac{3\pi}{8}$

and $\cos\left(\dfrac{\pi}{8}\right) = \sin\left(\dfrac{\pi}{2}-\dfrac{\pi}{8}\right) = \sin\dfrac{3\pi}{8}$

$$\therefore\ 1+\sin\frac{\pi}{8}+\cos\frac{\pi}{8} = 1+\cos\left(\frac{3\pi}{8}\right)+i\sin\frac{3\pi}{8}$$

$$= 2\cos^2\left(\frac{3\pi}{16}\right)+2i\sin\frac{3\pi}{16}\,.\,\cos\frac{3\pi}{16}$$

$$= 2\cos\frac{3\pi}{16}\left[\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right]$$

Similarly, $1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8} = 2\cos\frac{3\pi}{16}\left[\cos\frac{3\pi}{16}-i\sin\frac{3\pi}{16}\right]$

$$\therefore \quad \text{LHS} = \left[\frac{2\cos\frac{3\pi}{16}\left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)}{2\cos\frac{3\pi}{16}\left(\cos\frac{3\pi}{16}-i\sin\frac{3\pi}{16}\right)}\right]^{8/3}$$

$$= \left[\left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)\left(\cos\frac{3\pi}{16}-i\sin\frac{3\pi}{16}\right)^{-1}\right]^{8/3}$$

$$= \left[\left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)\left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)\right]^{8/3}$$

$$= \left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)^{\frac{16}{3}}$$

One of the values of this is $\cos\pi + i\sin\pi = -1$ = RHS.

Example 2.8 : If $\left(\sqrt{3}+i\right)^{100} = 2^{99}(a+ib)$, then show that $(a, b) = \left(-1, \sqrt{3}\right)$

Solution : We have $\sqrt{3}+i = 2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = 2(\cos 30^0 + i\sin 30^0)$

then $\left(\sqrt{3}+i\right)^{100} = 2^{100}(\cos 30^0 + i\sin 30^0)^{100}$

$$= 2^{100}(\cos 3000^0 + i\sin 3000)$$

$$= 2^{100}[\cos 120^0 + i\sin 120^0]$$

$$= 2^{100}\left[-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]$$

Given $\left(\sqrt{3}+i\right)^{100} = 2^{99}(a+ib) \Rightarrow 2^{100}\left[-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right] = 2^{99}(a+ib)$

$\Rightarrow 2^{99}\left(-1+i\sqrt{3}\right) = 2^{99}(a+ib) \Rightarrow a=-1, b=\sqrt{3}$

$\therefore \quad (a, b) = \left(-1, \sqrt{3}\right)$.

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EXERCISE 2.6

1. Simplify $\dfrac{(\cos 5\theta + i\sin 5\theta)^3(\cos 2\theta + i\sin 2\theta)^5}{(\cos 2\theta - i\sin 2\theta)^8(\cos 3\theta + i\sin 3\theta)^9}$

2. Simplify $\dfrac{(\cos\alpha + i\sin\alpha)^{12}(\cos 2\beta - i\sin 2\beta)^{10}}{(\sin\alpha + i\cos\alpha)^5}$

3. Find all the values of

   (i) $(1-i\sqrt{3})^{1/3}$ (ii) $(-i)^{1/6}$ (iii) $(1+i)^{2/3}$

   (iv) $(-16)^{1/4}$ (v) $(-32)^{1/5}$

4. If A, B, C are angles of a triangle such that x = cis A, y = cis B, z = dis C, then find the value of xyz.

5. If 1, w, w^2 are the cube roots of unity, then prove that

   (i) $\dfrac{1}{2+w} + \dfrac{1}{1+2w} = \dfrac{1}{1+w}$

   (ii) $(2-w)(2-w^2)(2-w^{10})(2-w^{11}) = 49$

   (iii) $(x+y+z)(x+yw+zw^2)(x+yw^2+zw) = x^3+y^3+z^3-3xyz$.

II.

1. Solve the follwing equations.

   (i) $x^4 - 1 = 0$ (ii) $x^5 + 1 = 0$

   (iii) $x^9 - x^5 + x^4 - 1 = 0$

2. Find all common roots of $x^{12} - 1 = 0$ and $x^4 + x^2 + 1 = 0$.

3. If the cube roots of units are 1, w, w^2, then find the roots of the equation $(x-1)^3 + 8 = 0$.

4. Find the product of all the values of $(1+i)^{4/5}$.

5. If $x+\frac{1}{x}=2\cos\alpha$ and $y+\frac{1}{y}=2\cos\beta$, then prove that

$xy+\frac{1}{xy}=2\cos(\alpha+\beta)$

6. If α and β are the roots of $x^2 - 2x + 4 = 0$, then prove that

$\alpha^n + \beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$.

KEY WORDS

- $z = a + bi$ is a complex number in the standard form where $a, b \in R$ and $i = \sqrt{-1}$.
- Any higher powers of'i' can be expressed in terms of one of the four value, i, -1, $-i$, 1.
- Conjugate of a complex number $z = a + bi$ is $a - bi$ and is denoted by $\bar{z}$.
- Modulus of a complex number $z = a + bi$ is $\sqrt{a^2+b^2}$ i.e., $|z| = |a+bi|$ $= \sqrt{a^2+b^2}$

  (a) $|z| = 0 \Leftrightarrow z = 0$ (b) $|z| = |\bar{z}|$ (c) $|z_1 + z_2| \le |z_1| + |z_2|$
- $z = r(\cos\theta + i\sin\theta)$ represents the polar form of a complex number $z = a + bi$.

  $r = \sqrt{a^2+b^2}$ is modulus and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ is its argument.
- Multiplicative inverse of a complex number $z = a + bi$ is $\frac{\bar{z}}{|z|^2}$.

(i) It is customary to write cis θ for $\cos\theta + i\sin\theta$.

Thus we may state that Demoivre's theorem as $(\text{cis}\theta)^n = \text{cis}\, n\theta$, if $n \in Z$.

(ii) $(\cos\theta + i\sin\theta)^{-n} = \cos(-n)\theta + i\sin(-n)\theta = \cos n\theta - i\sin n\theta$ provided 'n' is an integer.

(iii) $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta - i^2\sin^2\theta$

$$= \cos^2\theta + \sin^2\theta = 1$$

$$\therefore\ \cos\theta + i\sin\theta = \frac{1}{\cos\theta - i\sin\theta},\quad \cos\theta - i\sin\theta = \frac{1}{\cos\theta + i\sin\theta}$$

(iv) $(\cos\theta - i\sin\theta)^n = \left[\frac{1}{\cos\theta + i\sin\theta}\right]^n$

$$= (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$$

provided 'n' is an integer.

(v) $cis\theta\ .\ cis\ \phi = cis(\theta + \phi)$ for any $\theta,\ \phi \in R$

(vi) n^{th} root of a complex number : Let n be a positive integer and $z_0 \neq 0$ be a given complex number. Any complex number z satisfying $z^n = z_0$ is called an nth root of z_0 and is denoted by $z_0^{1/n}$ or $\sqrt[n]{z_0}$.

SUPORTIVE WEBSITES

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

PRACTICE EXERCISE

1. Find real and imaginary parts of each of the following:

(a) $2 + 7i$ (b) $3 + 0i$ (c) $-\frac{1}{2}$

(d) $5i$ (e) $\frac{1}{2+3i}$

2. Simplify each of the following:

(a) $\sqrt{-3}.\sqrt{-27}$ (b) $\sqrt{-3}.\sqrt{-4}\ .\ \sqrt{-72}$

(c) $3i^{15} - 5i^8 + 1$

3. Form complex number whose real and imaginary parts are given in the form of ordered pairs.

   (a) $z(3, -5)$ (b) $z(0, -4)$ (c) $z(8, \pi)$

4. Find the conjugate of each of the following:

   (a) $1 - 2i$ (b) $-1 - 2i$ (c) $6-\sqrt{2}i$

   (d) $4i$ (e) $-4i$

5. Find the modulus of each of the following:

   (a) $1 - i$ (b) $3 + \pi i$ (c) $-\frac{3}{2}i$

   (d) $-2+\sqrt{3}i$

6. Express $7i^{17} - 6i^6 + 3i^3 - 2i^2 + 1$ in the form of $a + bi$.

7. Find the values of x and y if:

   (a) $(x - yi) + 7 - 2i = 9 - i$

   (b) $2x + 3yi = 4 - 9i$

   (c) $x - 3yi = 7 + 9i$

8. Simplify each of the following:

   (a) $(3 + i) - (1 - i) + (-1 + i)$

   (b) $\left(\frac{1}{7}+i\right)-\left(\frac{2}{7}-i\right)+\left(\frac{3}{7}-2i\right)$

9. Write additive inverse and multiplicative inverse of each of the following:

   (a) $3 - 7i$ (b) $11 - 2i$ (c) $\sqrt{3}+2i$

   (d) $1-\sqrt{2}i$ (e) $\frac{1+5i}{1-i}$

10. Find the modulus of each of the following complex numbers:

    (a) $\frac{1+i}{3-i}$ (b) $\frac{5+2i}{\sqrt{2}+\sqrt{3}i}$ (c) $(3 + 2i)(1 - i)$

    (d) $(1 - 3i)(-2i^3 + i^2 + 3)$

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11. For the following pairs of complex numbers verify that $|z_1 z_2| = |z_1|\,|z_2|$

    (a) $z_1 = 3 - 2i, z_2 = 1 - 5i$

    (b) $z_1 = 3 - \sqrt{7}i,\ z_2 = \sqrt{3} - i$

12. For the following pairs of complex numbers verify that $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.

    (a) $z_1 = 1 + 3i, z_2 = 2 + 5i$

    (b) $z_1 = -2 + 5i,\ z_2 = 3 - 4i$

1. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$ then show that

   (i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

   (ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

   (iii) $\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0.$

2. If $1, w, w^2$ are the cube roots of unity, prove that

   (i) $(1 - w + w^2)^6 + (1 - w^2 + w)^6 = 128$

   $= (1 - w + w^2)^7 + (1 + w - w^2)^7$

   (ii) $(a + b)(aw + bw^2)(aw^2 + bw) = a^3 + b^3$

   (iii) $x^2 + 4x + 7 = 0$ where $x = w - w^2 - 2.$

3. If α, β are the roots of the equation $x^2 + x + 1 = 0$ then prove that $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = 0.$

4. If $z^2 + z + 1 = 0$, where z is a complex number, prove that

$$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+\left(z^4+\frac{1}{z^4}\right)^2+\left(z^5+\frac{1}{z^5}\right)^2+\left(z^6+\frac{1}{z^6}\right)^2=12$$

5. If $z = \cos\theta + i\sin\theta$, then show that $\dfrac{z^{2n}-1}{z^{2n}+1} = i\tan\theta$, n being an integer.

6. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$ and $a + b + c = abc$, then prove that
$\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) + 1 = 0$

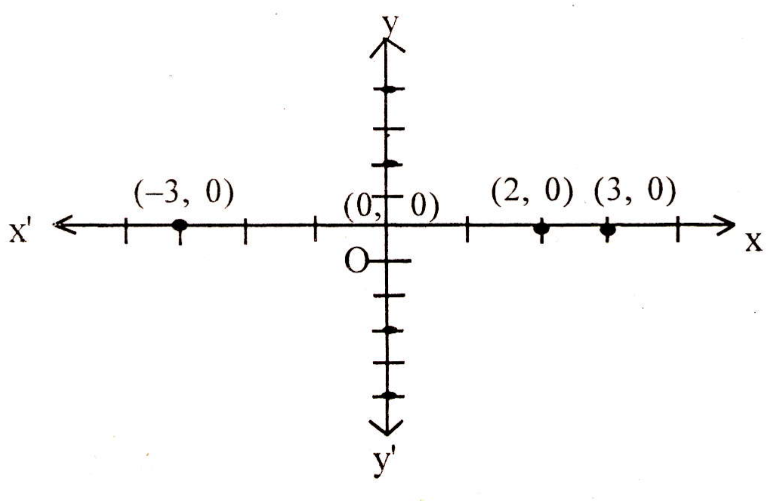
ANSWERS

EXERCISE 2.1

1. (a) $3\sqrt{3}i$ (b) $-3i$ (c) $\sqrt{13}i$
2. (a) $5+0i$ (b) $0-3i$ (c) $0+0i$
3. $12-4i$

EXERCISE 2.2

1. (a) $2i$ (b) $-5+3i$ (c) $-\sqrt{2}$ (d) $3+4i$
2. (a)



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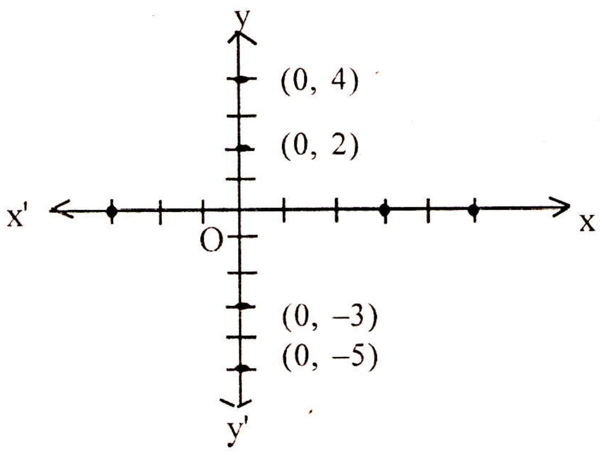
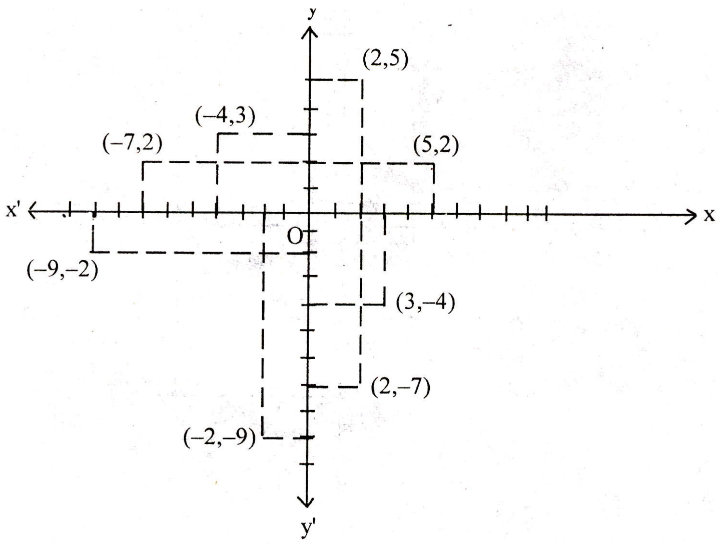
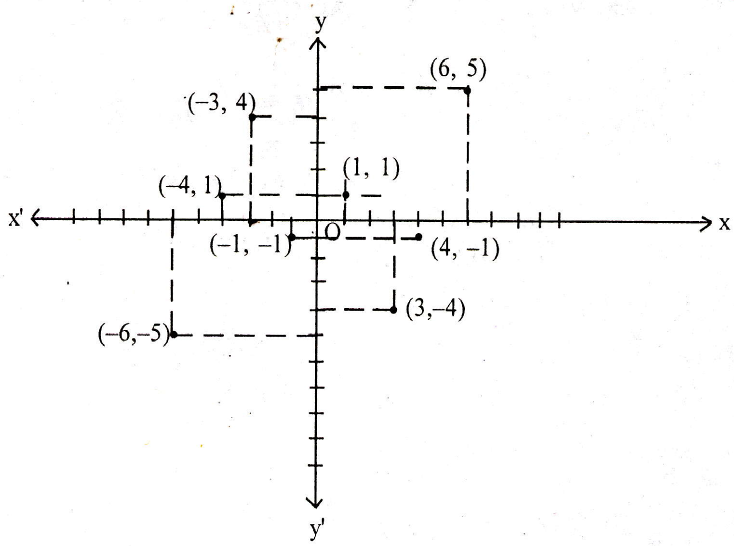
(b)



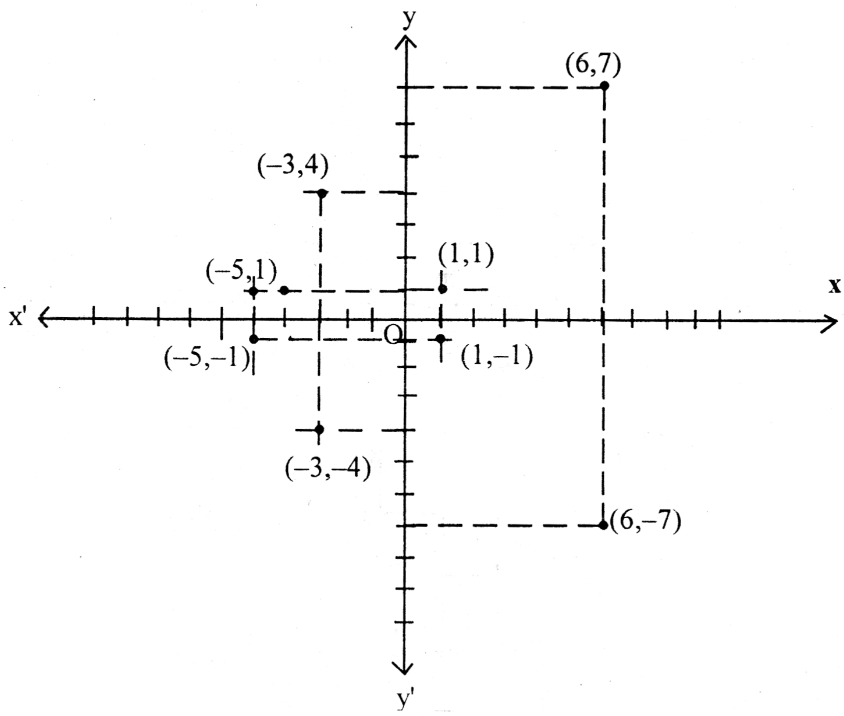
(c)



(d)



(e)



3. (a) (i) 3 (ii) $\sqrt{10}$ (iii) $\sqrt{13}$ (iv) $\sqrt{21}$

EXERCISE 2.3

1. (a) $\left(\sqrt{2}+\sqrt{5}\right)+\left(\sqrt{5}-\sqrt{2}\right)i$ (b) $\frac{1}{6}(6+i)$

   (c) $7i$ (d) $\sqrt{2}(\sqrt{2}+1)+(7-\sqrt{3})$

2. (a) $11 + 3i$ (b) $11 + 3i$ (c) Yes

   (d) $-1 - i$ (e) $1 + i$ (f) No

3. (a) $4 + 3i$ (b) $4 + 3i$ (c) Yes

   (d) $2 + 5i$ (e) $-2 - i$ (f) No

4. (a) $-12 + 7i$ (b) $-4 + 3i$

5. $18 - 6i$

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EXERCISE 2.4

1. (a) $\left(\sqrt{2}+2\right)+\left(2\sqrt{2}-1\right)i$ (b) $1+2\sqrt{2}i$

   (c) $-2 + 6i$ (d) $\frac{1}{5}(-4+7i)$

   (e) $\frac{1}{2}(3+i)$ (f) $\frac{1}{58}(3-7i)$

2. (a) $\frac{1}{25}(3+4i)$ (d) $\frac{1}{52}(\sqrt{3}-7i)$

   (c) $\frac{1}{34}(-9-19i)$

EXERCISE 2.5

I. 1. (i) -8 (ii) 16 (iii) 256 (iv) i

EXERCISE 2.6

I. 1. $\cos(14\theta) + i \sin(14\theta)$

2. $\sin(17\alpha - 20\beta) - i\cos(17\alpha - 20\beta)$

3. (i) $2^{1/3} cis(6k-1)\frac{\pi}{9},\ k=0,1,2$

   (ii) $cis(4k-1)\frac{\pi}{12},\ k=0,1,2,3,4,5$

   (iii) $2^{1/3} cis(4k+1)\frac{\pi}{6},\ k=0,1,2$

   (iv) $2\ cis(2k+1)\frac{\pi}{4},\ k=0,1,2,3$

   (v) $2\ cis(2k+1)\frac{\pi}{5},\ k=0,1,2,3,4$

4. -1

II.

1. (i) $\pm 1, \pm i$

   (ii) $cis\left(\frac{2k+1}{5}\right)\pi, \ k = 0,1,2,3,4$

   (iii) $\pm 1, \ \pm i, \ cis\left(\pm\frac{\pi}{5}\right), \ cis\left(\pm\frac{3\pi}{5}\right)$

2. $cis\frac{\pi}{3}, \ cis\ \frac{2\pi}{3}, \ cis\frac{4\pi}{3}, \ cis\frac{5\pi}{3}$

3. $-1, \ 1 - 2w, \ 1 - 2w^2$

4. -4

PRACTICE EXERCISE

1. (a) 2, 7 (b) 3, 0

   (c) $-\frac{1}{2}, 0$ (d) 0, 5

   (e) $\frac{2}{13}, -\frac{3}{13}$

2. (a) -9 (b) $-12\sqrt{6}i$

   (c) $-4 - 3i$

3. (a) $3 - 5i$ (b) $0 - 4i$

   (c) $8 + \pi i$

4. (a) $1 + 2i$ (b) $-1 + 2i$

   (c) $6 + \sqrt{2}i$ (d) $-4i$

   (e) $4i$

5. (a) $\sqrt{2}$ (b) $\sqrt{9+\pi^2}$

(c) $\frac{3}{2}$ (d) $\sqrt{7}$

6. $9 + 4i$

7. (a) $x = 2, \ y = -1$ (b) $x = 2, \ y = -3$

(c) $x = 7, \ y = -3$

8. (a) $1 + 3i$ (b) $\frac{2}{7}+0i$

9. (a) $-3+7i, \frac{1}{58}(3+7i)$

(b) $-11+2i, \frac{1}{125}(-11+2i)$

(c) $-\sqrt{3}-2i, \frac{1}{7}(\sqrt{3}-2i)$

(d) $-1+\sqrt{2}i, \frac{1}{3}(1+\sqrt{2}i)$

(e) $2-3i, \frac{1}{13}(2+3i)$

10. (a) $\frac{1}{\sqrt{5}}$ (b) $\frac{1}{5}\sqrt{145}$

(c) $\sqrt{26}$ (d) $4\sqrt{5}$

QUADRATIC EQUATIONS

Chapter 3

LEARNING OUTCOMES

After studying this lesson, you will be able to:

- solve a quadratic equation with real coefficients by factorization and by using quadratic formula;
- find relationship between roots and coefficients;
- form a quadratic equation when roots are given;

PREREQUISITES

- Real numbers
- Quadratic Equations with real coefficients.

INTRODUCTION

Any equation that can be expressed in the form $ax^2 + bx + c = 0$, where $a, b, c \in C$ with $a \neq 0$ is called a quadratic equation in x. Here a, b, c are called coefficients.

The form $ax^2 + bx + c = 0$ the standard form of quadratic equation in x.

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e.g. $5x^2 + 8x = 2x + 4$ which can be expressed as $5x^2 - 10x - 4 = 0$ is a quadratic equation.

In this lesson we will discuss how to solve quadratic equations with real and complex coefficients and establish relation between roots and coefficients. We will also find the sign of quadratic expressions, change in signs and Maximum and Minimum values.

3.1 ROOTS OF A QUADRATIC EQUATION

The value which when substituted for the variable in an equation, satisfies it, is called a root (or solution) of the equation.

If α be one of the roots of the quadratic equation

then, $$ax^2 + bx + c = 0, \; a \neq 0 \qquad ...(i)$$

$$a\alpha^2 + b\alpha + c = 0$$

In otherwords $x - \alpha$ is a factor of the quadratic equation (i)

In particular, consider a quadratic equation $x^2 + x - 6 = 0$ ... (ii)

If we substitute $x = 2$ in (ii) , we get

$$\text{L.H.S.} = 2^2 + 2 - 6 = 0$$

$\therefore$ L.H.S. = R.H.S

Again put $x = -3$ in (ii) we get

$$\text{L.H.S.} = (-3)^2 - 3 - 6 = 0$$

$\therefore$ L.H.S. = R.H.S

Again put $x = -1$in (ii) we get

$$\text{L.H.S.} = (-1)^2 + (-1) - 6 = -6 \neq 0 = \text{R.H.S}$$

$\therefore$ $x = 2$ and $x = -3$ are the only values of$x$ which satisfy the quadratic equation (ii)

There are no other values which satisfy (ii)

$\therefore$ $x = 2$ and $x = -3$ are the only two roots of the quadratic equation (ii)

Note: If α, β be two roots of the quadratic equation

$$ax^2 + bx + c = 0; \quad a \neq 0 \qquad ...(A)$$

then $(x - \alpha)$ and $(x - \beta)$ will be thefactors of (A). The given quadratic equation can be written in terms of these factors as

$$(x - \alpha) \; (x - \beta) = 0$$

3.2 SOLVING QUADRATIC EQUATION BY FACTORIZATION

Recall that you have learnt how to factorize quadratic polynomial of the form $P(x) = ax^2 + bx + c$. $a \neq 0$ by splitting the middle term and taking the common factors. Same method can be applied while solving quadratic equation by factorization.

If $x - \frac{p}{q}$ and $x - \frac{r}{s}$ are two factors of the quadratic equation

$ax^2 + bx + c = 0,\ a \neq 0$ then $\left(x - \frac{p}{q}\right)\left(x - \frac{r}{s}\right) = 0$.

$\therefore$ either $x = \frac{p}{q}$ or, $x = \frac{r}{s}$

$\therefore$ The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\frac{p}{q}, \frac{r}{s}$.

Example 3.1 : Using factorization method, solve the following quadratic equation:

$$6x^2 + 5x - 6 = 0.$$

Solution: The given quadratic equation is

$$6x^2 + 5x - 6 = 0 \qquad \text{.... (i)}$$

Splitting the middle term, we have

$$6x^2 + 9x - 4x - 6 = 0$$

or $\quad 3x(2x + 3) - 2(2x + 3) = 0$

or $\quad (2x + 3)(3x - 2) = 0$

$\therefore$ Either $2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$

or $3x - 2 = 0 \qquad \Rightarrow x = \frac{2}{3}$

$\therefore$ Two roots of the given quadratic equation are $\frac{-3}{2}, \frac{2}{3}$.

Example 3.2 : Using factorization method, solve the following quadratic equation:

$$3\sqrt{2}\,x^2 + 7x - 3\sqrt{2} = 0.$$

Solution: Splitting the middle term, we have

$$3\sqrt{2}\,x^2 + 9x - 2x - 3\sqrt{2} = 0$$

or $$3x(\sqrt{2}\,x + 3) - \sqrt{2}(\sqrt{2}x + 3) = 0$$

or $$(\sqrt{2}\,x + 3)\,(3x - \sqrt{2}) = 0$$

$\therefore$ Either $\sqrt{2}\,x + 3 = 0 \Rightarrow x = \dfrac{-3}{\sqrt{2}}$

or $3x - \sqrt{2} = 0 \Rightarrow x = \dfrac{\sqrt{2}}{3}$

$\therefore$ Two rootsof the given quaratic equation are $\dfrac{-3}{\sqrt{2}}, \dfrac{\sqrt{2}}{3}$.

Example 3.3: Using factorization method, solve the following quadratic equation:

$$(a + b)^2\ x^2 + 6(a^2 - b^2)\,x + 9(a - b)^2 = 0.$$

Solution: The given quadratic equation is

$$(a + b)^2 x^2 + 6(a^2 - b^2)\,x + 9(a - b)^2 = 0$$

Splitting the middle term, we have

$$(a + b)^2\ x^2 + 3(a^2 - b^2)\,x + 3(a^2 - b^2)\,x + 9(a - b)^2 = 0.$$

or, $(a + b)\,x\ \{(a + b)\,x + 3(a - b)\} + 3(a - b)\{(a + b)\,x + 3(a - b)\} = 0$

or, $\{(a + b)\,x + 3(a - b)\}\ .\ \{(a + b)\,x + 3(a - b)\} = 0$

$\therefore$ either $(a + b)\,x + 3(a - b) = 0 \Rightarrow x = \dfrac{-3(a-b)}{a+b} = \dfrac{3(b-a)}{a+b}$

or, $(a + b)\,x + 3(a - b) = 0 \Rightarrow x = \dfrac{-3(a-b)}{a+b} = \dfrac{3(b-a)}{a+b}$

$\therefore$ The equal roots of the given quadratic equation are

$$\frac{3(b-a)}{a+b}, \frac{3(b-a)}{a+b}$$

Alternative Method

The given quadratic equation is

$(a+b)^2 \; x^2 + 6(a^2 - b^2)\, x + 9(a-b)^2 = 0$

This can be rewritten as

$$\{(a+b)\,x\}^2 + 2(a+b)\,x \,.\, 3(a-b) + \{3(a-b)\}^2 = 0$$

or, $\{(a+b)\,x + 3(a-b)\}^2 = 0$

or, $x = \dfrac{-3(a-b)}{a+b} = \dfrac{3(b-a)}{a+b}$

$\therefore$ The quadratic equation have equal roots $\dfrac{3(b-a)}{a+b}, \dfrac{3(b-a)}{a+b}$

EXERCISE 3.1

1. Solve each of the following quadratic equations by factorization method:

(i) $\sqrt{3}x^2 + 10x + 8\sqrt{3} = 0$ (ii) $x^2 - 2ax + a^2 - b = 0$

(iii) $x^2 + \left(\dfrac{ab}{c} - \dfrac{c}{ab}\right)x - 1 = 0$ (iv) $x^2 - 4\sqrt{2}x + 6 = 0$

3.3 SOLVING QUADRATIC EQUATION BY QUADRATIC FORMULA

Recall the solution of a standard quadratic equation

$ax^2 + bx + c = 0,\ a \neq 0$ by the "Method of Completing Squares"

Roots of the above quadratic equation are given by

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{\mathrm{D}}}{2a} \qquad\qquad = \frac{-b-\sqrt{\mathrm{D}}}{2a}$$

where $\mathrm{D} = b^2 - 4\,ac$ is called the discriminant of the quadratic equation.

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For a quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$ if

(i) D > 0 the equation will have two real and unequal roots

(ii) D = 0 the equation will have two real and equal roots and both roots are equal to $\dfrac{-b}{2a}$.

(iii) D < 0 the equation will have two conjugate complex (imaginary) roots.

Example 3.4 : Examine the nature of roots in each of the following quadratic equations and also verify them by formula.

(i) $x^2 + 9x + 10 = 0$ (ii) $9y^2 - 6\sqrt{2}y + 2 = 0$

(iii) $\sqrt{2}t^2 - 3t + 3\sqrt{2} = 0$

Solution: (i) The given quadratic equation is $x^2 + 9x + 10 = 0$

Here, $a = 1,\ b = 9$ and $c = 10$

$\therefore \quad D = b^2 - 4ac = 81 - 4.1.10$

$= 41 > 0.$

$\therefore$ The equation will have two real and unequal roots.

Verification: By quadratic formula, we have

$$x = \frac{-9 \pm \sqrt{41}}{2}$$

$\therefore$ The two roots are $\dfrac{-9+\sqrt{41}}{2},\ \dfrac{-9-\sqrt{41}}{2}$ which are real and unequal.

(ii) The given quadratic equation is $9y^2 - 6\sqrt{2}y + 2 = 0$

Here $D = b^2 - 4ac$

$\therefore \quad = (-6\sqrt{2})^2 - 4(9)(2)$

$= 72 - 72 = 0.$

$\therefore$ The equation will have two real and equal roots.

Verification: By quadratic formula, we have

$$y = \frac{6\sqrt{2} \pm \sqrt{0}}{(2)(9)} = \frac{\sqrt{2}}{3}$$

$\therefore$ The two equal roots are $\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}$.

(iii) The given quadratic equation is $\sqrt{2}t^2 - 3t + 3\sqrt{2} = 0$

Here, $D = (-3)^2 - 4(\sqrt{2}) \cdot 3\sqrt{2}$

$= -15 < 0$

$\therefore$ The equation will have two conjugate complex roots.

Verification: By quadratic formula, we have

$$t = \frac{3 \pm \sqrt{-15}}{2\sqrt{2}}$$

$$= \frac{3 \pm \sqrt{15}i}{2\sqrt{2}}, \quad \text{where } i = \sqrt{-1}$$

$\therefore$ Two conjugate complex roots are $\frac{3+\sqrt{15}i}{2\sqrt{2}}, \frac{3-\sqrt{15}i}{2\sqrt{2}}$.

Example 3.5: Prove that the quadratic equation $x^2 + py - 1 = 0$ has real and distinct roots for all real values of p.

Solution: Here, $D = P^2 + 4$ which is always positive for all real values of p.

$\therefore$ The quadratic equation will have real and distinct roots for all real values of p.

Example 3.6: For what value ofk the quadratice equation

$(4k + 1)x^2 + (k + 1)x + 1 = 0$ will have equal roots?

Solution: The given quadratic equation is

$(4k + 1)x^2 + (k + 1)x + 1 = 0$

Here, $D = (k + 1)^2 - 4(4k + 1).1$

For equal roots, $D = 0$

$\therefore \quad (k + 1)^2 - 4(4k + 1) = 0$

$\Rightarrow \quad k^2 - 14k - 3 = 0$

$$\therefore k = \frac{14 \pm \sqrt{196 + 12}}{2}$$

or $k = \frac{14+\sqrt{208}}{2}$

$= 7\pm 2\sqrt{13}$ or $7+2\sqrt{13},\ 7-2\sqrt{13}$

which are the required values of k.

Example 3.7 : Prove that the roots of the equation

$x^2 (a^2 + b^2) + 2x(ac + bd) + (c^2 + d^2) = 0$ are imaginary. But if $ad = bc$, roots are real and equal.

Solution: The given equation is $x^2 (a^2 + b^2) + 2x(ac + bd) + (c^2 + d^2) = 0$

$$\begin{aligned}\text{Discriminant} &= 4(ac + bd)^2 - 4(a^2 + b^2)\ (c^2 + d^2)\\ &= 8abcd - 4(a^2 d^2 + b^2c^2)\\ &= -4(-2abcd + a^2d^2 + b^2c^2)\\ &= -4(ad - bc)^2\\ &< 0 \text{ for all } a, b, c, d.\end{aligned}$$

$\therefore$ The roots of the given equation are imaginary.

For real and equal roots, discriminant is equal to zero.

$\Rightarrow \quad -4(ad - bc)^2 = 0$

or $\quad ad = bc$

Hence, if ad=bc, the roots are real and equal.

EXERCISE 3.2

1. Solve each of the following quadratic equation by quadratic formula:

   (i) $2x^2 - 3x + 3 = 0$ (ii) $-x^2+\sqrt{2}x-1=0$

   (iii) $-4x^2+\sqrt{5}x-3=0$ (iv) $3x^2+\sqrt{2}x+5=0$

2. For what values of k will the equation

   $y^2 - 2(1 + 2k)\ y + 3 + 2k = 0$ have equal roots?

3. Show that the roots of the equation
$(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a)$ are always real and they can not be equal unless $a = b = c$.

3.4 RELATION BETWEEN ROOTS AND COEFFICIENTS OF A QUADRATIC EQUATION

You have learnt that, the roots of a quadratic equation

$$ax^2 + bx + c = 0,\ a \neq 0$$

are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

Let $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ .... (i) and $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ ... (ii)

Adding (i) and (ii), we have

$$\alpha+\beta = \frac{-2b}{2a} = \frac{-b}{a}$$

$$\therefore \quad \text{Sum of the roots} = \frac{\text{coefficient of } x}{\text{coefficient of } x^2} = -\frac{b}{a} \quad \text{... (iii)}$$

$$\alpha\cdot\beta = \frac{+b^2-(b^2-4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$= \frac{c}{a}$$

$$\therefore \quad \text{Product of the roots} = \frac{\text{constant term}}{\text{coefficient of } x^2} = \frac{c}{a} \quad \text{... (iv)}$$

(iii) and (iv) are the required relationships between roots and coefficients of a given quadratic equation. These relationships helps to find out a quadratic equation when two roots are given.

Example 3.8: If α, β are the roots of the equation $3x^2 - 5x + 9 = 0$ find the value of:

$(a)\ \alpha^2 + \beta^2 \qquad (b)\ \dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$

Solution:

(a) It is given that α, β are the roots of the quadratic equation

$$3x^2 - 5x + 9 = 0$$

$$\therefore\ \alpha + \beta = \frac{5}{3} \qquad \text{..... (i)}$$

$$\text{and}\quad \alpha \cdot \beta = \frac{9}{3} = 3 \qquad \text{..... (ii)}$$

$$\text{Now}\quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha \cdot \beta$$

$$= \left(\frac{5}{3}\right)^2 - 2.3 \qquad \text{[By (i) and (ii)]}$$

$$= -\frac{29}{9}$$

$\therefore$ The required value is $-\dfrac{29}{9}$

(b) Now, $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} = \dfrac{\alpha^2 + \beta^2}{\alpha^2 \beta^2}$

$$= \frac{-\frac{29}{9}}{9} \qquad \text{[By (i) and (ii)]}$$

$$= -\frac{29}{81}$$

Example 3.9: If α, β are the roots of the equation $3y^2 + 4y + 1 = 0$ form a quadratic equation whose roots are α^2, β^2.

Solution: It is given that α, β are two roots of the quadratic equation

$$3y^2 + 4y + 1 = 0.$$

$\therefore$ Sum of the roots

i.e., $\alpha + \beta = -\dfrac{\text{coefficient of } y}{\text{coefficient of } y^2}$

$$= -\frac{4}{3} \qquad \text{..... (i)}$$

Product of the roots i.e., $\alpha, \beta = \dfrac{\text{constant term}}{\text{coefficient of } y^2}$

$$= \frac{1}{3} \qquad \text{..... (ii)}$$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha \cdot \beta$

$$= \left(-\frac{4}{3}\right)^2 - 2\left(\frac{1}{3}\right) \qquad \text{[By (i) and (ii)]}$$

$$= \frac{16}{9} - \frac{2}{3}$$

$$= \frac{10}{9}$$

and $\alpha^2 \cdot \beta^2 = \dfrac{1}{9}$. [By (i)]

$\therefore$ The required quadratic equation is $y^2 - (\alpha^2 + \beta^2)y + \alpha^2 \cdot \beta^2 = 0$

or $\quad y^2 - \dfrac{10}{9}y + \dfrac{1}{9} = 0$

or $\quad 9y^2 - 10y + 1 = 0.$

Example 3.10: If one root of the equation $ax^2 + bx + c = 0$, $a \neq 0$ be the square of the other, prove that $b^3 + ac^2 + a^2c = 3abc$.

Solution: Let α, α^2 be two roots of the equation $ax^2 + bx + c = 0$

$$\therefore \quad \alpha + \alpha^2 = -\frac{b}{a} \qquad \text{... (i)}$$

and $\quad \alpha \cdot \alpha^2 = \dfrac{c}{a}$

i.e., $\quad \alpha^3 = \dfrac{c}{a}$ ... (ii)

From (i) we have

$$\alpha(\alpha+1) = -\frac{b}{a}$$

or, $$\{\alpha(\alpha+1)\}^3 = \left(-\frac{b}{a}\right)^3 = -\frac{b^3}{a^3}$$

or, $$\alpha^3(\alpha^3+3\alpha^2+3\alpha+1) = -\frac{b^3}{a^3}$$

or, $$\frac{c}{a}\left\{\frac{c}{a}+3\left(-\frac{b}{a}\right)+1\right\} = -\frac{b^3}{a^3} \qquad \text{...[By (i) and (ii)]}$$

or, $$\frac{c^2}{a^2}-\frac{3bc}{a^2}+\frac{c}{a} = -\frac{b^3}{a^3}$$

or, $$ac^2 - 3abc + a^2c = -b^3$$

or, $$b^3 + ac^2 + a^2c = 3abc.$$

which is the required result.

Example 3.11 : Find the condition that the roots of the equation

$ax^2 + bx + c = 0$ are in the ratio $m : n$

Solution: Let $m\alpha$ and $n\alpha$ be the roots of the equation

$ax^2 + bx + c = 0$

Now, $$m\alpha + n\alpha = -\frac{b}{a} \qquad \text{... (i)}$$

and $$m \cdot n \cdot \alpha^2 = \frac{c}{a} \qquad \text{... (ii)}$$

From (i) we have, $$\alpha(m+n) = -\frac{b}{a}$$

or, $$\alpha^2(m+n)^2 = \frac{b^2}{a^2}$$

or, $$\frac{c}{a}(m+n)^2 = mn\frac{b^2}{a^2} \qquad \text{[By (ii)]}$$

or, $$ac(m+n)^2 = mnb^2$$

which is the required condition.

EXERCISE 3.3

1. If α, β are the roots of the equation $ax^2 + bx + c = 0$ then find the value of:

   (i) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ (ii) $\dfrac{1}{\alpha^4}+\dfrac{1}{\beta^4}$

2. If α, β are the roots of the equation $5x^2 - 6x + 3 = 0$, form a quadratic equation whose roots are:

   (i) α^2, β^2 (ii) $\alpha^3\beta, \alpha\beta^3$

3. If the roots of the equation $ax^2 + bx + c = 0$ be in the ratio 3:4, prove that $12b^2 = 49ac$.

4. Find the condition that one root of the quadratic equation $px^2 - qx + p = 0$ may be 1 more than the other.

3.5 SOLUTION OF A QUADRATIC EQUATION WHEN D < 0

Let us consider the following quadratic equation :

(a) Solve for t:

$$t^2 + 3t + 4 = 0$$

$$\therefore t = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{-7}}{2}$$

Here, $D = -7 < 0$

$\therefore$ The roots are $\dfrac{-3+\sqrt{-7}}{2}$ and $\dfrac{-3-\sqrt{-7}}{2}$

or $\dfrac{-3+\sqrt{7}i}{2}, \dfrac{-3-\sqrt{7}i}{2}$

Thus, the roots are complex and conjugate .

(b) Solve for 'y'

$$-3y^2 + \sqrt{5}y - 2 = 0$$

$$\therefore\ y = \frac{-\sqrt{5} \pm \sqrt{5-4(-3)(-2)}}{2(-3)}$$

or $$y = \frac{-\sqrt{5} \pm \sqrt{-19}}{-6}$$

Here, $D = -19 < 0$

$\therefore$ The roots are $\dfrac{-\sqrt{5}+\sqrt{19}i}{-6}, \dfrac{-\sqrt{5}-\sqrt{19}i}{-6}$

Here, also roots are complex and conjugate. From the above examples, we can make the following conclusions:

(i) $D < 0$ in both the cases.

(ii) Roots are complex and conjugate to each other.

Is it always true that complex roots occur in conjugate pairs?

Let us form a quadratic equation whose roots are $2 + 3i$ and $4 - 5i$

The equation will be $\{(x - (2 + 3i)\}\{x - (4 - 5i)\} = 0$

or, $x^2 - (2 + 3i)\,x - (4 - 5i)x + (2 + 3i)\,(4 - 5i) = 0$

or, $x^2 - (-6 + 2i)x + 23 + 2i = 0$

which is an equation with complex coefficients.

Note: If the quadratic equation has two complex roots, which are not conjugate of each other, the quadratic equation is an equation with complex coefficients.

3.6 SIGN OF QUADRATIC EXPRESSIONS, CHANGE IN SIGNS

In this chapter, we discuss basic concepts of quadratic expressions, extreme values, changes in sign and magnitude.

The quadratic equation $ax^2 + bx + c = 0$ depemds on the coefficient 'a' of x^2 and the nature of the roots. Here a, b are non zero real numbers, we say that a and b have the same sign, if both a and b are positive or both of them are negative.

1. Theorem : The roots of $ax^2 + bx + c = 0$ (where $a, b, c \in R$ real coefficients, $a \neq 0$) are non real complex numbers if and only if $ax^2 + bx + c$ and a have the same sign for all $x \in R$.

Proof : The condition for the equation $ax^2 + bx + c = 0$ to have non-real complex roots is $b^2 - 4ac < 0$.

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

Now $$\frac{ax^2 + bx + c}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$$

Now the right hand expression becomes $\geq \frac{4ac - b^2}{4a^2}$.

Hence if $4ac - b^2 > 0$ the $\frac{ax^2 + bx + c}{a} > 0$ Hence for all real values of x, the expression $ax^2 + bx + c$ and a have the same sign.

Theorem 2. If the equation $ax^2 + bx + c = 0$ (where $a, b, c \in R$ and $a \neq 0$) has equal roots, then '$ax^2 + bx + c$' and a have the same sign for all real x, except for $x = -\frac{b}{2a}$.

Proof : The condition for having equal roots is $b^2 - 4ac = 0$.

$$\frac{ax^2 + bx + c}{a} = \left(x + \frac{b}{2a}\right)^2 > 0$$

$$= 0 \text{ if } \quad x = \frac{-b}{2a}.$$

$\therefore$ Hence, for all real x, except for $x = \frac{-b}{2a}$ the expression $ax^2 + bx + c$ and a have the same sign.

Theorem 3. α, β ($\alpha < \beta$) are the real roots of $ax^2 + bx + c = 0$

(i) for $\alpha < x < \beta$, $ax^2 + bx + c$ and a have opposite signs

(ii) for $x < \alpha$ or $x > \beta$, $ax^2 + bx + c$ and a have the same sign.

Proof : $ax^2 + bx + c = a(x - \alpha)(x - \beta)$

$$\frac{ax^2+bx+c}{a} = (x - \alpha)(x - \beta) \quad ...(1)$$

(i) When $\alpha < x < \beta$, we have $x - \alpha > 0$ and $x - \beta < 0$ so that, by (1)

$$\frac{ax^2+bx+c}{a} < 0.$$

Hence $ax^2 + bx + c$ and a have opposite signs.

(ii) When $x < \alpha$, $x - \alpha < 0$ and $x - \beta < 0$ ($\because$ $\alpha < \beta$),

so that by (1) $\frac{ax^2+bx+c}{a} > 0$.

when $x < \beta$, $x - \beta > 0$ and $x - \alpha > 0$ ($\because \beta > \alpha$) so that

by (1) $\frac{ax^2+bx+c}{a} > 0$

Thus for $x < \alpha$ or $x > \beta$, $ax^2 + bx + c$ and a have the same sign.

Example 3.12: When x is a real number ($x \in R$) discuss the sign of the expression $x^2 - 5x + 4$.

Solution: Comparing with $ax^2 + bx + c$, we have $a = 1$, $b = -5$, $c = 4$.

Discriminant $\Delta = b^2 - 4ac = 25 - 16 = 9 > 0$

The roots are real and distinct.

$x^2 - 5x + 4 = (x - 1)(x - 4) = 0$

roots are $x = 1, 4$

Coefficient of x^2 is $a = 1 > 0$

$\therefore$ $1 < x < 4$ then $x^2 - 5x + 4$ and the coefficent of x^2 have opposite signs.

Also, if $x < 1$ or $x > 4$, then $x^2 - 5x + 4$ and the coefficient of x^2 have the same sign.

(i) Hence, for case $1 < x < 4$, $x^2 - 5x + 4 < 0$.

(ii) for $x < 1$ or $x > 4$, $x^2 - 5x + 4 > 0$ [i.e., $x \in (-\infty, 1) \cup (4, \infty)$]

(iii) when $x = 1$ or $x = 4$, $x^2 - 5x + 4 = 0$.

Example 3.13 : Discuss the sign of the expression $x^2 - x + 3$.

Solution : The discriminant $\Delta = b^2 - 4ac = (-1)^2 - 4(1)(3)$

$$= 1 - 12 = -11 < 0.$$

$\because$ $\Delta = 0$ and coefficient of x^2 is $a = 1 > 0$ then $\forall\ x \in \mathrm{R}$, $x^2 - x + 3 > 0$

The roots of the equation $x^2 - x + 3 = 0$ are $\dfrac{1 \pm \sqrt{11}i}{2}$. These roots are non real, Therefore $x^2 - x + 3$ and the coefficient of x^2 have the same sign.

Hence $x^2 - x + 3 > 0$ for all real x.

3.7 MAXIMUM AND MINIMUM VALUES

To find the methods of the extreme values of a quadratic expression.

It depends on the sign of the coefficient of x^2.

Theorem 4: $f(x) = ax^2 + bx + c$ is a quadratic expression (where $a, b, c \in \mathrm{R}$, $a \neq 0$)

$x \in \mathrm{R}$, $\dfrac{4ac - b^2}{4a}$ is a number.

(i) If $a > 0$, then $f(x)$ has absolute minimum at $x = \dfrac{-b}{2a}$ and the minimum value is $\dfrac{4ac - b^2}{4a}$

(ii) If $a < 0$ then $f(x)$ has absolute maximum at $x = \frac{-b}{2a}$ and the maximum value is $\frac{4ac - b^2}{4a}$

Proof :

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \quad ...(1)$$

(i) Let $a > 0$, then $f(x) \le \frac{4ac - b^2}{4a}$ $\forall$ $x \in \mathrm{R}$ and when $x = \frac{-b}{2a}$,we have

$f(x) = \frac{4ac - b^2}{4a}$ by(1).

$\therefore$ for $a > 0$, $f(x)$ has absolute minimum at $x = \frac{-b}{2a}$ and the minimum value is $\frac{4ac - b^2}{4a}$.

(ii) If $a < 0$, then $f(x) \le \frac{4ac - b^2}{4a}$ $\forall$ $x \in \mathrm{R}$ and when $x = \frac{-b}{2a}$, we have

$f(x) = \frac{4ac - b^2}{4a}$, by (1)

$\therefore$ for $a < 0$, $f(x)$ has a absolute maximum at $x = \frac{-b}{2a}$ and

the maximum value is $\frac{4ac - b^2}{4a}$.

Example 3.13 : Find the value of x at which the following expressions have maximum or minimum

(i) $x^2 - x + 7$ (ii) $12x - x^2 - 32$

(iii) $ax^2 + bx + a$ (iv) $2x - 7 - 5x^2$

Sol : (i) Comparing $x^2 - x + 7$ with $ax^2 + bx + c$, we have $a = 1$, $b = -1$, $c = 7$

since $a = 1 > 0$, $x^2 - x + 7$ has absolute minimum value at $\frac{-b}{2a} = \frac{1}{2}$

The minimum value is $\frac{4ac - b^2}{4a} = \frac{4(1)(7) - (-1)^2}{4(1)} = \frac{27}{4}$.

(ii) Comparing $12x - x^2 - 32$

we have $a = -1$, $b = 12$, $c = -32$

$\because$ $a = -1 < 0$, the expression has absolute maximum value at $\frac{-b}{2a} = \frac{-12}{2(-1)} = 6$.

The maximum value is

$$= \frac{4ac - b^2}{4a} = \frac{4(-1)(-32) - (12)^2}{4(-1)} = \frac{128 - 144}{-4} = \frac{-16}{-4} = 4.$$

(iii) (i) If $a > 0$, the expression $ax^2 + bx + a$ have the minimum value.

$\therefore$ The minimum value = $\frac{4(a)(a) - b^2}{4a} = \frac{4a^2 - b^2}{4a}$

(ii) If $a < 0$, the expression $ax^2 + bx + a$ have the maximum value.

$\therefore$ The maximum value « $\frac{4(a)(a) - b^2}{4a} = \frac{4a^2 - b^2}{4a}$

(iv) Comparing with $ax^2 + bx + c = 0$

we have $a = -5$, $b = 2$, $c = -7$

so $\frac{4ac - b^2}{4a} = \frac{4(-5)(-7) - (2)^2}{4(-5)} = \frac{140 - 4}{-20} = \frac{+136}{-20} = \frac{-34}{5}$.

and $\frac{-b}{2a} = \frac{-2}{-10} = \frac{1}{5}$.

Since $a < 0$, $2x - 7 - 5x^2$ has absolute maximum at $x = \frac{1}{5}$

$\therefore$ The maximum value is = $\frac{-34}{5}$

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3.8 CHANGES IN THE MAGNITUDE OF A QUADRATIC EXPRESSION

Now we observe the changes in (magnitude) the value of the quadratic expression $ax^2 + bx + c$ when the value of x varies in R.

$f(x) = ax^2 + bx + c$ can be written as

$$y = f(x) = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a}$$

If $a > 0$	**If $a < 0$**
1. If "x approaches $-\infty$", then $f(x)$, "approaches $+\infty$"	1. If x, "approaches $-\infty$" then $f(x)$, "approaches $-\infty$"
2. If "x approaches $+\infty$" then $f(x)$, " approaches $+\infty$"	2. If x, "approaches $+\infty$" then $f(x)$, "approaches $-\infty$".
3. If $x = \frac{-b}{2a}$ then $f(x) = \frac{4ac-b^2}{4a}$	3. If $x = \frac{-b}{2a}$, then $f(x) = \frac{4ac-b^2}{4a}$
4. When x increases from $-\infty$ to $\left(\frac{-b}{2a}\right)$ then $f(x)$ decreases from $+\infty$ to $\frac{4ac-b^2}{4a}$	4. when x increases from $-\infty$ to $\left(\frac{-b}{2a}\right)$, then $f(x)$, increases from $-\infty$ to $\frac{4ac-b^2}{4a}$
5. when x increases from $\frac{-b}{2a}$ to $+\infty$, then $f(x)$ increases from $\frac{4ac-b^2}{4a}$	5. when x increases from $\frac{-b}{2a}$ to $+\infty$, then $f(x)$, decreases from to $+\infty$. $\frac{4ac-b^2}{4a}$ to $-\infty$.

Example 3.14: Find the changes in the sign of $4x - 5x^2 + 2$ for $x \in R$ and find the extreme value.

Solution: Comparing the given expression with $ax^2 + bx + c$,

we have $a = -5 < 0$.

The roots of the equation $5x^2 - 4x - 2 = 0$ are $\dfrac{4 \pm \sqrt{16 - 4(5)(-2)}}{2(5)}$

$$= \frac{4 \pm \sqrt{16+40}}{10} = \frac{4 \pm 2\sqrt{14}}{10}$$

$$= \frac{2 \pm \sqrt{14}}{5}$$

$\therefore$ when $\dfrac{2-\sqrt{14}}{5} < x < \dfrac{2+\sqrt{14}}{5}$ the sign of

$4x - 5x^2 + 2$ is positive, and

when $x < \dfrac{2-\sqrt{14}}{5}$ or $x > \dfrac{2+\sqrt{14}}{5}$ the sign of $4x - 5x^2 + 2$ is negative.

Since $a < 0$, the maximum value of the given problem is

$$\frac{4ac - b^2}{4a} = \frac{4(-5)(2)-(4)^2}{4(-5)} = \frac{-56}{-20} = \frac{14}{5}$$

$\therefore$ The extreme value of the given expression is $= \dfrac{14}{5}$.

Example 3.15 : Find the changes in the sign of the expression $x^2 - 5x + 6$ and find their extreme values.

Solution: Comparing $x^2 - 5x + 6$ with $ax^2 + bx + c$, we have $a = 1 > 0$

The roots are 2 and 3, which are real.

$\therefore$ If $2 < x < 3$, then $x^2 - 5x + 6$ and the coefficient of x^2 have opposite signs.

Also, if $x < 2$ or $x > 3$ then $x^2 - 5x + 6$ and the coefficient of x^2 have the same sign.

Hence, for the case $2 < x < 3$, $x^2 - 5x + 6 < 0$ and for $x < 2$ or $x > 3$, $x^2 - 5x + 6 > 0$.

$$\text{Extreme value} = \frac{4ac-b^2}{4a} = \frac{4(1)(6)-(-5)^2}{4(1)} = \frac{24-25}{4} = \frac{-1}{4}.$$

Example 3.16 : Find the maximum or minimum value of the expression $3x^2 + 4x + 1$.

Sol : Comparing with $ax^2 + bx + c$, $a = 3$, $b = 4$, $c = 1$ since $a = 3 > 0$, $3x^2 + 4x + 1$has absolute minimum $-m$ value.

$$\text{So that minimum value} = \frac{4ac-b^2}{4a} = \frac{4(3)(1)-(4)^2}{4(3)} = \frac{-1}{3}.$$

There is no maximum value for this expression.

EXERCISE 3.4

1. For what values of x the expression $x^2 - 5x + 14$ is positive ?
2. For what values of x the expression $-6x^2 + 2x - 3$ is negative?
3. Find the value of$x$ at which the expression $x^2 + 5x + 6$ have maximum or minimum.
4. Find the value of x at which the expression $2x - x^2 + 7$ have maximum or minimum.
5. Find the maximum or minimum values of the expression. $3x^2 + 2x + 11$.
6. Find the changes in the sign of the expression and extreme values of the expression $-5x^2 + 4x + 2$, $(x \in \mathrm{R})$
7. Find the sign of the expression $x^2 + x + 1$ for $x \in \mathrm{R}$.
8. Find the maximum or minimum value of the qudratic expression.

   (i) $3x^2 + 4x + 1$ (ii) $4x - x^2 - 10$

   (iii) $2x - 7 - 5x^2$ (iv) $3x^2 + 2x + 11$
9. Find the changes in the sign of the expression $15 + 4x - 3x^2$ and find their extreme values.

Theory of Equations

In the earlier classes, we have studied the linear Equations in one and two variable, the rational integral function of x as a polynomial in one variable and the solutions of quadratic equations.

In the previous chapter. We have learnt about the quadratic expressions, quadratic equations more in detail. We have established certain relations between the roots and coefficients of quadratic equations. But in many problems that arise in Science and technology we encounter equations of degree higher than two. We now investigate the relations which hold between the roots and the coefficients of equations of the n^{th} degree and then discuss some elementary properties in the general theory of equations.

3.9 POLYNOMIAL FUNCTION

A function defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$, where $a_0 \neq 0, a_1, a_2, a_3 ... a_n$ are real or complex numbers and n is positive integer or zero (n is a non-negative integer or whole number) is called a polynomial function of degree n in x. $a_0, a_1, a_2 ... a_n$ are called the coefficients of f.

A polynomial is also denoted by $g(x)$, $h(x)$ etc.

Constant Polynomial : If $n = 0$, then the polynomial consists of just one term a_0. Such a polynomial is called a constant polynomial.

Zero Polynomial : The function which makes the number 0 (all of whose coefficients are zero) is called the zero polynomial. The degree of zero polynomial is not defined the domain of each is **R** and range is a subset of **R**.

Note:

1. The polynomial $f(x) = a_0x + a_1, \quad a_0 \neq 0$ is of degree one and is called a linear polynomial.

2. The polynomial $f(x) = a_0x^2 + a_1x + a_2, \quad a_0 \neq 0$ is of degree two is called a quadratic polynomial.

3. The polynomial $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3, \quad a_0 \neq 0$ is degree three and is called cubic polynomial.

4. The polynomial $f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4, \quad a_0 \neq 0$ is degree four and is called a biquadratic polynomial.

5. **Zero of a Polynomial:** A number α is called a zero of the polynomial $f(x)$ iff $f(\alpha) = 0$.

Polynomial Equation

If any two differently constituted polynomials are equal for some value of x, then such a relation is called a polynomial equation.

The general form of an n^{th} degree polynomial equations is $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0, \ a_0 \neq 0$ denoted by $f(x) = 0$ or $g(x) = 0$ or $h(x) = 0$ etc.

Degree of an equation: The exponent of the heightest power of x occurring in the equation $f(x) = 0$ is called the degree of an equation.

Division Algorithm : Let $f(x)$ and $g(x)$ be the polynomials of degree n and $m(<n)$ respectively. Then there exist a unique pair of polynomial $q(x)$ and $r(x)$ such that $f(x) = q(x) \, . \, g(x) + r(x)$ where either $r(x) = 0$ or degree of $r(x) < m$. The polynomial $q(x)$ has degree $(n - m)$.

Remainder Theorem: If a polynomial $f(x)$ is divided by $(x - \alpha)$ where α is any complex number, then the remainder is $f(\alpha)$.

Proof: When $f(x)$ is divided by $(x - \alpha)$ let $q(\alpha)$ be the quotient and $r(x)$ be the remainder.

By division algorithm : $f(x) = (x - \alpha) \quad q(\alpha) + r(x)$ ... (1)

where $r(x) = 0$ or degree of $r(x) < 1$.

Degree of $r(x) < 1 \Rightarrow$ degree of $r(x) = 0$. Hence $r(x) = r_0$ where r_0 is a complex number.

r_0 is a complex number.

Now, substituting $x = \alpha$ in (1), we have : $f(\alpha) = 0. q(\alpha) + r(\alpha) = r_0$.

Thus r_0, the remainder is $f(\alpha)$.

- **ROOT of an Equation:** Let $f(x) = 0$ be a polynomial equation and α be a number. α is called a root of $f(x) = 0$ iff $f(\alpha) = 0$.

- **Factor Theorem :** If α is a root of the equation $f(x) = 0$, then for $(x - \alpha)$ is a factor of $f(x)$ or α is a zero of $f(x)$. Conversely if $(x - \alpha)$ is a factor of $f(x)$, then α is a root of $f(x) = 0$.

Proof : Let α be a root of the equation $f(x) = 0$, then $f(\alpha) = 0$. ...(1)

By division algorithm and the remainder theorem, we have:

$$f(x) = (x - \alpha)\, q(x) + f(\alpha)$$

Hence from (1), we have $f(x) = (x - \alpha)\, q(x)$

$\Rightarrow \quad (x - \alpha)$ is a factor of $f(x)$.

- **Conversely:** Let $(x - \alpha)$ be a factor of $f(x)$. Then $(x - \alpha)$ must divide $f(x)$ exactly.

Hence $f(\alpha) = 0$.

$\Rightarrow$ By definition α is a root of $f(x) = 0$

Note: Every polynomial equation $f(x) = 0$ has a root real (or) imaginary.

- **Fundamental Theorem of Alzebra :** Every polynomial equation $f(x) = 0$ of degree $n \geq 1$ has atleast one root real (or) complex.
- Every polynomial equation of degree $n \geq 1$ has only n roots real or complex and no more.

The Relations Between the roots and the coeeficients.

Let us consider the n^{th} degree polynomial equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + ... + p_{n-1} x + p_n = 0$$

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be its roots.

Then we have

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + ... + p_{n-1} x + p_n$$

$$= (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)...(x - \alpha_n)$$

$$= x^n - (\alpha_1, \alpha_2 + ... + \alpha_n) x^{n-1}$$

$$+ (\alpha_1\alpha_2 + \alpha_2\alpha_3 + ... + \alpha_{n-1}\alpha_n) x^{n-2} ..... + (-1)^n \alpha_1\alpha_2 .... \alpha_n$$

On equating the coefficients of like power of x in this equation and denoting the sum of products of the roots taken r at a time by sr, we get

$$-p_1 = s_1 = \sum_{i=1}^{n} \alpha_i \text{ (sum of roots)}$$

$$p_2 = s_2 = \sum_{1 \leq i \leq j \leq n}^{n} \alpha_i \alpha_j \text{ (sum of the products of the roots taken two at a time)}$$

$$-p_3 = s_3 = \sum_{1 \leq i \leq j \leq n}^{n} \alpha_i \alpha_j \alpha_k \text{ (sum of the products of the roots taken three at a time)}$$

......

$$(-1)^n p_n = s_n = \alpha_1\alpha_2\alpha_3.....\alpha_n \text{ (Product of the roots)}$$

These equalities give the relations between the roots and the coefficients for any polynomial equation whose leading coeffcient is 1.

Note:

(i) If the leading coefficient in $f(x)$ is a_0 then on dividing each term of the equation $f(x) = 0$ by $a_0 \neq 0$, we get

$$x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + ... + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0$$

whose roots coincide with those of $f(x) = 0$.

In this case, the above relations reduce to

$$s_1 = \frac{-a_1}{a_0},\ s_2 = \frac{a_2}{a_0}, ...s_n = (-1)^n \frac{a_n}{a_0};\ s_r = (-1)^r \frac{a_r}{a_0} \quad \text{for} \quad 1 \leq r \leq n$$

(ii) We recall that in the case of quadratic equation $ax^2 + bx + c = 0$ with roots α and β. $\alpha + \beta = \frac{-b}{a}$, $\alpha\beta = \frac{c}{a}$

(iii) For $n = 3$, we get a cubic equation $x^3 + p_1x^2 + p_2x^3 + p_3 = 0$

Let α_1 α_2 and α_3 be its roots.

Then $S_1 = \alpha_1 + \alpha_2 + \alpha_3 = -p_1$

$S_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = p_2$

and $S_3 = \alpha_1\alpha_2\alpha_3 = -p_3$

(iv) For $n = 4$, we get a biquadralic equation $x^4 + p_1x^3 + p_2x^3 + p_3x + p_4 = 0$

Let α_1 α_2 and α_3 be its roots.

Then $S_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -p_1$

$S_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = p_2$

$S_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 = -p_3$

$S_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = p_4$

MODULE - I
Algebra

Notes

Example 1 : Form the polynomial equation of degree 3 whose roots are 2, 3 and 6.

Solution: The required polynomial equation is $(x - 2)(x - 3)(x - 6) = 0$

$$\Rightarrow x^3 - 11x^2 + 36x - 36 = 0$$

Example 2 : Find the relations between the roots and the coefficients of the cubic equation $3x^3 - 10x^2 + 7x + 10 = 0$.

Solution : Given cubic equation is $3x^3 - 10x^2 + 7x + 10 = 0$

On dividing the euqation by 3, we get

$$x^3 - \frac{10}{3}x^2 + \frac{7}{3}x + \frac{10}{3} = 0 \qquad \text{...(i)}$$

On comparing (1) with $x^3 - p_1x^2 + p_2x^2 + p_2x + p_3 = 0$, we have

$$p_1 = -\frac{10}{3},\ p_2 = \frac{7}{3},\ p_3 = \frac{10}{3}$$

Let α, β, γ be the roots of (1). Then

$$\Sigma\alpha = -p_1 = -\left(-\frac{10}{3}\right) = \frac{10}{3}$$

$$\Sigma\alpha\beta = p_2 = \frac{7}{3}$$

and $$\alpha\beta\gamma = -p_3 = -\frac{10}{3}.$$

Example 3 : If 1, 2, 3 and 4 are the roots of $x^4 + ax^3 + bx^2 + cx + d = 0$ then find the value of a, b, c and d.

Solution: Given that the roots of the polynomial equation are 1, 2, 3 and 4 then

$$x^4 + ax^3 + bx^2 + cx + d = (x - 1)(x - 2)(x - 3)(x - 4) = 0$$

i.e., $x^4 + ax^3 + bx^2 + cx + d = x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$

On equating the coefficients of like powers of x, we obtain

$$a = -10, \quad b = 35, \quad c = -50 \quad \text{and } d = 24.$$

Example 4: Find the sum of the suqares and the sum of the cubes of the roots of the equation $x^3 - px^2 + qx - r = 0$ in terms of p, q, r.

Solution: Let a, b, c be the roots of the given equation.

Then $a + b + c = p$, $ab + bc + ca = q$ and $abc = r$.

Sum of the squares of the roots is

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = p^2 - 2q$$

Sum of the cubes of the roots is

$$a^3 + b^3 + c^3 = (a + b + c)\,(a^2 + b^3 + c^2 - ab - bc - ca) + 3abc$$

$$= p(p^2 - 2q - q) + 3r = p(p^2 - 3q) + 3r.$$

Example 5: Let α, β, γ be the roots of $x^3 + px^2 + qx + r = 0$. Then find

(i) $\Sigma\alpha^2$ (ii) $\Sigma\frac{1}{\alpha}$ if α, β, γ are non zero.

(iii) $\Sigma\alpha^3$

Solution : Since α, β, γ are the roots of the given equation.

we have $\alpha + \beta + \gamma = -p$ ...(1)

$\alpha\beta + \beta\gamma + \gamma\alpha = q$ ...(2)

$\alpha\beta\gamma = -\gamma$ ...(3)

(i) To find $\Sigma\alpha^2$

On squaring equation (1), we get

$$\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha\beta + \beta\gamma + \gamma) = p^2$$

i.e., $\Sigma\alpha^2 + 2\Sigma\alpha\beta = p^2$

i.e., $\Sigma\alpha^2 = p^2 - 2q$

(ii) To find $\Sigma\frac{1}{\alpha}$

$$\Sigma\frac{1}{\alpha}=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$$

$$=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}=-\frac{q}{r}$$

(iii) To find $\Sigma\alpha^3$

We know that

$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$$

$$=(\alpha+\beta+\gamma)[(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha)]$$

i.e., $\Sigma\alpha^3-3r=-p(p^2-3q)$

$\therefore$ $\Sigma\alpha^3=3pq-p^3+3r.$

EXERCISE 3.5

I 1. Form polynomial equations of the lowest degree, with roots as given below.

(i) 1, −1, 3 (ii) 0, 0, 2, 2 −2, −2 (iii) $0, 1, \frac{-3}{2}, \frac{-5}{2}$

2. If α, β, γ are the roots of $4x^3-6x^2+7x+3=0$,

Then find the value of $\alpha\beta+\beta\gamma+\gamma\alpha$.

3. If $1, 1, \alpha$ are the roots of $x^3-6x^2+9x-4=0$ then find α.

4. If $-1, 2$ and α are the roots of $2x^3+x^2-7x-6=0$, then find α.

5. If $1, -2$ and 3 are the roots of $x^3-2x^2+ax+6=0$, then find a.

II. 1. Write down the relations between the roots and Coefficients of the biqua dratic equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ (Find s_1, s_2, s_3, s_4).

2. Write down the relations between the roots and coefficients of the equation $6x^4 + 15x^2 + 7x - 10 = 0$

KEY WORDS

- Roots of the quadratic equation $ax^2 + bx + c = 0$ are complex and conjugate of each other, when D < 0.
- If α, β be the roots of the quadratic equation

  $ax^2 + bx + c = 0$ then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Signs of quadratic expression

1. If the roots of $ax^2 + bx + c = 0$ are imaginary (complex roots) then for $x \in \mathrm{R}$, $ax^2 + bx + c$, and a have the same sign.
2. Let α, β $(\alpha < \beta)$ be the real roots of $ax^2 + bx + c = 0$. Then

   (i) $x \in \mathrm{R}$, $\alpha < x < \beta \Rightarrow ax^2 + bx + c$ and "a" have opposite signs.

   (ii) $x \in \mathrm{R}$, $x < \alpha$ or $x > \beta \Rightarrow ax^2 + bx + c$, and "$a$" have same signs.
3. The expression $f(x) = ax^2 + bx + c$ and a number $\frac{4ac - b^2}{4a}$.

   (i) If $a > 0$, then f has minimum value.

   (ii) If $a < 0$, then f has maximum value.
4. (i) If $a > 0$ then $f(x)$ has absolute minimum value at $x = \frac{-b}{2a}$ and the minimum value is $\frac{4ac - b^2}{4a}$.

(ii) If $a < 0$, then $f(x)$ has absolute maximum value at $x = \frac{-b}{2a}$ and the maximum value is $\frac{4ac - b^2}{4a}$.

SUPPORTED WEB SITES

- http://www.wikipedia.org
- http://mathworld.wolfram.com

PRACTICE EXERCISE

1. Show that the roots of the equation

   $2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$ are imaginary,when $a \neq b$.

2. Show that the roots of the equation

   $bx^2 + (b - c)x = c + a - b$ are always real if those of

   $ax^2 + b(2x + 1) = 0$ are imaginary.

3. If α, β be the roots of the equation $2x^2 - 6x + 5 = 0$ find the equation whose roots are:

   (i) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (ii) $\alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha}$

   (iii) $\alpha^2 + \beta^2, \frac{1}{\alpha^2} + \frac{1}{\beta^2}$

ANSWERS

EXERCISE 3.1

1. (i) $-2\sqrt{3}, \frac{-4}{\sqrt{3}}$ (ii) $a - \sqrt{b}, a + \sqrt{b}$

   (iii) $-\frac{ab}{c}, \frac{c}{ab}$ (iv) $3\sqrt{2}, \sqrt{2}$

EXERCISE 3.2

1. (i) $\dfrac{3 \pm \sqrt{15}i}{4}$ (ii) $\dfrac{1 \pm i}{\sqrt{2}}$

   (iii) $\dfrac{\sqrt{5} \pm \sqrt{43}i}{8}$ (iv) $\dfrac{-\sqrt{2} \pm \sqrt{58}i}{6}$

2. $-1, \dfrac{1}{2}$

EXERCISE 3.3

1. (i) $\dfrac{b^2 - 2ac}{c^2}$ (ii) $\dfrac{(b^2 - 2ac)^2 - 2a^2c^2}{c^4}$

2. (i) $25x^2 - 6x + 9 = 0$ (ii) $625x^2 - 90x + 81 = 0$

4. $q^2 - 5p^2 = 0$

EXERCISE 3.4

1. $\forall x \in \mathrm{R}$

2. The coefficient of x^2 is negative

   for all values of x in **R**.

3. $x = \dfrac{-5}{2}$

4. $x = 1$

5. Minimum value $= \dfrac{32}{3}$

6. Maximum value $= \dfrac{14}{5}$

   Extreme value $= \dfrac{14}{5}$

   when $\dfrac{2-14}{5} < x < \dfrac{2+14}{5}$ the sign of $4x - 5x^2 + 2$ is positive.

   when $x < \dfrac{2-\sqrt{14}}{5}$ or, $x > \dfrac{2+\sqrt{14}}{5}$, the sign of $4x - 5x^2 + 2$ is negative.

7. $x^2 + x + 1$ and the coefficient of x^2 have the same sign and $\forall x$, $x^2 + x + 1 > 0$

8. (i) Maximum value = $\frac{-1}{3}$

(ii) Maximum value = -6

(iii) Maximum value = $\frac{-34}{3}$

(iv) Maximum value = $\frac{32}{3}$

9. when $\frac{-5}{3} < x < 3$, the expression is positive. when $x < \frac{-5}{3}$ or $x > 3$, the sign of the expression is negative and Extreme value = $\frac{49}{3}$.

EXERCISE 3.5

3. (i) $5x^2 - 8x + 5 = 0$ (ii) $10x^2 - 42x + 49 = 0$

(iii) $25x^2 - 116x + 64 = 0$

I. 1. (i) $x^3 - 3x^2 - x + 3 = 0$

(ii) $x^6 - 8x^4 + 16x^2 = 0$

(iii) $4x^4 + 12x^3 - x^2 - 15x = 0$

2. $\frac{7}{4}$ 3. 4 4. $-\frac{3}{2}$ 5. -5

II. 1. $s_1 = 2$, $s_2 = 4$, $s_3 = -6$, $s_4 = -21$

2. $s_1 = 0$, $s_2 = 15/6$, $s_3 = -7/6$, $s_4 = -10/6$

PRACTICE EXERCISE

3. (i) $5x^2 - 8x + 5 = 0$ (ii) $10x^2 - 42x + 49 = 0$

(iii) $25x^2 - 116x + 64 = 0$

Chapter 4

MATRICES

LEARNING OUTCOMES

After studying this lesson, student will be able to:

- Understand matrix notation.
- Identify the order of a matrix
- Define various types of matrices - square row, column, zero matrices.
- Calculate sum and product of matrices wehere possible.
- Define Transpose, symmetric and skew symmetry matrices.

PREREQUISITES

- System of linear equations, number system.

INTRODUCTION

The term matrix was apparents coined by sylvester about 1850, but was introduced by Cayley in 1860. By a 'matrix' we mean an "arrangement" or "rectangular array" of numbers. Matrices (Plural of matrix) find applications in solution of system of linear equations, mathematical economics, quantam mechanics, transportation problems, frame works in mechanics. Matrices an easily an unable for computers.

In this chapter we will discuss variable types of matrices algebraic operations on matrices and then use the theory to find the solution of simultaneous linear equations.

4.1 DIFINITION (MATRIX)

An ordered rectangular array of elements is a called a martix. Usually, we denote a matrix by a capital letter of English alphabets, i.e. A, B, X, etc. Thus, to represent the above information in the form of a matrix, we write

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 3 \end{pmatrix}.$$

- In the above example the horizontal arragement of elements are called rows and vertical arrangement of elements are called columns.

Note: Plural of matrix is matrices.

4.1.1 Order of a Matrix Observe the following matrices (arrangement of numbers):

$$\text{(a)} \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & i \\ i & 1+i \\ 1+i & 1 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 0 & -1 & -5 \\ 2 & 3 & 4 & 5 \\ 4 & -1 & -2 & 0 \end{bmatrix}$$

In matrix (a), there are two rows and two columns, this is called a 2 by 2 matrix or a matrix of order 2×2. This is written as 2×2 matrix. In matrix (b), there are three rows and two columns. It is a 3 by 2 matrix or a matrix of order 3×2. It is written as 3×2 matrix. The matrix (c) is a matrix of order 3×4.

Note that there may be any number of rows and any number of columns in a matrix. If there are m rows and n columns in matrix A, its order is $m \times n$ and it is read as an $m \times n$ matrix.

Use of two suffixes i and j helps in locating any particular element of a matrix. In the above $m \times n$ matrix, the element a_{ij} belongs to the i^{th} row and j^{th} column.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

A matrix of order $m \times n$ can also be written as

$$A = [a_{ij}], \quad i = 1, 2, ..., m, \text{ and}$$

$$j = 1, 2, ... n$$

Example 4.1: Write the order of each of the following matrices:

(i) $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 7 \end{bmatrix}$ (iv)

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 10 \end{bmatrix}$

Solution:

(i) Since the martix has 2 rows and 2 columns, The order of the martix is 2×2

(ii) Since the marix has 3 rows and 1 column the order of the matrix in 3×1

(iii) Since the matrix has 1 row and 3 columns, the order of the martix is 1×3

(iv) Since the matrix has 2 rows and 3 columns, the order of the matrix is 2×3

Example 4.2: For the following matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ 0 & 3 & 2 & 5 \\ 3 & 2 & 3 & 6 \end{bmatrix}$$

(i) find the order of A.

(ii) write the total number of elements in A.

(iii) write the elements a_{23}, a_{32}, a_{14} and a_{34} of A.

(iv) express each element 3 in A in the form a_{ij} .

Solution: The order of the matrix

(i) Since A has 3 rows and 4 columns, A is of order 3×4.

(ii) number of elements in A = $3 \times 4 = 12$

(iii) $a_{23} = 2$, $a_{32} = 2$, $a_{14} = 4$ and $a_{34} = 6$.

(iv) a_{22}, a_{31} and a_{33}.

Example 4.3: If the element in the i th row and j th column of a 2×3 matrix A is given by $\frac{i+2j}{2}$, write the matrix A.

Solution: Here, $a_{ij} = \frac{i+2j}{2}$ (Given)

$$a_{11} = \frac{1+2\times1}{2} = \frac{3}{2}; \qquad a_{12} = \frac{1+2\times2}{2} = \frac{5}{2}; \qquad a_{13} = \frac{1+2\times3}{2} = \frac{7}{2}$$

$$a_{21} = \frac{2+2\times1}{2} = 2; \qquad a_{22} = \frac{2+2\times2}{2} = 3; \qquad a_{23} = \frac{2+2\times3}{2} = 4$$

Thus $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ 2 & 3 & 4 \end{bmatrix}$.

Example 4.4: There are two stores A and B. In store A, there are 120 shirts, 100 trousers and 50 cardigans; and in store B, there are 200 shirts, 150 trousers and 100 cardigans. Express this information in tabular form in two different ways and also in the matrix form.

Solution:

Tabular Form 1 **Matrix Form**

	Shirts	**Trousers**	**Cardigans**
Store A	120	100	50
Store B	200	150	100

$$\Rightarrow \begin{bmatrix} 120 & 100 & 50 \\ 200 & 150 & 100 \end{bmatrix}$$

Tabular Form 2 **Matrix Form**

	Store A	**Store B**
Shirts	120	200
Trousers	100	150
Cardigans	50	100

$$\Rightarrow \begin{bmatrix} 120 & 200 \\ 100 & 150 \\ 50 & 100 \end{bmatrix}$$

EXERCISE 4.1

1. Marks scored by two students A and B in three tests are given in the adjacent table.

   Represent this information in the matrix form, in two ways

	Test 1	Test 2	Test 3
A	56	65	71
B	29	37	57

2. Three firms X, Y and Z supply 40, 35 and 25 truck loads of stones and 10, 5 and 8 truck loads of sand respectively, to a contractor. Express this infonnation in the matrix form in two ways.

3. In family P, there are 4 men, 6 women and 3 children; and in family Q, there are 4 men, 3 women and 5 children. Express this infonnation in the form of a matrix of order 2×3.

4. How many elements in all are there in a

(a) 2×3 matrix (b) 3×4 matrix (c) 4×2 matrix

(d) 6×2 matrix (e) $a \times b$ matrix (f) $m \times n$ matrix

5. What are the possible orders of a matrix if it has

(a) 8 elelnents (b) 5 elements (c) 12 elements (d) 16 elements

6. In the matrix A,

$$A = \begin{bmatrix} 5 & 1 & 8 & 0 & 5 \\ 7 & 6 & 7 & 4 & 6 \\ 3 & 9 & 3 & -3 & 9 \\ 4 & 4 & 8 & 5 & 1 \end{bmatrix}$$

find:

(a) a) number of rows;

(b) number of columns;

(c) the order of the matrix A;

(d) the total number of elements in the matrixA;

(e) a_{14}, a_{23}, a_{34}, a_{45} and a_{33}.

7. Construct a 3×3 matrix whose elements in the ith row and *jth* column is given by

(a) $i - j$ (b) $\frac{i^2}{j}$ (c) $\frac{(i+2j)^2}{2}$ (d) $3j - 2i$

8. Construct a 3×2 matrix whose elements in the ith row and *jth* column is given by

(a) $i + 3j$ (b) $5\,ij$ (c) i^j (d) $i + j - 2$

4.2 TYPES OF MATRICES

Row Matrix : A matrix is said to be a *row matrix* if it has only one row, but may have any number of columns, e.g. the matrix [1 6 0 1 2] is a row matrix.

The order of a row matrix is $1\times$ n.

Column Matrix : A matrix is said to be a *column matrix* if it has only one column, but may have any number of rows, e.g. the matrix $\begin{bmatrix}2\\3\\0\\7\end{bmatrix}$ is a column matrix.

Square Matrix: A matrix is said to be a *square matrix* if number of rows is equal to the number of columns, e.g. the matrix $\begin{bmatrix}1&2&3\\0&6&1\\3&4&2\end{bmatrix}$ having 3 rows and 3 columns is a square matnx.

The diagonal of a square matrix from the top extreme left element to the bottom extreme right element is said to be the principal diagonal. The principal diagonal of the matrix

$\begin{bmatrix}2&3&5\\4&1&7\\3&8&9\end{bmatrix}$ contains elements 2, 1 and 9.

Note : In any given matrix $A = [a_{ij}]$ of order $m \times n$ the elements of the principal diagonal are $a_{11}, a_{22}, a_{33}, ..., a_{nn}$.

Rectangular Matrix : A matrix is said to be a *rectangular matrix* if the number of rows is not equal to the number of columns, e.g. the matrix $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 0 \\ -1 & 2 & 1 & 3 \end{bmatrix}$ having 3 rows and 4.

columns is a rectangular matrix. It may be noted that a row matrix of order $1 \times n$ $(n \neq 1)$ and a column matrix of order $m \times 1$ $(m \neq 1)$ are rectangular matrix.

Zero or Null : A matrix each of whose element is zero is called *azero or null matrix,* e.g. each of the matrix.

$$[0 \quad 0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

is a zero matrix. Zero matrix is denoted by o.

Note **:** A zero matrix may be of any order $m \times n$.

Diagonal Matrix : A square matrix is said to be a *diagonal matrix,* if all elements other than those occuring in the principal diagonal are zero, i.e., if $A = [a_{ij}]$ is a square matrix of order
$m \times n$. then it is said to be a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$.

For example,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$ are diagonal matrices.

Note : A diagonal matrix $A = [a_{ij}]_{m \times n}$ is also written as $A = [a_{11}, a_{12}, a_{13}....., a_{nm}]$.

Scalar Matrix : A diagonal matrix is said to be a *scalar matrix* if all the elements in its principal diagonal are equal to some non-zero constant, say k e.g., the matrix

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$ is a scalar matrix.

Note: A square zero matrix is not a scalar matrix.

Unit or Identity Matrix : A scalar matrix is said to be a *unit or identity matrix,* if all of its elements in the principal diagonal are unity. It is denoted by I_n, if it is of order n e.g., the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Note: A square matrix A = $[a_{ij}]$ is a unit matrix if

$$a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ 1, & \text{when } i = j \end{cases}$$

Equal Matrices : Two matrices are said to be equal if they are of the same order and if their corresponding elements are equal.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ then

A = B if $a_{ij} = b_{ij}$ for $i = 1, 2, ...$ $j = 1, 2, 3.$

Two matrices X and Y given below are not equal, since they are of different orders, namely 2×3 and 3×2 respectively.

$$X = \begin{bmatrix} 7 & 1 & 3 \\ 2 & 1 & 5 \end{bmatrix}, \quad Y = \begin{bmatrix} 7 & 2 \\ 1 & 1 \\ 3 & 5 \end{bmatrix}$$

Also, the two matrices P and Q are not equal, since some elements of P are not equal to the corresponding elements of Q.

$$P = \begin{bmatrix} -1 & 3 & 7 \\ 0 & 1 & 2 \end{bmatrix}, \ Q = \begin{bmatrix} -1 & 3 & 6 \\ 0 & 2 & 1 \end{bmatrix}$$

Example 4.5 : Find whether the following matrices are equal or not:

(i) $A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}$

(ii) $P = \begin{bmatrix} 0 & 1 & 7 \\ 2 & 3 & 5 \end{bmatrix}, \ Q = \begin{bmatrix} 0 & 1 & 7 \\ 2 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $X = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 6 \\ 7 & 1 & 0 \end{bmatrix}, \ Y = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 6 \\ 7 & 1 & 0 \end{bmatrix}$

Solution : (i) Matrices A and $B$are of the same order 2×2. But some of their corresponding elements are different. Hence, $A \neq B$. .

(ii) Matrices P and Q are of different orders, So, $P \neq Q$.

(iii) Matrices X and $Y$are of the same order 3×3, and their corresponding elements are also equal.

So, $X = Y$.

Example 4.6: Determine the values of x and y, if

(i) $\begin{bmatrix} x & 5 \end{bmatrix} = \begin{bmatrix} 2 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ y \end{bmatrix}$

(iii) $\begin{bmatrix} x & 2 \\ 3 & -y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

Solution: Since the two matrices are equal, their corresponding elements should be equal.

(i) $x = 2$

(ii) $x = 4, y = 3$

(iii) $x = 1, y = -5$

Example 4.7: For what values of *a, b,* c and *d,* are the following matrices equal?

(i) $A = \begin{bmatrix} a & 2b \\ 3 & d \end{bmatrix}, B = \begin{bmatrix} a & 4 \\ 5c & 2 \end{bmatrix}$

(ii) $P = \begin{bmatrix} a & b-2d \\ -3 & 2b \\ a+c & 7 \end{bmatrix}, Q = \begin{bmatrix} 5 & 1 \\ -3 & 6 \\ 4 & 7 \end{bmatrix}$

Solution: (i)The given matrices A and B will be equal only if their corresponding elements are equal, i.e. if

$a = 1,\ 2b = 4,\ 3 = 5c,$ and $d = 2$

$\Rightarrow a = 1,\ b = 2,\ c = \frac{3}{5}$ and $d = 2$

Thus, for $a = 1,\ b = 2,\ c = \frac{3}{5}$ and $d = 2$ matrices A and B are equal.

(ii) The given matrices P and Q will be equal if their corresponding elements are equal, i.e. if

$a = 5,\ b - 2d = 1,\ 2b = 6$ and $a + c = 4$

$\Rightarrow a = 5,\ b = 3,\ c = -1$ and $d = 1$

Thus, for $a = 5, b = 3, c = -1$ and $d = 1$matrices P and Q are equal.

EXERCISE 4.2

1. Which of the following matrices are
   (a) row matrices (b) column matrices (c) square matrices
   (d) diagonal matrices (e) scalar matrices (f) identity matrices and
   (g) zero matrices

$$A=\begin{bmatrix}4&0&0\\0&5&0\\0&0&6\end{bmatrix},\ B=\begin{bmatrix}2\\7\\8\\0\end{bmatrix},\ C=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix},\ D=\begin{bmatrix}2&0\\0&2\end{bmatrix},$$

$$E=\begin{bmatrix}1&2&4\\3&9&8\\1&0&2\end{bmatrix},\ F=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$G=\begin{bmatrix}3&4&10&8\end{bmatrix},\ H=\begin{bmatrix}2&3&7\\1&4&9\end{bmatrix},\ I=\begin{bmatrix}2&-1\\3&2\\-1&0\end{bmatrix}$$

2. Find the values of *a, b,* c and *d* if

(a) $\begin{bmatrix}b&2c\\b+d&c-2a\end{bmatrix}=\begin{bmatrix}10&12\\8&2\end{bmatrix}$ (b) $\begin{bmatrix}a+2&4\\b+3&25\end{bmatrix}=\begin{bmatrix}4&2c\\6&5d\end{bmatrix}$

(c) $\begin{bmatrix}2a&b\\-4&6\end{bmatrix}=\begin{bmatrix}3&-2\\d&3c\end{bmatrix}$

3. Can a matrix of order 1×2 be equal to a matrix of order 2×1 ?
4. Can a matrix of order 2×3 be equal to a matrix of order 3×3 ?

Let us consider the following situation:

The marks obtained by three students in English, Hindi, and Mathematics are as follows:

	English	Hindi	Mathematics
Elizabeth	20	10	15
Usha	22	25	27
Shabnam	17	25	21

It is also given that these marks are out of 30 in each case. In matrix form, the above information can be written as

$$\begin{bmatrix} 20 & 10 & 15 \\ 22 & 25 & 27 \\ 17 & 25 & 21 \end{bmatrix}$$ (It is understood that rows correspond to the names and columns correspond to the subjects)

If the maximum marks are doubled in each case, then the marks obtained by these girls will also be doubled. In matrix form, the new marks can be given as:

$$\begin{bmatrix} 2\times 20 & 2\times 10 & 2\times 15 \\ 2\times 22 & 2\times 25 & 2\times 27 \\ 2\times 17 & 2\times 25 & 2\times 21 \end{bmatrix} \text{ which is equal to } \begin{bmatrix} 40 & 20 & 30 \\ 44 & 50 & 54 \\ 34 & 50 & 42 \end{bmatrix}$$

So, we write that

$$2\times\begin{bmatrix} 20 & 10 & 15 \\ 22 & 25 & 27 \\ 17 & 25 & 21 \end{bmatrix} = \begin{bmatrix} 2\times 20 & 2\times 10 & 2\times 15 \\ 2\times 22 & 2\times 25 & 2\times 27 \\ 2\times 17 & 2\times 25 & 2\times 21 \end{bmatrix} = \begin{bmatrix} 40 & 20 & 30 \\ 44 & 50 & 54 \\ 34 & 50 & 42 \end{bmatrix}$$

Now consider another matrix

$$A = \begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 6 \end{bmatrix}$$

Let us see what happens, when we multiply the matrixA by 5

$$\text{i.e., } 5\times A = 5A = 5\times\begin{bmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 5\times 3 & 5\times 2 \\ 5\times(-2) & 5\times 0 \\ 5\times 1 & 5\times 6 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ -10 & 0 \\ 5 & 30 \end{bmatrix}$$

When *a* matrix is multiplied by a scalar, then each of its element is multiplied by the same scalar.

For example,

$$\text{if } A = \begin{bmatrix} 2 & -1 \\ 6 & 3 \end{bmatrix} \text{ then, } kA = \begin{bmatrix} k\times 2 & k\times(-1) \\ k\times 6 & k\times 3 \end{bmatrix} = \begin{bmatrix} 2k & -k \\ 6k & 3k \end{bmatrix}$$

$$\text{when } k = -1,\ kA = (-1)\ A = \begin{bmatrix} -2 & 1 \\ -6 & -3 \end{bmatrix}$$

So, $(-1)\,A = -A$

Thus, if $A = \begin{bmatrix} 2 & -1 \\ 6 & 3 \end{bmatrix}$, then $-A = \begin{bmatrix} -2 & 1 \\ -6 & -3 \end{bmatrix}$.

Example 4.8: If $A = \begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix}$, find

(i) 2A (ii) $\frac{1}{2}A$ (iii) $-A$ (iv) $\frac{2}{3}A$

Solution:

(i) Here.

$$2A = 2\times\begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\times(-2) & 2\times3 & 2\times4 \\ 2\times(-1) & 2\times0 & 2\times1 \end{bmatrix} = \begin{bmatrix} -4 & 6 & 8 \\ -2 & 0 & 2 \end{bmatrix}$$

(ii) $$\frac{1}{2}A = \frac{1}{2}\times\begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\times(-2) & \frac{1}{2}\times3 & \frac{1}{2}\times4 \\ \frac{1}{2}\times(-1) & \frac{1}{2}\times0 & \frac{1}{2}\times1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{3}{2} & 2 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(iii) $$-A = (-1)\times\begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -4 \\ 1 & 0 & -1 \end{bmatrix}$$

(iv) $$\frac{2}{3}A = \frac{2}{3}\times\begin{bmatrix} -2 & 3 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & 2 & \frac{8}{3} \\ -\frac{2}{3} & 0 & \frac{2}{3} \end{bmatrix}$$

EXERCISE 4.3

1. If $A = \begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}$, find :

(a) 4A (b) $-A$ (c) $\frac{1}{2}A$ (d) $-\frac{3}{2}A$

2. If $A = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$, find :

(a) $5A$ (b) $-3A$ (c) $\frac{1}{3}A$ (d) $-\frac{1}{2}A$

3. If $A = \begin{bmatrix} -1 & 0 \\ 4 & 2 \\ 0 & -1 \end{bmatrix}$, find $(-7)A$

4. If $X = \begin{bmatrix} 3 & 0 & 1 \\ 4 & -2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$, find :

(a) $5X$ (b) $-4X$ (c) $\frac{1}{3}X$ (d) $-\frac{1}{2}X$

4.5 ADDITION OF MATRICES

Two students A and B compare their performances in two tests in Mathematics, Physics and English. The maximum marks in each test in each subject are 50. The marks scored by them are as follows:

First Test

$$\begin{array}{c c} & \begin{array}{ccc} M & P & E \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 50 & 38 & 33 \\ 47 & 40 & 36 \end{bmatrix} \end{array}$$

Second Test

$$\begin{array}{c c} & \begin{array}{ccc} M & P & E \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 45 & 32 & 30 \\ 42 & 30 & 39 \end{bmatrix} \end{array}$$

How can we find their total marks in each subject in the two tests taken together?

Observe that the new matrix giving the combined information of two matrices

$$\begin{array}{c c} & \begin{array}{ccc} M & P & E \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 50+45 & 38+32 & 33+30 \\ 47+42 & 40+30 & 36+39 \end{bmatrix} \end{array} \quad \begin{array}{c c} & \begin{array}{ccc} M & P & E \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 95 & 70 & 63 \\ 89 & 70 & 75 \end{bmatrix} \end{array}$$

This new matrix is called the *sum* of the given matrices.

If A and B are any two given matrices of the same order, then their sum is defined to be a matrix C whose respective elements are the sum of the corresponding elements of the matrices A and B and we write this as C = A + B.

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ then $A + B = C = [c_{ij}]_{m\times n}$

where $c_{ij} = a_{ij} + b_{ij}$.

Note: 1. The order of the matrix C will also be the same as that of A andB.

2. It is not possible to add two matrices of different orders.

Example 4.9: $A=\begin{bmatrix}1 & 3\\ 4 & 2\end{bmatrix}$ and $B=\begin{bmatrix}5 & 2\\ 1 & 0\end{bmatrix}$, then find A + B.

Solution: Since the given matrices A and B are of the same order, i.e. 2×2, we can add them. So,

$$A+B=\begin{bmatrix}1+5 & 3+2\\ 4+1 & 2+0\end{bmatrix}$$

$$=\begin{bmatrix}6 & 5\\ 5 & 2\end{bmatrix}$$

Example 4.10: If $A=\begin{bmatrix}0 & 1 & -1\\ 2 & 3 & 0\end{bmatrix}$ and $B=\begin{bmatrix}3 & 0 & 4\\ 1 & 2 & 1\end{bmatrix}$ then find $A + B$.

Solution: Since the given matrices A andB are of the same order, i.e. 2×2, we can add them. So,

$$A+B=\begin{bmatrix}0+3 & 1+0 & -1+4\\ 2+1 & 3+2 & 0+1\end{bmatrix}$$

$$=\begin{bmatrix}3 & 1 & 3\\ 3 & 5 & 1\end{bmatrix}.$$

4.5.1 Properties of Addition of matrices

Let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ be matrices of the same order. Then the addition of matrices satisfies the following propertices.

(i) Commutative property

$$A + B = B + A$$

Now $\quad A + B = [a_{ij}] + [b_{ij}]$

$$= [a_{ij} + b_{ij}]$$

$$= [b_{ij} + a_{ij}]$$

$$= [b_{ij}] + [a_{ij}]$$

$$= B + A.$$

For any two matrices A and B of the same order, $A + B = B + A$

i.e., matrix addition is commutative.

(ii) Associative property

$$A + (B + C) \quad = (A + B) + C$$

Now $(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$$= [a_{ij} + b_{ij}] + [c_{ij}]$$

$$= [(a_{ij} + b_{ij}) + c_{ij}]$$

$$= [a_{ij} + b_{ij} + c_{ij}]$$

$$= [a_{ij} + (b_{ij} + c_{ij})]$$

$$= [a_{ij}]+ [b_{ij} + c_{ij}]$$

$$= [a_{ij}]+ ([b_{ij}] + [c_{ij}])$$

$$= A + (B + C)$$

For any three matrices A, B and C of the same order,

$A + (B + C) = (A + B) + C$ i.e., matrix addition is associative.

(iii) Additive identity

If A is an $m \times n$ matrix and O is the $(m \times n)$ null matrix, A + O = O + A = A. We call O the additive identity in the set of all $m \times n$ matrices.

Addiitve identity is a zero matrix, which when added then a given matrix, gives the same give matrix

i.e., A + O = O + A = A.

Example 4.11: If $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, then

find (a) $A + B$ (b) $B + C$ (c) $(A + B) + C$ (d) $A + (B + C)$

Solution:

(a) $A + B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2+(-3) & 0+1 \\ 1+1 & 3+2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 5 \end{bmatrix}$$

(b) $B + C = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} (-3)+(-1) & 1+0 \\ 1+0 & 2+3 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 1 & 5 \end{bmatrix}$$

(c) $(A + B)+C = \begin{bmatrix} -1 & 1 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ ...[From (a)]

$$= \begin{bmatrix} (-1)+(-1) & 1+0 \\ 2+0 & 5+3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & 8 \end{bmatrix}$$

(d) $A + (B+C) = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} -4 & 1 \\ 1 & 5 \end{bmatrix}$ ... ...[From (b)]

$$= \begin{bmatrix} 2+(-4) & 0+1 \\ 1+1 & 3+5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & 8 \end{bmatrix}$$

Example 4.12: If $A = \begin{bmatrix} -2 & 3 & 5 \\ 1 & -1 & 0 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then find

(a) A + O (b) O + A

What do you observe?

Solution:(a) $A + O = \begin{bmatrix} -2 & 3 & 5 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} -2+0 & 3+0 & 5+0 \\ 1+0 & -1+0 & 0+0 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 5 \\ 1 & -1 & 0 \end{bmatrix}$$

(b) $O + A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 5 \\ 1 & -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0+(-2) & 0+3 & 0+5 \\ 0+1 & 0+(-1) & 0+0 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 5 \\ 1 & -1 & 0 \end{bmatrix}$$

From (a) and (b), we see that

4.5 SUBTRACTION OF MATRICES

Let A and B two matrices of t he same order. Then the matrix A – B is defined as the subtraction of B from A. A – B is obtained by subtracting corresponding elements of B from the corresponding elements of A.

We can write A – B = A + (–B)

Note: A – B and B – A do not denote the same matrix, except when A = B.

Example 4.13: If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ then find

(a) A – B (b) B – A

(a) We know that

$$A - B = A + (-B) \qquad ...(i)$$

Since $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, we have $-B = \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}$

Substituting it in (i), we get

$$A - B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ -1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+(-3) & 0+(-2) \\ 2+(-1) & (-1)+(-4) \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}$$

(b) Similarly,

$$B - A = B + (-A)$$

$$B - A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+(-1) & 2+0 \\ 1+(-2) & 4+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$$

Remark: To obtain $A - B$, we can subtract directly the elements of B from the corresponding elements of A. Thus,

$$A - B = \begin{bmatrix} 1-3 & 0-2 \\ 2-1 & -1-4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}$$

and $$B - A = \begin{bmatrix} 3-1 & 2-0 \\ 1-2 & 4-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$$

Example 4.14: Find (a) A – B (b) B – A, for the matrices *A* andB defined as under:

$$A = \begin{bmatrix} 2 & 4 \\ 5 & 7 \\ -1 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 3 \\ 5 & -2 \\ 3 & 1 \end{bmatrix}$$

Solution: (a) $A - B = \begin{bmatrix} 2 & 4 \\ 5 & 7 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 5 & -2 \\ 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2-0 & 4-3 \\ 5-5 & 7-(-2) \\ -1-3 & 4-1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 9 \\ -4 & 3 \end{bmatrix}$$

$$\text{(b) } B - A = \begin{bmatrix} 0 & 3 \\ 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 7 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0-2 & 3-4 \\ 5-5 & -2-7 \\ 3-(-1) & 1-4 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & -9 \\ 4 & -3 \end{bmatrix}$$

Remarks: From above examples we can conclude that the matrix subtraction is not commutative.

Example 4.15: If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$; $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and A+B = O, find B.

Solution: Here, it is given that A + B = 0

$$\therefore \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2+a & 3+b \\ -1+c & 4+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad 2 + a = 0 \quad ; \quad 3 + b = 0$$

$$-1 + c = 0 \quad ; \quad 4 + d = 0$$

$$\Rightarrow \quad a = -2; \quad b = -3; \quad c = 1 \quad \text{and} \quad d = -4$$

$$\therefore B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & -2 \end{bmatrix}$$

Remarks: In Example 4.15 the elements of *B* are the additive inverse of the corresponding elements *of A*. We, therefore, call *B* is the additive inverse of the matrixA. Further,

$$B = \begin{bmatrix} -2 & -3 \\ 1 & -4 \end{bmatrix} = (-1) \times \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = (-1) \times A = -A$$

In general, given a matrix A, there exists another matrix $B = (-1)\ A$ such that $A + B = 0$, then such a matrix B is called the additive inverse of the matrix of A.

EXERCISE 4.4

1. If $A = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$ then find :

   (a) A + B (b) 2A + B (c) A + 3B (d) 2A + 3B

2. If $P = \begin{bmatrix} 2 & 5 & 3 \\ -1 & 4 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 2 & -3 \\ 4 & 1 & -5 \end{bmatrix}$, then find :

   (a) P − Q (b) Q − P (c) P − 2Q (d) 2Q − 3P

3. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & -1 & 2 \\ 4 & 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -4 & 0 \\ 1 & 6 & 1 \\ 2 & 0 & 7 \end{bmatrix}$, then find :

   (a) A + B (b) A − B (c) − A + B (d) 3A + 2B

4. If $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ -1 & 1 \end{bmatrix}$, find the zero matrix 0 satisfying $A + 0 = A$.

5. If $A = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 2 & 3 \\ -4 & 0 & 1 \end{bmatrix}$ then find :

   (a) −A (b) A +(− A) (c) (−A) + A

6. If $A = \begin{bmatrix} 1 & 9 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 7 & 9 \end{bmatrix}$ then find :

   (a) 2A (b) 3B (c) 2A + 3B

   (d) 2A + 3B + 5X = 0, what is X?

7. If $A=\begin{bmatrix}1 & 4\\ -2 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix}$ and $C=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$, then find :

(a) A –B (b) B – C (c) A – C (d) 3B – 2C

(e) A – B – C (f) 2A – B – 3C

4.6 MULTIPLICATION OF MATRICES

Salina and Rakhi are two friends. Salina wants to buy 17 kg wheat, 3 kg pulses and 250gm ghee; while Rakhi wants to buy 15 kg wheat, 2kg pulses and 500 gm ghee. The prices of wheat, pulses and ghee per kg respectively are Rs. 8.00, Rs. 27.00 and Rs. 90.00.How much money will each spend? Clearly, the money needed by Salina and Rakhi will be :

Salina

Cost of 17 kg wheat $\Rightarrow 17 \times$ Rs.8 = Rs. 136.00

Cost of 3 kg pulses $\Rightarrow 3 \times$ Rs. 27 = Rs.81.00

Cost of 250 gm ghee $\Rightarrow \frac{1}{4} \times$ Rs. 90 = Rs. 22.50

Total = Rs. 239.50

Cost of 15 kg wheat $\Rightarrow 15 \times$ Rs.8 = Rs. 120.00

Cost of 2 kg pulses $\Rightarrow 2 \times$ Rs. 27 = Rs.54.00

Cost of 500 gm ghee $\Rightarrow \frac{1}{2} \times$ Rs. 90 = Rs. 45.00

Total = Rs. 219.00

In matrix form, the above information can be represented as follows:

Requirements Price Money Needed

$$\begin{bmatrix}\text{wheat} & \text{pulses} & \text{ghee}\\ 17 & 3 & 0.250\\ 15 & 2 & 0.500\end{bmatrix}\begin{bmatrix}8\\ 27\\ 90\end{bmatrix}\quad\begin{bmatrix}17\times8+3\times27+0.250\times90\\ 15\times8+2\times27+0.500\times90\end{bmatrix}=\begin{bmatrix}239.50\\ 219.00\end{bmatrix}$$

Another shop in the same locality quotes the following prices.

Wheat: Rs. 9 per kg.; pulses: Rs.26 per kg; ghee : Rs. 100 per kg.

The money needed by Salina and Rakhi to buy the required quantity of articles from this shop will be

Salina

17 kg wheat $\Rightarrow 17 \times$ Rs.9 = Rs. 153.00

3 kg pulses $\Rightarrow 3 \times$ Rs. 26 = Rs.78.00

250 gm ghee $\Rightarrow \frac{1}{4} \times$ Rs. 100 = Rs. 25.00

Total = Rs. 256.00

Rakhi

15 kg wheat $\Rightarrow 15 \times$ Rs.9 = Rs. 135.00

2 kg pulses $\Rightarrow 2 \times$ Rs. 26 = Rs.52.00

500 gm ghee $\Rightarrow \frac{1}{2} \times$ Rs. 100 = Rs. 50.00

Total = Rs. 237.00

In matrix form, the above information can be written as follows:

Requirements Price Money Needed

$$\begin{bmatrix} 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} \begin{bmatrix} 9.00 \\ 26.00 \\ 100.00 \end{bmatrix} = \begin{bmatrix} 17\times 9+3\times 26+0.250\times 100 \\ 15\times 9+2\times 26+0.500\times 100 \end{bmatrix} = \begin{bmatrix} 256.00 \\ 237.00 \end{bmatrix}$$

To have a comparative study, the two information can be combined in the following way:

$$\begin{bmatrix} 17 & 3 & 0.250 \\ 15 & 2 & 0.500 \end{bmatrix} \begin{bmatrix} 8.00 & 9.00 \\ 27.00 & 26.00 \\ 90.00 & 100.00 \end{bmatrix} = \begin{bmatrix} 239.50 & 256.00 \\ 219.00 & 237.00 \end{bmatrix}$$

Let us see how and when we write this product:

(i) The three elements of first row of the first matrix are multiplied respectively by the corresponding elements of the first column of the second matrix and added to give element of the first row and the first column of the product matrix. In the same way, the product of the elements of the second row of the first matrix to the corresponding elements of the first column of the second matrix on being added gives the element of the second row and the first column of the product matrix; and so on.

(ii) The number of column of the first matrix is equal to the number of rows of the second matrix so that the first matrix is compatible for multiplication with the second matrix.

Thus, If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ and $B = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$, then

$$AB = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \times \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 \end{bmatrix}$$

Definition: If A and B are two matrices of order $m \times p$ and $p \times n$ respectively, then their product will be a matrix C of order $m \times n$; and if a_{ij}, b_{ij} and c_{ij} are the elements of the i^{th} row and j^{th} column of the matrices A, B and C respectively, then

$$c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj}$$

Example 4.16: If $A = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$, then find:

(a) AB (b) BA Is AB = BA ?

Solution: Order of A is 1×3

Order of B is 3×1

$\therefore$ Number of columns of A = Number of rows of B

$\therefore$ AB exists

Now, $AB = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

$= [1 \times (-2) + (-1) \times 0 \{ 2 \times 2] = [-2 + 0 + 4] = [2]$

Thus, AB = [2], a matrix of order 1×1.

Again, number of columns of B = number of rows of A.

$\therefore$ BA exists.

Now,

$$BA = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2\times1 & (-2)\times(-1) & (-2)\times2 \\ 0\times1 & 0\times(-1) & 0\times2 \\ 2\times1 & 2\times(-1) & 2\times2 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$$

Thus, $BA = \begin{bmatrix} -2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & -2 & 4 \end{bmatrix}$ a matrix of order 3×3.

From the above, we find that AB $\neq$ BA.

Example 4.17: If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, then find AB.

Solution: Here, number of columns of A = number of rows of B

$\therefore$ AB exists.

Now $AB = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}_{2\times2} \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{2\times1}$

$$= \begin{bmatrix} 1\times1+2\times1 \\ 3\times3+(-1)\times1 \end{bmatrix}_{2\times1} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

Remarks: Can we find BA? The answer is no, because the number of columns of B is not equal to the number of rows of A.

Thus, in Example 4.17, we find thatAB exists, but BA does not exist.

Example 4.18: FindAB and BA, if possible for the matrices A and B:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}; \; B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Solution: Here, Number of columns of $A \neq$ Number of rows of B

$\therefore$ AB does not exist.

Further, Number of columns of $B \neq$ Number of rows of **A**

$\therefore$ BA does not exist.

Example 4.19: If $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$, then fmd AB and BA. Also fmd if AB=BA.

Solution: Here, Number of columns of A = Number of rows of B

$\therefore$ AB exists.

Further,Number of columns ofB = Number of rows of A

$\therefore$ BA also exists.

Now $$AB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times2+2\times2 & 1\times1+2\times2 \\ -1\times2+0\times2 & -1\times1+0\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4 & 1+4 \\ -2+0 & -1+0 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -2 & -1 \end{bmatrix}_{2\times2}$$

and $$BA = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times1+1\times(-1) & 2\times2+1\times0 \\ 2\times1+2\times(-1) & 2\times2+2\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 4+0 \\ 2-2 & 4+0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix}_{2\times2}$$

Thus AB $\neq$ BA.

Remarks: We observe that AB and BA are of the same order 2×2, but still AB $\neq$ BA.

Example 4.20 : If $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, find AB and BA. Is AB = BA?

Solution : Here, both A and B are of order 2×2. So, both AB and BA exist. Now

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}_{2\times2} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}_{2\times2} = \begin{bmatrix} 8+0 & 0+0 \\ 0+0 & 0-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -3 \end{bmatrix}_{2\times2} \text{ and}$$

$$BA = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 8+0 & 0+0 \\ 0+0 & 0-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & -3 \end{bmatrix}_{2\times2}$$

Here, both AB and BA are of the same order and AB = BA.

Hence, if two matrciesA andB are multiplied, then the following five cases arise:

(i) BothAB andBA exist, but are of different orders.

(ii) Only one of the productsAB or BA exists.

(iii) Neither AB nor BA exist.

(iv) Both AB and BA exist and are of the same order, but AB $\neq$ BA.

(v) Both AB and BA exist and are of the same order. Also, AB= BA.

Example 4.21 : If $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ verify that $A^2 - 2A - 3I = 0$.

Solution: Here,

$$A^2 = AA = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 9+0 & 0+0 \\ 0+0 & 0+9 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

and $$3I = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\therefore\ A^2 - 2A - 3I = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \left\{\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 9-9 & 0-0 \\ 0-0 & 9-9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

Hence, verified.

Example 4.22 : Solve the matrix equation:

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Solution: Here,

$$\text{L.H.S.} = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x-3y \\ x+y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x-3y \\ x+y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad 2x - 3y = 1; \quad x + y = 3$$

Solving these equations, we get

$$x = 2 \text{ and } y = 1.$$

Example 4.23 : If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ then find AB.

Solution : Here,

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1\times(-1)+1\times 1 & 1\times 1+1\times(-1) \\ 1\times(-1)+1\times 1 & 1\times 1+1\times(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -1+1 & 1-1 \\ -1+1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Remarks: From Example 4.23, we find that the product of two non-zeromatrices may be a zero matrix, i.e. $A \neq 0$ and $B \neq 0$ may imply $AB = 0$.

Hence, we conclude that the product of two non-zero matrices can be a zero matrix, whereas in numbers, the product of two non-zero numbers is always non-zero.

Example 4.24 : If $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ find A^2 .

Solution : $A^2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Remarks: From Example 4.24, we find that the square of a non-zero matrix may be a zero matrix, i.e., $A \neq 0$ may imply $A^2 = 0$.

Example 4.25: For $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, find

(a) (AB)C (b) A(BC) Is (AB)C = A(BC)?

Solution: (a) $(AB)C = \left\{ \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} \right\} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2 & 0-4 \\ 12-5 & 0+10 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -4 \\ 7 & 10 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6+0 & 0-12 \\ -7+0 & 0+30 \end{bmatrix} = \begin{bmatrix} -6 & -12 \\ -7 & 30 \end{bmatrix}$$

(b) $A(BC) = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \left\{ \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \right\}$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -4+0 & 0+0 \\ 1+0 & 0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4-2 & 0-12 \\ -12+5 & 0+30 \end{bmatrix} = \begin{bmatrix} -6 & -12 \\ -7 & 30 \end{bmatrix}$$

From (a) and (b), we find that (AB)C = A(BC), i.e., matrix multiplication is associative.

EXERCISE 4.5

1. If $A = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$, find AB and BA. Is AB = BA?

2. If $A = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & -2 \end{bmatrix}$ find AB and BA. Is AB = BA?

3. If $A = \begin{bmatrix} a \\ b \end{bmatrix}$ and $B = \begin{bmatrix} x & y & z \end{bmatrix}$ find AB and BA, whichever exists.

4. If $A = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$ find BA. Does AB exists?

5. If $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

   (a) Does AB exist? Why? (b) Does BA exist? Why?

6. If $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 2 & 5 \end{bmatrix}$ find AB and BA. Is AB = BA?

7. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 5 & 4 \\ 5 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ find AB and BA. Is AB = BA?

8. If $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ find AB and BA. Is AB = BA?

9. Find the values of x and y if

   (a) $\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

10. $A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$ verify that AB = 0.

11. For $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, verify that $A^2 - 5A + I = 0$,where I is aunit matrix of order 2.

12. If $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -3 \\ -2 & 3 \end{bmatrix}$, find

(a) A(BC) (b) (AB)C (c) (A + B)C

(d) AC + BC (e) $A^2 - B^2$ (f) (A – B) (A + B)

13. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, find : (a) AC (b) BC

Is AC = BC ? What do you conclude?

14. If $A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 8 \\ 7 & -1 \end{bmatrix}$ find

(a) B + C (b) A(B+C) (c) AB

(d) AC (e) AB + AC

What do you observe?

15. For matices $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 0 \end{bmatrix}$, verify that $(AB)' = B'A'$.

16. If $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, find X such that AX = B.

17. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that $A^2 - (a+d)A = (bc - ad)I$

18. If $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is it true that

(a) $(A + B)^2 = A^2 + B^2 + 2AB$?

(b) $(A - B)^2 = A^2 + B^2 - 2AB$?

(c) $(A + B)(A - B) = A^2 - B^2$?

MODULE - I
Algebra

Notes

4.7 TRANSPOSE OF A MATRIX

In this section we define the transpose of a matrix and study its properties. We also define symmetric and skew symmetic matrices.

Definition (Transpose of a matrix)

If $A = [a_{ij}]$ is an $m \times n$ matrix then the matrix obtained by intechanging the rows and columns of A is called the transpost of A. Transpose of the matrix a is denoted by A' or A^T. In other words if , then $A = \left[a_{ij}\right]_{m\times n}$, then $A' = \left[a_{ij}\right]_{n\times m}$

For example it

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 0 & 7 \end{bmatrix}, \quad \text{then} \quad A' = \begin{bmatrix} 3 & 4 & 0 \\ 2 & 1 & 7 \end{bmatrix}$$

4.7.1 Properties of transpose of matrices

We now state the following properties of transpose of matrices without proof. These may be verified by taking suitable examples.

For any two matrices A, B of suitables orders we have

(i) $(A')' = A$ (ii) $(kA)' = kA'$ (k is a constant)

(iii) $(A+B)' = A' + B'$ (iv) $(AB)' = B'A'$

Example 4.26 : If $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$

Verify that (i) $(A')' = A$ (ii) $(A + B)' = A' + B'$

(iii) $(5B)' = 5(B)'$

Solution:

(i) We have $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$

$$\Rightarrow \quad A' = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$\Rightarrow \quad (A')' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} = A.$$

(ii) $$A + B = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 8 & 7 \\ 6 & 3 & 7 \end{bmatrix}$$

$$\therefore \ (A+B)' = \begin{bmatrix} -2 & 6 \\ 8 & 3 \\ 7 & 7 \end{bmatrix}$$

$$A' + B' = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 4 \\ 4 & -2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 6 \\ 8 & 3 \\ 7 & 7 \end{bmatrix} = (A+B)'$$

(iii) We have $$5B = 5\begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 20 & 0 \\ 20 & -10 & -5 \end{bmatrix}$$

$$(5B)' = \begin{bmatrix} -15 & 20 \\ 20 & -10 \\ 0 & -5 \end{bmatrix}$$

$$= 5\begin{bmatrix} -3 & 4 \\ 4 & -2 \\ 0 & -1 \end{bmatrix}$$

$= 5B'$.

Example 4.27: If $A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}$ then verity that

$(AB)' = B'A'$

Solution : We have $AB = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 2+3+10 & -4+0+8 \\ 1-9-20 & -2+0-16 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 4 \\ -28 & -18 \end{bmatrix}$$

$$\therefore\ (AB)' = \begin{bmatrix} 15 & -28 \\ 4 & -18 \end{bmatrix}$$

Now $A' = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & -4 \end{bmatrix}$ and $B' = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix}$

$$\therefore\ B'A' = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3+10 & 1-9-20 \\ -4+0+8 & -2+0-16 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -28 \\ 4 & -18 \end{bmatrix}$$

Hence $(AB)' = B'A'$.

4.7.2 Symmetric Matrix

A square matrix A is said to be symmetric if $A' = A$

For example,

$$\text{If } A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -3 & -1 \\ 0 & -1 & 4 \end{bmatrix} \quad \text{then} \quad A' = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -3 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

Since $A' = A$, A is symmetric matrix.

Note : 1. In a symmetric matrix $A = [a_{ij}]_{n\times n}$ $a_{ij} = a_{ji}$ for all i and j.

2. The zero matrix $O_{n\times n}$, any diagonal matrix and the unit matrix $I_{n\times n}$ are symmetric.

3. If A is a square matrix, then $A + A'$ is a symmetric matrix.

4. A rectangular matrix can ever be symmetric.

4.7.3 Skew - Symmetric Matrix.

A square matrix A is said to be skew symmetric matrix, if $A' = A$.

For example

$$\text{If } A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}, \quad \text{then} \quad A' = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -4 \\ -2 & 4 & 0 \end{bmatrix}$$

$$\text{But } -A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -4 \\ -2 & 4 & 0 \end{bmatrix}, \text{ which is same as } A'$$

$$A' = -A.$$

Hence, A is a skew symmetric matrix.

Note : 1. The zero matrix $0_{m\times n}$ is skew - symmetric.

2. If A is a square matrix, then $A - A'$ is a skew - symmetric matrix.

3. In a skew - symmetric matrix $A = [a_{ij}]_{n \times n}$, $a_{ij} = 0$, for i = j i.e., all elements in the pricipla diagonal of a skew symmetric matrix are zeroes.

Example 4.28 : If $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$ is a symmetric matrix, then find x.

Solution: Since A is a symmetric matrix A' = A.

$$\Rightarrow \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & x \\ 3 & 6 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$$

$$\Rightarrow \quad x = 6.$$

Example 4.29: If $A = \begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & 8 \\ 2 & -8 & x \end{bmatrix}$ is a skew symmetric matrix, find the value of x.

Solution : A is a skew symmetric matrix and$x$ is an element of the diagonal.

Hence $x = 0$.

Example 4.30 : For any $n \times n$ matrix A, prove that a can be uniquely expressed as a sum of a symmetric matrix and a skew symmetric matrix.

Solution : A + A' is symmetric and A – A' is skew symmetric matrix and

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

To prove uniqueness, let B be a symmetric matrix and C be a skew symmetric matrix such that A = B + C

Then A' = (B + C)' = B' + C' = B – C

and A + A' = B + C + B – C

$$\Rightarrow B = \frac{1}{2}(A + A')$$

and $C = \frac{1}{2}(A - A')$.

EXERCISE 4.7

1. If $A = \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}$ then find 2A + B' and 3B' − A.

2. If $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$ is a skew symmetric matrix, then find x.

3. If $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$ then find A + A' and AA'.

4. If $A = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 4 & 0 \\ 3 & -1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -2 & 5 \\ 1 & 2 & 0 \end{bmatrix}$ then find 3A − 4B'.

5. If $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -4 & 0 \end{bmatrix}$, then find :

   (a) A' (b) B' (c) $A + B$ (d) $(A + B)'$

   (e) $A' + B'$

6. Find A' (transpose of A) :

   (a) $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 4 & 10 & 9 \\ 6 & 8 & 7 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & -2 \\ 4 & -1 \\ -6 & 9 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. For any matrix A, prove that $(A')' = A$.

8. Show that each of the following matrices is a symmetric matrix :

(a) $\begin{bmatrix} 2 & -4 \\ -4 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -3 \\ 2 & -3 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

9. Show that each of the following matrices is a skew symmetric matrix :

(a) $\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & i & 4 \\ -i & 0 & 2-i \\ -4 & -2+i & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 & 7 \\ 1 & 0 & 5 \\ -7 & -5 & 0 \end{bmatrix}$

4.8 SOLUTIONS OF NON HOMOGENEOUS SYSTEM OF EQUATIONS

We consider solving the following system of 3 equations in unknowns

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

This system can be represented by a matrix equation AX = D where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$ is the coefficient matrix,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ is the variable matrix,

$$D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$ is the constant matrix,

$$[AD] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$ is the augmented matrix

4.8.1 Gauss - Jordan method

In this method we try to transform the augmented matrix $\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$

to the form $\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{bmatrix}$

by using elementary row transformations, so that the solution is completely visible that is $x = \alpha$, $y = \beta$ $z = \gamma$.

Note :

For solving a system of three linear equations in three unknowns by Gauss-Jordan method, elementary row operations are performed on the augmented matric as indicated below.

Step 1

(i) Transform the element in (1, 1) position to 1, by a suitable elementary row transformation using the element at (2, 1) or (3, 1) prosition or other wise.

(ii) Transform the non-zero elements, If any at (2, 1) or (3, 1) positions as zeros (other elements of the first column) by using the element 1 at (1, 1) position.

If, at the end of step 1, there is a non-zero element at (2, 2) or (3, 2) position, go to step 2 otherwise skip it.

Step 2

(i) Transform the element in (2, 2) position to 1, by a suitable elementary row transformation using the element at (3, 2) position or otherwise.

(ii) Transform the non-zero elements, It any, of the second column (i.e. the non-zero elements, if any, at (1, 2) or (3, 2) positions) as zeros, by using the element 1 at (2, 2) positon. At the end of step 2, or after skipping it for reasons specified above, examine the element at (3, 3) position, If it is non zero, go to step 3. Otherwise, stop.

Step 3

(i) Transform the element in (3, 3) postion to 1, by dividing R_3 with a suitable number.

(ii) Transform the other non-zero elements if any of the third column (that is, the non-zero elements, if any, at (1, 3) or (2, 3) positions) as zeros, by using the 1 present at (3, 3) position.

Example 4.31 : Solve the following equations by Gauss-Jordan method

$$3x + 4y + 5z = 18$$

$$2x - y + 8z = 13$$

$$5x - 2y + 7z = 20$$

Sol: The augmented matrix is $\begin{bmatrix} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$

on applying $R_1 \to R_1 - R_2$ we get

$$\sim \begin{bmatrix} 1 & 5 & -3 & 5 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$$

on applying $R_2 \to R_2 - 2R_1,\ R_3\ R_3 - 5R_1,$

$$\sim \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & -11 & 14 & 3 \\ 0 & -27 & 22 & -5 \end{bmatrix}$$

On applying $R_2 \to -5R_2 + 2R_3$, we get

$$\sim \begin{bmatrix} 1 & 5 & -3 & 5 \\ 0 & 1 & -26 & -25 \\ 0 & -27 & 22 & -5 \end{bmatrix}$$

On applying $R_1 \to R_1 - 5R_2,\ R_3 \to R_3 + 27R_2$

$$\sim \begin{bmatrix} 1 & 0 & 127 & 130 \\ 0 & 1 & -126 & -25 \\ 0 & 0 & -680 & -680 \end{bmatrix}$$

On applying $R_3 \to R_3 \div (-680)$, we get

$$\sim \begin{bmatrix} 1 & 0 & 127 & 130 \\ 0 & 1 & -126 & -25 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

On applying $R_1 \to R_1 - 127R_3,\ R_2 \to R_2 + 26R_3$, we get

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence the solution is $x = 3,\ y = 1,\ z = 1.$

Example 4.32 : By using Gauss - Jordan method, show that the following system has no solution

$$2x + 4y - z = 0, \qquad x + y + 2z = 5, \qquad 3x + 6y - 7z = 2$$

Sol: $A = \begin{bmatrix} 2 & 4 & -1 \\ 1 & 2 & 2 \\ 3 & 6 & -7 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$

The Augmented matrix is

$$[AD] = \begin{bmatrix} 2 & 4 & -1 & 0 \\ 1 & 2 & 2 & 5 \\ 3 & 6 & -7 & 2 \end{bmatrix}$$

On interchanging R_1 and R_2, we get

$$\sim \begin{bmatrix} 1 & 2 & 2 & 5 \\ 2 & 4 & -1 & 0 \\ 3 & 6 & -7 & 2 \end{bmatrix}$$

On applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ we get

$$\sim \begin{bmatrix} 1 & 2 & 2 & 5 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & -13 & -13 \end{bmatrix}$$

On applying $R_2 \to R_2 \div (-5)$, $R_3 \to R_3 \div (-13)$ we get

$$\sim \begin{bmatrix} 1 & 2 & 2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

On applying $R_3 \to R_3 - R_2$ we get

$$\sim \begin{bmatrix} 1 & 2 & 2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Hence the given system of equations is equivalent to the following system of equations

$$x + y + 2z = 5, \qquad z = 2, \quad 0(x) + 0(y) + 0(z) = -1$$

Clearly no x, y, z satisfy the last equation in the above system.

Hence the given system has no solution.

EXERCISE 4.7

Solve the following systems of equations by using Gauss-Jordan method.

1. $x + y + z = 1$

   $2x + 2y + 3z = 6$

   $x + 4y + 9z = 3$

2. $x - y + 3z = 5$

   $4x + 2y - z = 0$

   $-x + 3y + z = 5$

3. $2x - y + 3z = 9$

   $x + y + z = 6$

   $x - y + z = 2$

4. $2x - y + 8z = 13$

   $3x + 4y + 5z = 18$

   $5x - 2y + 7z = 20$

4.8.2 Definition (Consistent and inconsistent systems)

We say that a system of linear equations is

(i) consistent if it has a solution

(ii) inconsistent if it has no solution

4.8.3 Solution of non-homogeneous system of equations

We consider soluving the following system of 3 equations in 3 unkowns

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

This system can be represented by a matrix equation

AX = D, where

$$[AD] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times3}$$ is the coefficient matrix,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1}$$ is the variable matrix,

$$D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}_{3\times1}$$ is the constant matrix,

$$[AD] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}_{3\times4}$$ is the augmented matrix.

Theorem 1

The system of three equations in three unknowns AX = D has

(i) a uniqu solutin if rank (A) = rank([AD]) = 3

(ii) infinitely many solutions if rank (A) = rank ([AD]) < 3

(iii) no solution if rank (A) $\neq$ rank ([AD])

Note that the system is consistent if and only if $\Leftrightarrow$ rank (A) = rank ([AD]).

Example 4.33 : Show that the system of equations given below is not consistant

$$2x + 6y = -11$$

$$6x + 20y - 6z = -3$$

$$6y - 18z = -1$$

Solution : The given system of equations can be written in the form AX = Dm where

$$A = \begin{bmatrix} 2 & 6 & 10 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad D = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}$$

Consider the augmented matrix

$$[AD] = \begin{bmatrix} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}$$

on applying $R_2 \to R_2 - 3R_1$, we get

$$[AD] = \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{bmatrix}$$

on applying $R_3 \to R_3 - 3R_2$, we get

$$AD \sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & 0 & -91 \end{bmatrix}$$

is non-singular its determinant is $-91(6)(-6) \neq 0$

But the rank of the coefficient matrix is not 3 because

$$\begin{bmatrix} 6 & 0 & -11 \\ 2 & -6 & 30 \\ 0 & 0 & -91 \end{bmatrix},$$

$$\det\begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ rank of (A) $\neq$ rank ([AD])

Hence the given system is inconsistent.

Example 4.34: Why do we use only elementary row transformations?

Let us apply elementary column transformations to the augmented matrix of example 3.

Sol: $[AD] \sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}$

on applying $C_2 \to C_2 - 3C_1$, we get

$$[AD] \sim \begin{bmatrix} 2 & 0 & 0 & -11 \\ 6 & 2 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}$$

on applying $C_1 \to C_1 - 3C_2$, C_3 and $C_3 + 3C_2$ we get

$$AD \sim \begin{bmatrix} 2 & 0 & 0 & -11 \\ 0 & 2 & 0 & -3 \\ -18 & 6 & 0 & -1 \end{bmatrix}$$

Now we can easily observe that the rank of the coefficient matrix is $\neq 3$, as

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -18 & 6 & 0 \end{bmatrix}, \text{ is singular}$$

The rank of the augmented matrix is 3, since the sub matrix

$$\begin{bmatrix} 2 & 0 & -11 \\ 0 & 2 & -3 \\ -18 & 6 & -1 \end{bmatrix} \text{is non-singular}$$

(determinant $2(-2+18) - 11(36) = 32 - 11\times 36 \neq 0$)

Hence rank (A) $\neq$ rank ([AD]).

Hence the system is inconsistent.

Thus, we can use either row trnasformations of column transformations to find whether a system is consistent or inconsistent.

Example 4.36 : Apply the test of rank to examine whether the following equations are consistent

$2x - y + 3z = 8$

$-x + 2y + z = 4$

$3x + y - 4z = 0$

and if consistent, find the complete solution.

Sol: The augmented matrix is $[AD] = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 8 \\ 3 & 1 & -4 & 0 \end{bmatrix} \text{(on interchanging } R_1 \text{ and } R_2\text{)}$$

we transform the above matrix into an upper triangular matrix.

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{bmatrix} \text{(on applying } R_2 \to R_2 + 2R_1,\ R_3 \to R_3 + 3R_1\text{)}$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -38 & -76 \end{bmatrix} \text{(on applying } R_3 \to 3R_3 - 7R_2\text{)}$$

Now $\det \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & -38 \end{bmatrix} = (-1)(3)(-38) = 114$

Hence rank (A) = rank ([AD]) = 3

$-x + 2y + z = 4$

$3y + 5z = 16$

$-38z = -76$

The system has a unique solution.

$\therefore\ z = 2, y = 2, x = 2$ is the solution..

Example 4.37: Show that the following system of equations is consistent and solve it completely :

$x + y + z = 3$

$2x + 2y - z = 3$

$x + y - z = 1$

Sol: The given equations are equivalent to the equation AX = D, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } D = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

Augmented matrix $[AD] \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & -1 & 3 \\ 1 & 1 & -1 & 1 \end{bmatrix}$

On applying $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$ we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

On applying $R_3 \to 3R_3 - 2R_2$ we get

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly all the submatrices of order 3 of the above matrix are singular

Hence rank $[A] \neq 3$ and rank $([AD]) \neq 3$.

Now the non singular matrix $\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$ is a submatrix of both A and [AD]

Hence rank (A) = rank ([AD]) = 2

Hence by theorem 1, the system is consistent and has infinitely many solutions.

we now write the equivalent set of equations.

$x + y + z = 3$

$-3z = -3$

Hence $z = 1$, $x + y = 2$.

Hence $x = k$, $y = 2 - k$, $z = 1$, $k \in R$ is the solution set.

EXERCISE 4.8

I. Examine whether the following systems of equations are consistent or inconsistent and it fonsistent find the complete solutions.

1. $x + y + z = 1$

   $2x + y + z = 2$

   $x + 2y + 2z = 1$

2. $x - 3y - 8z = -10$

   $3x + y - 4z = 0$

   $2x + 5y + 6z = 13$

3. $x + y + 4z = 6$

   $3x + 2y - 2z = 9$

   $5x + y + 2z = 13$

KEY WORDS

- A rectangular array of numbers, arranged in the form of rows and columns is called a *matrix*. Each number is called an *element* of the matrix.
- The *order* of a matrix having 'm' rows and 'n' columns is $m \times n$.
- If the number of rows is equal to the number of columns in a matrix, it is called a square matrix.
- A diagonal matrix is a square matrix in which all the elements, except those on the diagonal, are zeroes.
- A unit matrix of any order is a diagonal matrix of that order whose all the diagonal elements are 1.
- Zero matrix is a matrix whose all the elements are zeroes.
- Two matrices are said to be equal if they are of the same order and their corresponding elements are equal.
- A transpose of a matrix is obtained by interchanging its rows and columns.
- Matrix A is said to be symmetric if $A' = A$ and skew symmetric if $A' = -A$
- Scalar multiple of a matrix is obtained by multiplying each elements of the matrix by the scalar.
- The sum of two matrices (of the same order) is a matrix obtained by adding corresponding elements of the given matrices
- Difference of two matrices A and B is nothing but the sum of matrix A and the negative of matrix B.
- Product of two matrices A of order $m \times n$ and B of order $n \times p$ is a matrix of order $m \times p$, whose elements can be obtained by multiplying the rows of A with the columns of B element wise and then taking their sum.

1. A system of linear equations is

   (i) consistent if it has a solution

   (ii) inconsistent if it has no solution.

2. Non homogeneous system

   $a_1x + b_1y + c_1z = d_1$

   $a_2x + b_2y + c_2z = d_2$

   $a_3x + b_3y + c_3z = d_3$

   The above system of equations has

   (i) a unique solution if rank (A) = rank ([AD]) = 3

   (ii) infinitely many solutions if rank (A) = rank ([AD]) < 3

   (iii) no solution if rank (A) $\neq$ rank ([AD]) .

SUPPORTIVE WEB SITES

http : //www.wikipedia.org

http:// math world . wolfram.com

PRACTICE EXERCISE

1. How many elements are there in a matrix of order

   (a) 2×1 (b) 3×2 (c) 3×3 (d) 3×4

2. Construct a matrix of order 3×2 whose elements aij are given by

   (a) $a_{ij} = i - 2j$ (b) $a_{ij} = 3i - j$ (c) $a_{ij} = i + \frac{3}{2}j$

3. What is the order of the matrix?

   (a) $A = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$

   (c) $C = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $D = \begin{bmatrix} 2 & -1 & 5 \\ 7 & 6 & 1 \end{bmatrix}$

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4. Find the value of x, y and z if

(a) $\begin{bmatrix} x & y \\ z & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} x+y & z \\ 6 & x-y \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 6 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} x-2 & 3 \\ 0 & y+5 \end{bmatrix} = \begin{bmatrix} 1 & z \\ y+z & 2 \end{bmatrix}$ (d) $\begin{bmatrix} x+y & y-z \\ z-2x & y-x \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & -2 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & 4 \end{bmatrix}$, find :

(a) $A+B$ (b) $2A$ (c) $2A-B$

6. Find X. if

(a) $\begin{bmatrix} 4 & 5 \\ -3 & 6 \end{bmatrix} + X = \begin{bmatrix} 10 & -2 \\ 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -3 & 2 \\ 2 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} + X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7. Find the values of a and b so that

$$\begin{bmatrix} 3 & -2 & 2 \\ 1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} a-b & 2 & -2 \\ 4 & a & b \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 2a+b & 5 \end{bmatrix}$$

8. For matrices A, Band C

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 5 & 7 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 4 \\ 3 & 7 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 6 \\ 7 & 1 \\ 4 & 1 \end{bmatrix}$$

verify that $A+(B+C) = (A + B) + C$

9. If $A = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 & 5 \end{bmatrix}$ find AB and BA. Is AB= BA?

10. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$ find AB and BA. Is AB = BA?

11. If $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$, find A^2.

12. Find A(B + C), if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \text{and } C = \begin{bmatrix} -2 & 0 & 3 \\ 4 & 0 & -3 \end{bmatrix}$$

13. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} x & 1 \\ y & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$ find the values of x and y.

14. Show that $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$ satisfies the matrix equation $\mathbf{A}^2 + 4A - 2I = 0$.

ANSWERS

EXERCISE 4.1

1. $\begin{bmatrix} 56 & 65 & 71 \\ 29 & 37 & 57 \end{bmatrix}; \begin{bmatrix} 56 & 29 \\ 65 & 37 \\ 71 & 57 \end{bmatrix}$

2. $\begin{bmatrix} 40 & 35 & 25 \\ 10 & 5 & 8 \end{bmatrix}; \begin{bmatrix} 40 & 10 \\ 35 & 5 \\ 25 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 6 & 3 \\ 4 & 3 & 5 \end{bmatrix}$

4. (a) 6 (b) 12 (c) 8 (d) 12 (e) ab (f) *mn*

5. (a) 1 × 8 2 × 4 4 × 2 8 × 1 (b) 1 × 5 5 × 1
   (c) 1 × 12 2 × 6 3 × 4 4 × 3 6 × 2 12 × 1
   (d) 1 × 16 2 × 8 4 × 4 8 × 2 16 × 1

6. (a) 4 (b) 5 (c) 4×5 (d) 20
   (e) $a_{14} = 0$; $a_{23} = 7$; $a_{34} = -3$; $a_{45} = 1$ and $a_{33} = 3$

7. (a) $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 4 & 2 & \frac{4}{3} \\ 9 & \frac{9}{2} & 3 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} & \frac{49}{2} \\ 8 & 18 & 32 \\ \frac{25}{2} & \frac{49}{2} & \frac{81}{2} \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 4 & 7 \\ -1 & 2 & 5 \\ -3 & 0 & 3 \end{bmatrix}$

8. (a) $\begin{bmatrix} 4 & 7 \\ 5 & 8 \\ 6 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 10 \\ 10 & 20 \\ 15 & 30 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix}$

EXERCISE 4.2

1. (a) G (b) B (c) A, D, E and F
   (d) A, D and F (e) D and F (f) F
   (g) C
2. (a) $a = 2,\ b = 10,\ c = 6,\ d = -2$
   (b) $a = 2,\ b = 3,\ c = 2,\ d = 5$
   (c) $a = \frac{3}{2},\ b = -2,\ c = 2,\ d = -4$
3. No
4. No

EXERCISE 4.3

1. (a) $\begin{bmatrix} 28 & 8 \\ 8 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} -7 & -2 \\ -2 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{7}{2} & 1 \\ 1 & \frac{3}{2} \end{bmatrix}$

   (d) $\begin{bmatrix} \frac{-21}{2} & -3 \\ -3 & \frac{-9}{2} \end{bmatrix}$

2. (a) $\begin{bmatrix} 0 & -5 & 10 \\ 15 & 5 & 20 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 3 & -6 \\ -9 & -3 & -12 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & \frac{-1}{3} & \frac{2}{3} \\ 1 & \frac{1}{3} & \frac{4}{3} \end{bmatrix}$ (d) $\begin{bmatrix} 0 & \frac{1}{2} & -1 \\ \frac{-3}{2} & \frac{-1}{2} & -2 \end{bmatrix}$

3. $\begin{bmatrix} 7 & 0 \\ -28 & -14 \\ 0 & 7 \end{bmatrix}$

4. (a) $\begin{bmatrix} 15 & 0 & 5 \\ 20 & -10 & 0 \\ -5 & 0 & 25 \end{bmatrix}$ (b) $\begin{bmatrix} -12 & 0 & -4 \\ -16 & 8 & 0 \\ 4 & 0 & -20 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ \frac{4}{3} & \frac{-2}{3} & 0 \\ \frac{-1}{3} & 0 & \frac{5}{3} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-3}{2} & 0 & \frac{-1}{2} \\ -2 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{-5}{2} \end{bmatrix}$

EXERCISE 4.4

1. (a) $\begin{bmatrix} 3 & -2 \\ 8 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & -3 \\ 13 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -4 \\ 14 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 6 & -5 \\ 19 & 10 \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & 3 & 6 \\ -5 & 3 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -3 & -6 \\ 5 & -3 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 9 \\ -9 & 2 & 10 \end{bmatrix}$ (d) $\begin{bmatrix} -4 & -11 & -15 \\ 11 & -10 & -10 \end{bmatrix}$

3. (a) $\begin{bmatrix} 0 & -6 & 3 \\ 5 & 5 & 3 \\ 6 & 5 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 & 3 \\ 3 & -7 & 1 \\ 2 & 5 & -7 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & -2 & -3 \\ -3 & 7 & -1 \\ -2 & -5 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -14 & 9 \\ 14 & 9 & 8 \\ 16 & 15 & 14 \end{bmatrix}$

4. (a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

5. (a) $\begin{bmatrix} 2 & 1 & 0 \\ -1 & -2 & -3 \\ 4 & 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6. (a) $\begin{bmatrix} 2 & 18 \\ 6 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 15 & 3 \\ 21 & 27 \end{bmatrix}$ (c) $\begin{bmatrix} 17 & 21 \\ 27 & 31 \end{bmatrix}$

(d) $\begin{bmatrix} \frac{-17}{5} & \frac{-21}{5} \\ \frac{-27}{5} & \frac{-31}{5} \end{bmatrix}$

7. (a) $\begin{bmatrix} 0 & 2 \\ -5 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 3 \\ 0 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 5 \\ -5 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 8 \\ 3 & -7 \end{bmatrix}$ (e) $\begin{bmatrix} -2 & 3 \\ -8 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} -5 & 9 \\ -16 & -3 \end{bmatrix}$

EXERCISE 4.5

1. $AB = [-6]$; $BA = \begin{bmatrix} 0 & 0 & 0 \\ -4 & -6 & 0 \\ 2 & 3 & 0 \end{bmatrix}$ $AB \neq BA$

2. $AB = \begin{bmatrix} 2 & 2 \\ 1 & -6 \end{bmatrix}$; $BA = \begin{bmatrix} -3 & 13 & -4 \\ 1 & -1 & -2 \\ 2 & -6 & 0 \end{bmatrix}$ $AB \neq BA$

3. $AB = \begin{bmatrix} ax & ay & az \\ bx & by & bz \end{bmatrix}$; BA does not exist.

4. $BA = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$; AB does not exist.

5. Both AB and BA do not exist. AB does not exist since the number of columns of A is not equal to the number of rows of B. BA also does not exist since number of coluumns of B is not equal to the number of rows of A.

6. $AB = \begin{bmatrix} 0 & 5 \\ 6 & 15 \end{bmatrix}$; $BA = \begin{bmatrix} -2 & -1 \\ 4 & 17 \end{bmatrix}$; $AB \neq BA$

7. $AB = \begin{bmatrix} 4 & -3 & 7 \\ 3 & 17 & 24 \\ 14 & -13 & 17 \end{bmatrix}$; $BA = \begin{bmatrix} 16 & -8 & -11 \\ 16 & 11 & 3 \\ 10 & 21 & 11 \end{bmatrix}$; $AB \neq BA$

8. $AB = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$; $BA = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$; $AB \neq BA$.

9. (a) $x = 3,\ y = -1$ (b) $x = -1,\ y = 2$

12. (a) $\begin{bmatrix} -14 & 18 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} -14 & 18 \\ 2 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 5 & 0 \\ 7 & 8 \end{bmatrix}$ (f) $\begin{bmatrix} -2 & -3 \\ 9 & 15 \end{bmatrix}$

13. (a) $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$; $AC = BC$.

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Here, $A \neq B$, and $C \neq 0$, yet $AC = BC$.

i.e. cancellation law does not hold good for matrices.

14. (a) $\begin{bmatrix} 4 & 7 \\ 9 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & -7 \\ -14 & 9 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 1 \\ -3 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} -3 & -8 \\ -11 & 10 \end{bmatrix}$ (e) $\begin{bmatrix} -4 & -7 \\ -14 & 9 \end{bmatrix}$

We observe that $A(B+C) = AB + AC$.

16. $X = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ 18. (a) No (b) No (c) No

EXERCISE 4.7

1. $\begin{bmatrix} -6 & 6 \\ 13 & 0 \\ -1 & 10 \end{bmatrix}, \begin{bmatrix} -4 & 11 \\ 4 & 0 \\ 4 & 2 \end{bmatrix}$

2. $x = 2$

3. $\begin{bmatrix} 4 & -9 \\ -9 & 6 \end{bmatrix}, \begin{bmatrix} 20 & -22 \\ -22 & 34 \end{bmatrix}$

4. $\begin{bmatrix} -5 & 15 & 5 \\ 10 & 20 & -8 \\ 9 & -23 & -15 \end{bmatrix}$

5. (a) $\begin{bmatrix} 2 & 4 \\ 0 & 3 \\ -1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 \\ 0 & -4 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 6 & -1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 6 \\ 0 & -1 \\ 0 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 6 \\ 0 & -1 \\ 0 & 2 \end{bmatrix}$

We observe that $(A + B)' = B' + A'$

6. (a) $\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 10 & 8 \\ 9 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 4 & -6 \\ -2 & -1 & 9 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

EXERCISE 4.7

1. $x = 7, y = -10, z = 4$ unique solution
2. $x = 0, y = 1, z = 2$ unique solution
3. $x = 1, y = 2, z = 3$ unique solution
4. $x = 3, y = 1, z = 1$ unique solution

EXERCISE 4.8

1. Consistent, Infinitely many solution:

   Solution set « $\{(x, y, z) : x = 1, y + z = 0\}$
2. Consistent, Infinitly many solutions;

   $x = -1 + 2k, \ y = 3 - 2k, z = k \ k$ is a scalar
3. Consistent; unique solution; $x = 2, \ y = 2, z = 1/2$.

PRACTICE EXERCISE

1. (a) 2 (b) 6 (c) 9 (d) 12

2. (a) $\begin{bmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 5 & 4 \\ 8 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{5}{2} & 4 \\ \frac{7}{2} & 5 \\ \frac{9}{2} & 6 \end{bmatrix}$

3. (a) 3×1 (b) 1×3 (c) 3×2 (d) 2×3

4. (a) $x = 1, y = 2, z = 3$ (b) $x = 5, y = 1, z = 5$

(c) $x = 3, y = -3\ z = 3$ (d) $x = 2, y = 1, z = 5$

5. (a) $\begin{bmatrix} 3 & 2 \\ 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -4 \\ 8 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -8 \\ 9 & 0 \end{bmatrix}$

6. (a) $\begin{bmatrix} 6 & -7 \\ 4 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 4 & -3 \\ -3 & 0 & -1 \end{bmatrix}$

7. $a = \frac{3}{2}$ $b = -\frac{3}{2}$

9. $AB = \begin{bmatrix} 13 & 11 \\ 38 & 43 \end{bmatrix}$; $BA = \begin{bmatrix} 5 & 10 & 17 \\ 6 & 14 & 24 \\ 4 & 21 & 37 \end{bmatrix}$; $AB \neq BA$.

10. $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $AB = BA$.

11. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 9 & 1 & 1 \\ -1 & -4 & 10 \end{bmatrix}$

13. $x = 1, \ y = -4.$

Chapter 5

DETERMINANTS AND THEIR APPLICATIONS

LEARNING OUTCOMES

After studying this chapter, student will be able to :

- Compute determinant of square matrix.
- State the properties of determinants
- Evaluate a given determent by using properties of determinants
- Solve a system of linear equations by applying Cramer's rule.

PREREQUISITES

- System of linear equations, Number system including complex number.

INTRODUCTION

The determinant of a matrix is a number that is specially defined only for square matrices. Determinants are very useful in the analysis and solution of systems of linear equations. Determinants also have wide applications in engineerging, science and social science as well. In this chapter we will study various properties of deterinants and also learn to solve system of linear equations by Cramer's rule.

5.1 DETERMINANT OF ORDER 2

Let us consider the following system of linear equations:

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

On solving this system of equations for x and y, we get

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \quad \text{and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \text{ provided } a_1b_2 - a_2b_1 \neq 0.$$

The number $a_1b_2 - a_2b_1$ determines whether the values of x and y exist or not.

The number $a_1b_2 - a_2b_1$ is called the value of the determinant, and we write

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

i.e., a_{11} belongs to the 1st row and 1st column

a_{12} belongs to the 1st row and 2nd column

a_{21} belongs to the 2nd row and 1st column

a_{22} belongs to the 2nd row and 2nd column

5.2 EXPANSION OF A DETERMINANT OF ORDER 2

A formal rule for the expansion of a determinant of order 2 may be stated as follows:

In the determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

write the elements in the following manner:

$$\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}$$

Multiply the elements by the arrow. The sign of the arrow going downwards is positive, i.e., $a_{11}\ a_{22}$ and the sign of the arrow going upwards is negative, i.e., $-\ a_{21}\, a_{22}$.

Add these two products, i.e., $a_{11}.a_{22} + (-a_{21}.a_{12})$ or $a_{11}a_{22} - a_{21}a_{12}$ which is the required value of the determinant.

Example 5.1: Evaluate:

(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix}$ (iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} 6 & 4 \\ 8 & 2 \end{vmatrix} = (6 \times 2) - (8 \times 4) = 12 - 32 = -20$

(ii) $\begin{vmatrix} a+b & 2b \\ 2a & a+b \end{vmatrix} = (a+b)(a+b) - (2a)(2b)$

$= a^2 + 2ab + b^2 - 4ab$

$= a^2 + b^2 - 2ab$

$= (a - b)^2$

(iii) $\begin{vmatrix} x^2+x+1 & x+1 \\ x^2-x+1 & x-1 \end{vmatrix} = (x^2 + x + 1)\,(x - 1) - (x^2 - x + 1)(x + 1)$

$= (x^3 - 1) - (x^3 + 1)$

$= -2.$

Example 5.2 : Find the value of x if

(i) $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = 6$ (ii) $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = 0$

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Solution:

(i) Now $\begin{vmatrix} x-3 & x \\ x+1 & x+3 \end{vmatrix} = (x-3)(x+3) - x(x+1)$

$= (x^2 - 9) - x^2 - x = -x - 9$

According to the question,

$= -x - 9 = 6$

$\Rightarrow \quad x = -15$

(ii) Now $\begin{vmatrix} 2x-1 & 2x+1 \\ x+1 & 4x+2 \end{vmatrix} = (2x-1)(4x+2) - (x+1)(2x+1)$

$= 8x^2 + 4x - 4x - 2 - 2x^2 - x - 2x - 9$

$= 6x^2 - 3x - 3 = 3(2x^2 - x - 1)$

According to the question,

$3(2x^2 - x - 1) = 0$

or, $2x^2 - x - 1 = 0$

or, $2x^2 - 2x + x - 1 = 0$

or, $2x(x - 1) + 1(x - 1) = 0$

or, $(2x + 1)(x - 1) = 0$

or, $x = 1, -\frac{1}{2}$.

5.3 DETERMINANT OF ORDER 3

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ contains nine quantities $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3 aranged in 3 rows and 3 columns, is called determinant of order 3 (or a determinant of third order). A determinant of order 3 has (3)2 = 9 elements.

Using double subscript notations, viz., $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ for the elements $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3$ and c_3. we write a determinant of order 3 as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Usually a determinant, whether of order 2 or 3, is denoted by Δ or $|A|, |B|$ .....etc.,

$$\Delta = |a_{ij}|, \text{ where } i = 1, 2, 3 \text{ and } j = 1, 2, 3.$$

5.4 DETERMINANT OF A SQUARE MATRIX

With each square matrix of numbers (we associate) a "determinant of the matrix". With the 1×1 matrix $[a]$, we associate the determinant of order 1 and with the only element a. The value of the determinant is a.

If $|A| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a square matrix of order 2, then the expression $a_{11}a_{22} - a_{21}a_{12}$ is defined as the determinant of order 2. It is denoted by

$$|A| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

With the 3×3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, we associate the determinant,

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and its value is defined to be

$$a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

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Example 5.3: If $A = \begin{bmatrix} 3 & 6 \\ 1 & 5 \end{bmatrix}$, find $|A|$.

Solution: $A = \begin{vmatrix} 3 & 6 \\ 1 & 5 \end{vmatrix} = 3 \times 5 - 1 \times 6 = 15 - 6 = 9$

Example 5.4: If $A = \begin{bmatrix} a+b & a \\ b & a-b \end{bmatrix}$ find $|A|$

Solution: $|A| = \begin{vmatrix} a+b & a \\ b & a-b \end{vmatrix} = (a+b)(a-b) - b \times a = a^2 - b^2 - ab$

Note: 1. The determinant of a unit matrix I is 1.

2. A square matrix whose determinant is zero, is called the singular matrix.

5.5 EXPANSION OF A DETERMINANT OF ORDER 3

In Section 4.4, we have written

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \times \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)a_{12} \times \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \times \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

which can be further expanded as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23})$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

We notice that in the above method of expansion, each element of first row is multiplied by the second order determinant obtained by deleting the row and column in which the element lies.

Further, mark that the elements all a_{11}, a_{12} have been assigned positive, negative and positive signs, respectively. In other words, they are assigned positive and negative signs, alternatively, starting with positive sign. If the sum of the subscripts of the elements is an even number, we assign positive sign and if it is an odd number, then we assign negative sign.

Therefore, a_{11} has been assigned positive sign.

Note: We can expand the determinant using any row or column. The value ofthe determinant will be the same whether we expand it using first row or first column or any row or column, taking into consideration rule of sign as explained above.

Example 5.5: Expand the determinant, using the first row

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix}$$

Solution: $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 5 \end{vmatrix} = 1 \times \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$

$$= 1 \times (20 - 2) - 2 \times (10 - 3) + 3 \times (4 - 12)$$

$$= 18 - 14 - 24$$

$$= -20.$$

Example 5.6 : Expand the determinant, by using the second column.

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-1) \times 2 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + (-1)3 \times \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix}$

$$= -2 \times (3 - 4) \{ 1 \times (1 - 6) – 3 \times (2 - 9)$$

$$= 2 - 5 + 21$$

$$= 18.$$

EXERCISE 5.1

1. Find |A| if

(a) $A = \begin{bmatrix} 2+\sqrt{3} & 5 \\ 2 & 2-\sqrt{3} \end{bmatrix}$ (b) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

(c) $A = \begin{bmatrix} \sin\alpha+\cos\beta & \cos\beta+\cos\alpha \\ \cos\beta-\cos\alpha & \sin\alpha-\sin\beta \end{bmatrix}$ (d) $\begin{bmatrix} a+bi & c+di \\ c-di & a-bi \end{bmatrix}$

2. Find which of the following matrices are singular matrices :

(a) $\begin{bmatrix} 5 & 5 & 1 \\ -5 & 1 & 1 \\ 10 & 7 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 & 0 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$

3. Expand the determinant by using first row

(a) $\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 1 & -5 \\ 0 & -3 & 0 \\ 4 & 2 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}$

(d) $\begin{vmatrix} x & y & z \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}$

5.6 MINORS AND COFACTORS

5.6.1 Minor of a_{ij} in |A|

To each element of a determinant, a number called its minor is associated.

The minor of an element is the value of the determinant obtained by deleting the row and column containing the element.

Thus, the minor of an element a_{ij} in |A| is the value of the determinant obtained by deleting the i^{th} row and j^{th} column of |A| and is denoted by M_{ij}. For example, minor of 3 in the determinant $\begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix}$ is 7.

Example 5.7 : Find the minors of the elements of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution : Let M_{ij} denote the minor of a_{ij}. Now, a_{11} occurs in the 1st row and 1st column. Thus to find the minor of a_{11}, we delete the 1 st row and 1 st column of |A|.

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

Similarly, the minor M_{12} of a_{12} is given by

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31};$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{32}a_{13};$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{31}a_{13}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$$

Similarly we can find M_{31}, M_{32} and M_{33}.

5.6.2 *Cofactors of a_{ij} in $|A|$*

The cofactor of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$ It is usually denoted by C_{ij}, Thus,

Cofactor of $a_{ij} = C_{ij} = (-1)^{i+j}\ M_{ij}$.

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Example 5.8: Find the cofactors of the elements a_{11}, a_{12} and a_{21} of the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution: The cofactor of any element a_{ij} is $(-1)^{i+j}$ M_{ij}, then

$$C_{11} = (-1)^{1+1} M_{11} = (-1)^2 (a_{22} a_{33} - a_{32} a_{23})$$

$$= (a_{22} a_{33} - a_{32} a_{23})$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(a_{21} a_{33} - a_{31} a_{23}) = (a_{31}a_{23} - a_{21}a_{33})$$

and $$C_{21} = (-1)^{2+1} M_{21} = -M_{21} = (a_{32} a_{13} - a_{12} a_{33})$$

$$= (a_{12} a_{33} - a_{32} a_{13}).$$

Example 5.9: Find the minors and cofactors of the elements of the second row in the determinant

$$|A| = \begin{vmatrix} 1 & 6 & 3 \\ 5 & 2 & 4 \\ 7 & 0 & 8 \end{vmatrix}$$

Solution: The elements of the second row are $a_{21} = 5;\ a_{22} = 2;\ a_{23} = 4.$

$$\text{Minor of } a_{21} \text{ (i.e., 5)} = \begin{vmatrix} 6 & 3 \\ 0 & 8 \end{vmatrix} = 48 - 0 = 48$$

$$\text{Minor of } a_{22} \text{ (i.e., 2)} = \begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} = 8 - 21 = -13$$

$$\text{and Minor of } a_{23} \text{ (i.e., 4)} = \begin{vmatrix} 1 & 6 \\ 7 & 0 \end{vmatrix} = 0 - 42 = -42$$

The corresponding cofactors will be

$$C_{21} = (-1)^{2+1} M_{21} = (-48) = -48$$

$$C_{22} = (-1)^{2+2} M_{22} = +(-13) = -13$$

and $$C_{23} = (-1)^{2+3} M_{23} = -(-42) = 42$$

5.7 VALUE OF A DETERMINANT USING COFACTOR

With the cofactor notation, we can give expansion of a determinant in terms of the elements of anyone row or column and their corresponding cofactors.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

where C_{ij} is the cofactor of the element a_{ij}

Example 5.10 : Evaluate $\Delta = \begin{vmatrix} 3 & 1 & 7 \\ -6 & 2 & -3 \\ 8 & 4 & 5 \end{vmatrix}$ in terms of the elements of fIrst row and their corresponding cofactors, and then in terms of the elements of first column and their corresponding cofactors. Verify that both the results are same.

Solution: Expanding the determinant in terms of the elements of first row and their corresponding cofactors, we have the value of Δ, if we expand by R_1,

$$\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

First we need to evaluate C_{11}, C_{12} and C_{13}

$$C_{11} = (-1)^{1+1}M_{11} = (-1)^2 \begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}$$

$$= 10 + 12 = 22$$

$$C_{12} = (-1)^{1+2}M_{12} = (-1) \begin{vmatrix} -6 & -3 \\ 8 & 5 \end{vmatrix}$$

$$= -(-30 + 24) = 6$$

$$C_{13} = (-1)^{1+3}M_{13} = (-1)^{1+3} \begin{vmatrix} -6 & 2 \\ 8 & 4 \end{vmatrix}$$

$$= -24 - 16 = -40$$

$$\therefore \Delta = 3 \times 22 + 1 \times 6 + 7 \times (-40) = 66 + 6 - 280 = -208$$

Expanding the deteminant Δ in terms of the elements of first column and their corresponding cofactors, we have

$$\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Now, $C_{11} = (-1)^{1+1}M_{11} = (-1)^2 \begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix} = 10 + 12 = 22$

$$C_{21} = (-1)^{2+1}M_{21} = (-1)^3 \begin{vmatrix} 1 & 7 \\ 4 & 5 \end{vmatrix} = -(5-28) = 23$$

$$C_{31} = (-1)^{3+1}M_{31} = (-1)^4 \begin{vmatrix} 1 & 7 \\ 2 & -3 \end{vmatrix} = (-3-14) = -17$$

$$\therefore \Delta = 3 \times 22 + (-6) \times 23 + 8 \times (-17) = 66 - 138 - 136 = -208$$

The value of the determinant by both the methods of expansion is -208. Thus, we verify that both the results are same.

Example 5.11: Show that the sum of the products of the elements of one row (or column) by the cofactors of another row (or column) is always zero.

Solution:

Consider, $\Delta = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$

Here, $C_{11} = 5$ and $C_{12} = -4$

Let us find out $a_{21} \times C_{11} + a_{22} \times C_{12}$

$$= 4 \times 5 + 5 \times (-4)$$

$$= 20 - 20 = 0$$

Similarly, if we consider

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

Here $\quad C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -7$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -(1-6) = 5$$

Let us find out

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$$

$$= 1 \times (-1) \{ 2(-7) \{ 3 \times (5)$$

$$= -1 - 14 + 15 = 0$$

We can verify this property for any other row or column.

5.8 SARRUS DIAGRAM FOR THE EXPANSION OF DETERMINANT OF ORDER 3

Another simple method for evaluating a determinant of order 3 is Sarrus diagram.

Consider the detenninant :

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

First of all we write down the three columns of the detenninant, and then write the first row columns again as shown below

$$\begin{array}{ccccccccc} a_{11} & & a_{12} & & a_{13} & & a_{11} & & a_{12} \\ a_{21} & \searrow & a_{22} & \nearrow\!\!\!\!\!\searrow & a_{23} & \nearrow\!\!\!\!\!\searrow & a_{21} & \nearrow & a_{22} \\ a_{31} & \nearrow & a_{32} & \nearrow\!\!\!\!\!\searrow & a_{33} & \nearrow\!\!\!\!\!\searrow & a_{31} & \searrow & a_{32} \end{array}$$

From the sum oftenns connected with downward arrows, we subtract the tenns connected with upwards arrows. Thus we get

$$a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32}$$

$$-a_{31}\,a_{22}\,a_{13} - a_{32}\,a_{23}\,a_{11} - a_{33}\,a_{21}\,a_{12}$$

$$= a_{11}(a_{22}\,a_{33} - a_{32}\,a_{23}) - a_{12}(a_{21}\,a_{33} - a_{23}a_{31}) + a_{13}(a_{21}\,a_{32} - a_{31}a_{22})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

which is equal to the value of given detenninant.

Example 5.12: Using Sarrus diagram, find the value of the detenninant

$$\Delta = \begin{vmatrix} -1 & 2 & 3 \\ 7 & 5 & 0 \\ 3 & -2 & -4 \end{vmatrix}$$

Solution:

$$\begin{array}{ccccc} -1 & 2 & 3 & -1 & 2 \\ 7 & 5 & 0 & 7 & 5 \\ 3 & -2 & -4 & 3 & -2 \end{array}$$

The value of the detenninant

= (–1) (5) (–4) { 2.0.3 { 3.7(–2) – 3.5.3–(–2).0. (–1) – (–4).7.2

= 20 + 0 – 42 – 45 – 0 + 56

= – 11.

EXERCISE 5.2

1. Find the minors and cofactors of the elements of the second row of the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$$

2. Find the minors and cofactors of the elements of the third column of the determinat

$$\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

3. Evaluate each of the following determinants using cofactors:

(a) $\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 3 & -4 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ (e) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ (f) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

4. Using Sarrus diagram, evaluate each of the following determinants:

(a) $\begin{vmatrix} 2 & 1 & -3 \\ 1 & 1 & -2 \\ 2 & -2 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 2 & 7 \\ 1 & 2 & 5 \\ -1 & 2 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} -1 & -a & b \\ a & -1 & c \\ b & c & 1 \end{vmatrix}$

5. Using Sarrus diagram, show that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

6. Solve for x, the following equations:

(a) $\begin{vmatrix} x & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} x & 3 & 3 \\ 3 & 3 & x \\ 2 & 3 & 3 \end{vmatrix} = 0$ (c) $\begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 28$

5.9 PROPERTIES OF DETERMINANTS

(i) If each element of a row (or column) of a square matrix is zero, then the determinant of that matrix is zero.

The value of the determinant of such a matrix can be easily found to be zero by expanding it along a row (column) containing zeros.

(ii) If two rows (or columns) of a square matrix are interchanged, then the sign of the determinant changes.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

(B is obtained by interchanging first and second rows of A)

$$\det B = a_1(-1)^{2+1}(b_2c_3 - b_3c_2) + b_1(-1)^{2+2}(a_2c_3 - a_3c_2) + c_1(-1)^{2+3}(a_2b_3 - a_3b_2)$$

$$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$

$$= -\det A.$$

(iii) If each element of a row (or column) of a square matrix is multiplied by a number k, then the determinant of the matrix obtained is k times the determinant of the given matrix.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad B = \begin{bmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{bmatrix}$$

(B is obtained by multiplying the elements of first column of A by k)

If the cofactors of a_1, a_2, a_3 in A are A_1, A_2, A_3 then the cofactors of ka_1, ka_2, ka_3 in B are also A_1, A_2, A_3 respectively. Hence

$$\det B = ka_1 A_1 + ka_2 A_2 + ka_3 A_3$$

$$= k(a_1 A_1 + a_2 A_2 + a_3 A_3)$$

$$= k(\det A).$$

(iv) If A is square matrix of order 3 and k is a scalar, then $|kA| = k^3|A|$. By applying property (iii), three times, we get the result.

(v) If two rows (or columns) of a square matrix are identical, then the determinant of that matrix is zero.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

(second and third rows are identical)

$$\text{Then } \det A = a_1 A_1 + b_1 B_1 + c_1 C_1$$
$$= a_1(0) + b_1(0) + c_1(0) = 0.$$

(vi) If the corresponding elements of two rows (or columns) of a square matrix are in the same ratio, then the determinant of that matrix is zero.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Then

$$\det A = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{by property (iii)}$$

$$= k(0) \quad \text{by property (v)}$$
$$= 0.$$

(vii) If each element in a row (or column) of a square matrix is the sum of two numbers, then its determinant can be expressed as the sum of the determinants of two square matrices as shown below.

$$\text{Let } A = \begin{bmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{bmatrix}, \; B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \; C = \begin{bmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{bmatrix}$$

If in A, the cofactors of $a_1 + x_1,\ a_2 + x_2, a_3 + x_3$ are A_1, A_2, A_3 then the cofactors of

$a_1,\ a_2, a_3$ in B and of x_1, x_2, x_3 in C are also A_1, A_2, A_3 respectively.

Now,

$$\det A = (a_1 + x_1)A_1 + (a_2 + x_2)\ A_2 + (a_3 + x_3)A_3$$

$$= (a_1A_1 + a_2A_2 + a_3A_3) + (x_1A_1 + x_2A_2 + x_3A_3)$$

$$= \det B + \det C.$$

$$\therefore \quad \begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}.$$

(viii) If each element of a row (or column) of a square matrix is multiplied by a number k and added to the corresponding element of another row (or column) of the matrix, then the determinant of the resultant matrix is equal to the determinant of the given matrix.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 + ka_1 & b_2 + kb_1 & c_2 + kc_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

(B is obtained from A by multiplying each element of the 1st row of A by k and then adding them to the corresponding elements of the 2nd row of A)

$$\det B = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ by property (vii)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0 \quad \text{by property (vi)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \det A.$$

(ix) The sum of the products of the elements of a row (or column) with the cofactors of the corresponding elements of another row (or column) of a square matrix is zero.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

Consider the sum of the products of the elements of the second row with the cofactors of the corresponding elements of the first row.,

$$\text{i.e., } a_2 A_1 + b_2 B_1 + c_2 C_1$$

$$= a_2 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_2 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad \text{by property (v).}$$

(x) If the elements of a square matrix are polynomials in x and its determinant is zero when $x = a$, then $x - a$ is a factor of the determinant of the matrix.

$$\text{Let } A(x) = \begin{bmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{bmatrix}.$$

Now det [A(x)] is a polynomial in x.

If det $[A(a)] = 0$ then by Remainder theorem, $x - a$ is a factor of det [A(x)].

(xi) For any square matrix A, det A = det (A').

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

The values of the cofactors of a_1, b_1, c_1, are same in both A and A'.

Hence det $A = a_1 A_1 + b_1 B_1 + c_1 C_1 = \det A'$.

(xii) Det (AB) = (det A) (det B) for matirces A, B of order 2.

Consider the matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$,

$$\det A = a_{11}a_{22} - a_{21}a_{12};\ \det B = b_{11}b_{22} - b_{21}b_{12}.$$

$$\text{Now AB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$\det(AB) = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{21}b_{11} + a_{22}b_{21})(a_{11}b_{12} + a_{12}b_{22})$$

$$= a_{11}a_{21}b_{11}b_{12} + a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} + a_{12}a_{22}b_{21}b_{22}$$
$$- a_{11}a_{21}b_{11}b_{12} - a_{12}a_{21}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{22}b_{21}b_{22}$$

$$= a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21}$$

$$= a_{11}a_{22}(b_{11}b_{22} - b_{12}b_{21}) - a_{12}a_{21}(b_{11}b_{22} - b_{12}b_{21})$$

$$= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$$

$$= (\det A)(\det B).$$

If A and B are matrices of order three then also in a similar manner we can show that

$$\det(AB) = (\det A)(\det B).$$

This is true in general, for all matrices of order *n;* the proof of this is beyond the scope of this book.

(xiii) For any positive integer *n,* $\det(A^n) = (\det A)^n$.

(xiv) If A is a triangular matrix (upper or lower), then determinant of A is the product of the diagonal elements.

5.4.9 Notation

While evaluating determinants, we use the following notations.

(i) $R_1 \leftrightarrow R_2$, to mean that the rows R_1 and R_2 are interchanged.

(ii) $R_1 \to kR_1$, to mean that the elements of R_1 are multiplied by *k.*

(iii) $R_1 \to R_1 + kR_2$ to mean that the elements of R_1 are added with *k* times the corresponding elements of R_2.

Similar notation is used for other rows and columns.

Example 5.13: Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & a & b^2 \\ 1 & a & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

Solution : L.H.S. = $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

On applying $R_2 \to (R_2 - R_1)$; $R_3 \to (R_3 - R_1)$ on LHS we get

$$\text{L.H.S.} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

On expanding the det. along the first column, we get

$$= 1.\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a) = \text{R.H.S} .$$

5.10 EVALUATION OF A DETERMINANT USING PROPERTIES

Now we are in a position to evaluate a detenninant easily by applying the aforesaid properties. The purpose of simplification of a detenninant is to make maximum possible zeroes in a row (or column) by using the above properties and then to expand the detenninant by that row (or column). We denote 1st, 2nd and 3rd row by R_1, R_2, and R_3 respectively and 1st, 2nd and 3rd column by C_1, C_2 and C3 respectively.

Example 5.13: Show that $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$

where w is a non-real cube root of unity.

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Solution: $\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

Add the sum of the 2nd and 3rd column to the 1st column. We write this operation as $C_1 \to C_1 + (C_2 + C_3)$

$$\therefore\ \Delta = \begin{vmatrix} 1+w+w^2 & w & w^2 \\ w+w^2+1 & w^2 & 1 \\ w^2+1+w & 1 & w \end{vmatrix} = \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix} = 0 \text{ (on expanding by } C_1)$$

(since w is a non-real cube root of unity, therefore, $1 + w + w^2 = 0$).

Example 5.14: Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution: $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$$= \begin{vmatrix} 0 & a-c & bc-ab \\ 0 & b-c & ca-ab \\ 1 & c & ab \end{vmatrix}$$ [$R_1 \to R_1 - R_3$=°i†Çò $R_2 \to R_2 - R_3$]

$$= \begin{vmatrix} 0 & a-c & b(c-a) \\ 0 & b-c & c(a-a) \\ 1 & c & ab \end{vmatrix} = (a-c)(b-c) = \begin{vmatrix} 0 & 1 & b \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix}$$

Expanding by C_1 we have

$$\Delta = (a-c)(b-c)\begin{vmatrix} 1 & -b \\ 1 & -a \end{vmatrix}$$

$$= (a-c)(b-c)\,(b-a)$$

$$= (a-b)(b-c)\,(c-a).$$

Example 5.15: Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Solution: $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$$= \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \quad [R_1 \to R_1 - (R_2 + R_3)]$$

Expanding by R_1, we get

$$= 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} - 2b\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$= 2c[b(a+b) - bc] - 2b[bc - c(c+a)]$

$= 2bc[a+b-c] - 2bc[b-c-a]$

$= 2bc[(a+b-c) - (b-c-a)]$

$= 2bc\ [a+b-c-b+c+a]$

$= 4abc.$

Example 5.16 : Evaluate

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Solution : $\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

= 0 on expanding by C_1.

Example 5.17 : Prove that

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Solution: $\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

$$= \begin{vmatrix} 1 & bc & bc+ab+ac \\ 1 & ca & ca+bc+ba \\ 1 & ab & ab+ca+cb \end{vmatrix} \quad [C_3 \to C_2 + C_3]$$

$$= (ab+bc+ca) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

$= (ab + bc + ca) = 0$ (by property 3)

$= 0.$

Example 5.18: Show that

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution : $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

$$= abc \begin{vmatrix} -a & b & c \\ 0 & 0 & 2c \\ 0 & 2b & 0 \end{vmatrix} \begin{bmatrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{bmatrix}$$

$$= abc\,(-a) \begin{vmatrix} 0 & 2c \\ 2b & 0 \end{vmatrix} \quad \text{(on expanding by } C_1\text{)}$$

$$= abc(-a)\,(-4bc)$$

$$= 4a^2b^2c^2.$$

Example 5.19 : Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^2(a+3)$

Solution : $\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$

$$= \begin{vmatrix} a+3 & 1 & 1 \\ a+3 & 1+a & 1 \\ a+3 & 1 & 1+a \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= (a+3) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= (a+3) \begin{vmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 1 & 0 & a \end{vmatrix} \begin{bmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix}$$

$$= (a+3)\times(1)\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix}$$

$$= (a+3)(a^2)$$

$$= a^2(a+3).$$

EXERCISE 5.3

1. Show that $\begin{vmatrix} x+3 & x & x \\ x & x+3 & x \\ x & x & x+3 \end{vmatrix} = 27(x+1)$

2. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

3. Show that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = bc+ca+ab+abc$

4. Show that $\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$

5. Show that $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$

6. Show that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

7. Evaluate

(a) $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$ (b) $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

8. Solve for 'x' :

$$\begin{vmatrix} 3x-8 & 3 & x \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

5.11 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS BY CRAMER'S RULE

Consider the system of lin ear equations

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

In this section, we shallieam to solve simultaneous linear equations in two unknowns and three unknowns with the help of determinants.

A pair of values of x and y which satisfies both the equations simultaneously is called a solution of the given system of linear equations and then the system is said to be consistent.

When the constants c_1 and c_2 on the R.H.S. of the equations are both zero, the system is said to be homogeneous system: otherwise, the system is said to be non-homogeneous. When the system is homogeneous, it has always a solution $x = y = 0$. It is called a trival solution.

If the determinant $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ then the system of homogeneous equations has only one solution, i.e., the trival solution $x = y = 0$. However, if the detenninant D $= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ then the system forms a pair of dependent equations and in that case the system has infinite number of solutions, i.e., the system has also non-trival solutions. Detenninants are used mainly to solve non-homogeneous linear equations in two or more variables.In this section, we shallieam to solve simultaneous linear equations in two/three unknowns with the help of determinants. The method of solving simultaneous linear equations by determinants is commonly known as Cramer's Rule.

5.11.1 Solving a system of Linear Equation in two variables

Consider the following system of linear equations:

$$2x + 3y - 5 = 0 \qquad ...(i)$$

$$3x + 5y - 7 = 0 \qquad ...(ii)$$

Usually, to find solutions of such a system, we apply the method of elimination. Thus, first of all solving for y, we get

$$3(2x + 3y - 5) = 0$$

$$2(3x + 5y - 7) = 0$$

$$3(2x) + 3(3y) = 3(5) \qquad ...(iii)$$

$$2(3x) + 2(5y) = 2(7) \qquad ...(iv)$$

Subtracting (iii) from (iv), we get

$$[2(5) - 3(3)]\, y = 2(7) - 3(5)$$

i.e., $\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} y = \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}$

$$\Rightarrow y = \frac{\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} \qquad ...(A)$$

Observe that in the detenninant of the numerator, the 1st column consists of the coefficeints of x and the 2[nd] columns consist of the constant tenns.

And, in the detenninant of the denominator of y, the 1[st] column and the 2[nd] column consist of the coefficients of x and y respectively.

Similarly, solving (i) and (ii) for x, we get

$$5(2x + 3y - 5) = 0$$

$$3(3x + 5y - 7) = 0$$

$$\Rightarrow \begin{cases} 5(2x) + 5(3y) - 5(5) = 0 & ...(v) \\ 3(3x) + 3(5y) - 3(7) = 0 & ...(vi) \end{cases}$$

Subtracting (vi) from (v), we get

$$\{5(2) - 3(3)\}\ x = \{5(5) - 3(7)\}$$

i.e., $\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} x = \begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix}$

$$\Rightarrow x = \frac{\begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} \quad ...(B)$$

Again observe that in the determinant of the numerator of x, the 1[st] column consists of the constant terms and the 2[nd] column consists of the coefficients of y.

And, in the determinant of the denominator of x, the 1[st] column and the 2[nd] column consist of the coefficients of x and y respectively.

Thus, $x = \dfrac{\begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} = \dfrac{D_1}{D}$ (say)

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and $y = \dfrac{\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} = \dfrac{D_2}{D}$ (say)

are the solutions of the given system of equations in the determinant form.

Consider another system of equations:

$x + y = 3$ ...(i)

$2x - 3y = 1$ ...(ii)

Again, solving (i) and (ii) by elimination method, for x, we get

$-3\ (x + y) = 3(3)$

$1(2x - 3y) = 1$

$$\Rightarrow \begin{cases} -3(x) - 3(y) = -3(3) & \text{...(iii)} \\ 1(2x) - 1(3y) = 1(1) & \text{...(vi)} \end{cases}$$

Subtracting (iv) from (iii), we get

$$-3(x) - 1(2x) = -3(3) - 1(1)$$

$$\Rightarrow \quad \{-3(1) - 1(2)\}x = -3(3) - 1(1)$$

i.e., $\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} x = \begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}$

$$\Rightarrow x = \dfrac{\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}} = \dfrac{D_1}{D} \text{ (say)}$$

Again, we observe that in the determinant D_1, the 1^{st} column consists of the constant terms and the 2^{nd} column consists of the coefficient of y. Also, in D, the 1^{st} and the 2^{nd} columns consist of the coefficients of x and y respectively.

Similarly, solving (i) and (ii) for y, we get

$2(x) + 2(y) = 2(3)$ ...(v)

$1(2x) + 2(-3y) = 1(1)$ ...(vi)

Subtracting (v) from (vi), we get

$\{1(-3) - 2(1)\}\, y = 1(1) - 2(3)$

i.e., $\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} y = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$

$$\Rightarrow y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}} = \frac{D_2}{D} \quad \text{(say)}$$

Again, we observe that the determinant D is nothing but the determinant of the coefficient of x and y and the determinant D_2 is obtained by replacing the coefficient of y by the constant terms.

$$\text{Thus,} \quad x = \frac{\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}}$$

are the solutions of the given equation

Therefore, if $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_1}{D}$$

and $y = \dfrac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \dfrac{D_2}{D}$

provided $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

5.11.2 Solution of a System of Linear Equations in Three Variables

Consider the following system of equations in three variables

$x + 2y + 3z = 4$

$2x + y + z = 1$ ...(C)

$3x + 3y + 5z = 3$

Let $D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix}$

i.e., D is the determinant of the coefficients of x, y and z.

Then, $xD = \begin{vmatrix} x & 2 & 3 \\ 2x & 1 & 1 \\ 3x & 3 & 5 \end{vmatrix}$

Multiplying the 2nd column by y and the 3rd column by z and adding these to the 1st column, we get

$$xD = \begin{vmatrix} x+2y+3z & 2 & 3 \\ 2x+y+z & 1 & 1 \\ 3x+3y+5z & 3 & 5 \end{vmatrix}$$

From (C), we get

$$xD = \begin{vmatrix} 4 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix} = D_1 \text{ (say)}$$

i.e., $xD = D_1 \Rightarrow x = \dfrac{D_1}{D}$

Similarly, we will get

$$D_2 = \begin{vmatrix} 1 & 4 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 5 \end{vmatrix} \text{ and } D_3 = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 1 \\ 3 & 3 & 3 \end{vmatrix}$$

Then, as before, we can see that

$yD = D_2$ and $zD = D_3$

Thus, $x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$, $z = \dfrac{D_3}{D}$ where $D \neq 0$

are the solutions of the given system of linear equations in three variables.

Therefore, if

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

is a given system of linear equations in three variables, then

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_1}{D}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_2}{D}$$

and $$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_3}{D}$$

are the solutions of the given system, provided

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

The method used for solving the system of equations in three variables can be used in exactly the same way to solve a system *of 'n'* equations in *'n'* unknowns. The method discussed above is commonly known as Cramer's rule, after the Swiss Mathematician Gabriel Cramer (1704-1752).

Note: Cramer's Rule does not apply if $D = 0$

Example 5.20: Solve the following system of equations by Cramer's Rule:

$2x + 3y = 5$

$3x + 5y = 7$

Solution: Now,

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1$$

$$D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix} = 25 - 21 = 4$$

and $$D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix} = 14 - 15 = -1$$

Thus, by Cramer's Rule $x = \frac{D_1}{D} = \frac{4}{1} = 4$

and $y = \frac{D_2}{D} = \frac{-1}{1} = -1$

are the solutions of the given system of equations.

Example 5.21: Solve the following system of equations by Cramer's Rule:

$2x + y - 3z = 3$

$x + 2y + z = 7$

$3x - 5y + 2z = 1$

Solution: We have,

$$D = \begin{vmatrix} 2 & 1 & -3 \\ 1 & 2 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 2(4+5) - 1(2-3) - 3(-5-6) = 18 + 1 + 33 = 52 \neq 0$$

Also, to find D_1, the 1st column will be replaced by constants.

$$D_1 = \begin{vmatrix} 3 & 1 & -3 \\ 5 & 2 & 1 \\ 1 & -5 & 2 \end{vmatrix} = 3(4+5) - 1(10-1) - 3(-25-2) = 27 - 9 + 81 = 99$$

To find, D_2 2nd column will be replaced by constants.

$$D_2 = \begin{vmatrix} 2 & 3 & -3 \\ 1 & 5 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 2(10-1) - 3(2-3) - 3(1-15) = 18 + 3 + 42 = 63$$

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To find D_3, 3^{rd} column will be replaced by constants

$$D_3 = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & 5 \\ 3 & -5 & 1 \end{vmatrix} = 2(2+25)-1(1-15)+3(-5-6) = 35$$

Thus, by Cramer's Rule

$$x = \frac{D_1}{D} = \frac{99}{52}$$

$$y = \frac{D_2}{D} = \frac{63}{52}$$

and $z = \dfrac{D_3}{D} = \dfrac{35}{52}$

5.12 CONDITION FOR A SYSTEM OF LINEAR EQUATIONS TO HAVE A UNIQUE SOLUTION

Consider the system of equations

$$2x + 3y = 4$$

$$x - 2y = 3$$

Now $D = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -4-3 = -7 \neq 0$

Also $D_1 = \begin{vmatrix} 4 & 3 \\ 3 & -2 \end{vmatrix} = -8-9 = -17$

and $D_2 = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 6-4 = 2.$

By Cramer's rule

$$x = \frac{D_1}{D} = \frac{-17}{-7} = \frac{17}{7}$$

and $y = \frac{D_2}{D} = \frac{2}{-7} = \frac{-2}{7}$

Thus, we find that for $D \neq 0$ and $D_1 \neq 0, D_2 \neq 0$ the given system of equations have non-zero, unique solution $x = \frac{17}{7}$ and $y = \frac{-2}{7}$.

In this case, we say that the given system of equations is consistent.

Now, consider the system of equations

$x + 2y = 0$

$-2x + 3y = 0$

Here, $D = \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} = 3 + 4 = 7 \neq 0$

Also, $D_1 = \begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} = 0 - 0 = 0$

and $D_2 = \begin{vmatrix} 1 & 0 \\ -2 & 0 \end{vmatrix} = 0 - 0 = 0$

Hence, $x = \frac{D_1}{D} = \frac{0}{7} = 0$

and $y = \frac{D_2}{D} = \frac{0}{7} = 0$

Thus, we find that for $D \neq 0$ and $D_1 = D_2 = 0$the given system of equations will have only the trivial solution $x = y = 0$.

We already know that Cramer's rule does not apply if $D = 0$. The two cases arise namely system of equations having (i) no solution, (ii) infinitely many solutions.

Consider the system of equations

$2x + 4y = 5$

$x + 2y = 3$

Here, $D = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 4-4=0$

Since $D=0$ Cramer's rule does not apply here.

Now $D_1 = \begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix} = 10-12=-2$

and $D_2 = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = 6-5=1$

Hence, if $D=0$ and $D_1 \neq 0, D_2 \neq 0$ the equations will have no solution.

Similar is the case for a system of three equations in three variables.For that, consider the system of equations

$x+y+z=2$

$x+2y+3z=3$

$x+3y+5z=5$ ...(i)

Here, $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{vmatrix} = 1(10-9)-1(5-3)+1(3-2)=1-2+1=0$

Now, $xD = \begin{vmatrix} x & 1 & 1 \\ x & 2 & 3 \\ x & 3 & 5 \end{vmatrix}$

Multiplying the 2nd column by y and the 3rd column by z and adding them to the 1 st column, we get

$$xD = \begin{vmatrix} x+y+z & 1 & 1 \\ x+2y+3z & 2 & 3 \\ x+3y+5z & 3 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 5 & 3 & 5 \end{vmatrix} = D_1 \text{ [using (1)]}$$

Thus, $D_1 = 2(10-9)-1(15-15)+1(9-10) = 2-0-1 = 1$

So, $xD = D_1 \Rightarrow xD = 1$

$\Rightarrow x = \frac{1}{D} = \frac{1}{0}$ which is undefined

Similarly, we get $yD = D_2$ and $zD = D_3$

where

$$D_2 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 5 & 5 \end{vmatrix} = 1(15-15) - 2(5-3) + 1(5-3) + 1(5-3) = 0 - 4 + 2 = -2$$

and $$D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{vmatrix} = 1(10-9) - 1(5-3) + 2(3-2) = 1 - 2 + 2 = 1$$

$yD = D_1 \Rightarrow y = \frac{D_2}{D} \Rightarrow y = \frac{-2}{0}$ which is undefined.

and $zD = D_1 \Rightarrow z = \frac{D_1}{D} \Rightarrow z = \frac{1}{0}$ which is undefined.

Thus, if $D = 0$ and $D_1 \neq 0$, $D_2 \neq 0$ and $D_3 \neq 0$then the system of equations will have no solution. In this case, we say that the system of equations is inconsistent.

Now, consider the system of equations

$$x - y + 3z = 6$$
$$x + 3y - 3z = -4$$
$$5x + 3y + 3z = 10$$

Here,

$$D = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{vmatrix} = 1(9+9) + 1(3+15) + 3(3-15) = 18 + 18 - 36 = 0$$

Also,

$$D_1 = \begin{vmatrix} 6 & -1 & 3 \\ -4 & 3 & -3 \\ 10 & 3 & 3 \end{vmatrix}$$

$$= 6(9+9)+1(-12+30)+3(-12-30) = 108+18-126 = 0$$

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 1 & -4 & -3 \\ 5 & 10 & 3 \end{vmatrix} = 1(-12+30)-6(3+15)+3(10+20) = 18-108+90 = 0$$

$$D_3 = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 3 & -4 \\ 5 & 3 & 10 \end{vmatrix} = 1(30+12)+1(10+20)+6(3-15) = 42+30 = 72 = 0$$

Thus, for $D = 0$ and $D_1 = D_2 = D_3 = 0$ for the given system of equations will have infmitely many solutions.

Consider the first two equations,

i.e., $x - y + 3z = 6$

$x + 3y - 3z = -4$

These can be written as

$x - y = 6 - 3z$

$x + 3y = -4 + 3z$

Solving these equations by determinants, we get

$$x = \frac{\begin{vmatrix} 6-3z & -1 \\ -4+3z & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}} = \frac{3(6-3z)+1(-4+3z)}{3+1} = \frac{18-9z-4+3z}{4} = \frac{14-6z}{4}$$

$$x = \frac{7-3z}{2}$$

and

$$y = \frac{\begin{vmatrix} 1 & 6-3z \\ 1 & -4+3z \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}} = \frac{1(-4+3z)-1(6-3z)}{3+1} = \frac{-4+3z-6+3z}{4} = \frac{-10+6z}{4}$$

$$y = \frac{-5+3z}{2}$$

Let $z = k$, where k is any number, then we get

$$x = \frac{7-3k}{2},\ y = \frac{-5+3k}{2} \text{ and } z = k$$

Thus, the system of equations has many solutions.

So, we conclude that for a given system of equations

(i) If $\mathrm{D} \neq 0$ and atlease on of $\mathrm{D}_1, \mathrm{D}_2, \ldots \mathrm{D}_n$ is not equal to zero, then the system will have non-zero, unique solution.

(ii) If $\mathrm{D} \neq 0$ and each $\mathrm{D}_i = 0$ then the system has only the trivial solution $x_1 = x_2 = ... = x_n = 0$.

(iii) If $\mathrm{D} = 0$ and some $\mathrm{D}_i \neq 0$ then the system has no solution.

(iv) If $\mathrm{D} = 0$ and each $\mathrm{D}_i = 0$ then the system has infinitely many solutions.

Example 5.22: Solve the following system of equations:

$x + y + z = 2$

$2x + 7y - 3z = 5$

$3x + 5y - z = 4.$

Solution: Here,

$$\mathrm{D} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 7 & -3 \\ 3 & 5 & -1 \end{vmatrix} = 1(-7+15) - 1(-2+9) + 1(10-21) = 8-7-11 = -10 \neq 0$$

$$\text{Now } D_1 = \begin{vmatrix} 2 & 1 & 1 \\ 5 & 7 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 2(-7+15)-1(-5+12)+1(25-28) = 16-7-3 = 6$$

$$D_2 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & -3 \\ 3 & 4 & -1 \end{vmatrix} = 1(-5+12)-2(-2+9)+1(8-15) = 7-14-7 = -14$$

$$D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 7 & 5 \\ 3 & 5 & 4 \end{vmatrix} = 1(28-25)-1(8-15)+2(10-21) = 3+7-22 = -12$$

Since $D \neq 0$ and $D_1 \neq 0$, $D_2 \neq 0$, $D_3 \neq 0$, therefore, the system of equations will have non-zero, unique solution.

Thus, by Cramer's Rule

$$x = \frac{D_1}{D} = \frac{6}{-10} = \frac{-3}{5}$$

$$y = \frac{D_2}{D} = \frac{-14}{-10} = \frac{7}{5}$$

$$z = \frac{D_3}{D} = \frac{-12}{-10} = \frac{6}{5}$$

are the solutions of the given system of equations.

Example 5.23: Detennine which of the following systems of equations will have a unique solution; and also find the solutions in each case:

(i) $$\begin{aligned} 2x-3y+4z &= -9 \\ -3x+4y+2z &= -12 \\ 4x-2y-3z &= -3 \end{aligned}$$

(ii) $$\begin{aligned} x+2y-z &= 0 \\ 2x+y+2z &= 0 \\ x-3y+z &= 0 \end{aligned}$$

(iii) $$\begin{aligned} x+2y+z &= 2 \\ 2x+y+2z &= 3 \\ x-3y+z &= 4 \end{aligned}$$

(iv) $$\begin{aligned} x+2y+3z &= 1 \\ 3x-y+2z &= 1 \\ 4x+y+5z &= 2 \end{aligned}$$

Solution:

(i) $2x - 3y + 4z = -9$

$-3x + 4y + 2z = -12$

$4x - 2y - 3z = -3$

Here, $D = \begin{vmatrix} 2 & -3 & 4 \\ -3 & 4 & 2 \\ 4 & -2 & -3 \end{vmatrix} = 2(-12+4) + 3(9-8) + 4(6-16)$

$$= -16 + 3 - 40 = -53 \neq 0$$

Also,

$D_1 = \begin{vmatrix} -9 & -3 & 4 \\ -12 & 4 & 2 \\ -3 & -2 & -3 \end{vmatrix} = -9(-12+4) + 3(36+6) + 4(24+12)$

$$= 72 + 126 + 144 = 342$$

$D_2 = \begin{vmatrix} 2 & -9 & 4 \\ -3 & -12 & 2 \\ 4 & -3 & 2 \end{vmatrix} = 2(36+6) + 9(9-8) + 4(9+48)$

$$= 84 + 9 + 228 = 321$$

and $D_3 = \begin{vmatrix} 2 & -3 & -9 \\ -3 & 4 & -12 \\ 4 & -2 & -3 \end{vmatrix} = 2(-12-24) + 3(9-48) - 9(6-16)$

$$= -72 + 171 + 90 = 189$$

Since $D \neq 0$ and $D_1 \neq 0$, $D_2 \neq 0$, $D_3 \neq 0$.

$\therefore$ The system of equations will have a non-zero unique solution which is

$$x = \frac{D_1}{D} = \frac{342}{-53} = \frac{-342}{53}$$

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$$y = \frac{D_2}{D} = \frac{321}{-53} = \frac{-321}{53}$$

and $$z = \frac{D_3}{D} = \frac{189}{-53} = \frac{-189}{53}$$

(ii) $x + 2y - z = 0$

$2x + y + 2z = 0$

$x - 3y + z = 0$

Here,

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 1(1+6) - 2(2-2) - 1(-6-1) = 7 + 7 = 14 \neq 0$$

Also,

$$D_1 = \begin{vmatrix} 0 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & -3 & 1 \end{vmatrix} = 0 \quad \text{(expanding by the 1}^{\text{st}}\text{ column)}$$

$$D_2 = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

and

$$D_3 = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & -3 & 0 \end{vmatrix} = 0$$

Thus, we find that $D \neq 0$ and $D_1 = D_2 = D_3 = 0$.

$\therefore$ The system of linear equations will not have a unique solution. In fact, it has a trivial solution $x = 0, y = 0, z = 0$.

(iii) $x+2y+z=2$

$2x+y+2z=3$

$x-3y+z=4$

Here $D=\begin{vmatrix}1 & 2 & 1\\ 2 & 1 & 2\\ 1 & -3 & 1\end{vmatrix}=0$

Also

$$D_1=\begin{vmatrix}2 & 2 & 1\\ 3 & 1 & 2\\ 4 & -3 & 1\end{vmatrix}=2(1+6)-2(3-8)+1(-9-4)=14+10-13=11$$

$$D_2=\begin{vmatrix}1 & 2 & 1\\ 2 & 3 & 2\\ 1 & 4 & 1\end{vmatrix}=0\ (\because C_1=C_3)$$

and $D_3=\begin{vmatrix}1 & 2 & 2\\ 2 & 1 & 3\\ 1 & -3 & 4\end{vmatrix}=1(4+9)-2(8-3)+2(-6-1)=13+10-14=-11$

Since $D=0$ and $D_1\neq 0$, $D_2=0$ and $D_3\neq 0$.

$\therefore$ The system of equations has no solution.

(iv) $x+2y+3z=1$

$3x-y+2z=1$

$4x+y+5z=2$

Here, $D=\begin{vmatrix}1 & 2 & 3\\ 3 & -1 & 2\\ 4 & 1 & 5\end{vmatrix}=1(-5-2)-2(5-4)+3(1+2)=-7-2+9=0$

Also, $D_1=\begin{vmatrix}1 & 2 & 3\\ 1 & -1 & 2\\ 2 & 1 & 5\end{vmatrix}=1(-5-2)-2(5-4)+3(1+2)=-7-2+9=0$

$$D_2 = \begin{vmatrix} 1 & 1 & 3 \\ 3 & 1 & 2 \\ 4 & 2 & 5 \end{vmatrix} = 1(5-4) - 1(15-8) + 3(6-4) = 1 - 7 + 6 = 0$$

and $$D_3 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -1 & 1 \\ 4 & 1 & 2 \end{vmatrix} = 1(-2-1) - 2(6-4) + 1(3+4) = -3 - 4 + 7 = 0$$

Therefore, the given system of equations will have infmitely many solutions.

EXERCISE 5.4

1. Solve the following systems of equations by Cramer's Rule

   (a) $2x - 4y = 3$, $3x + y = 5$
   (b) $x + 2y = 1$, $2x + 5y = 3$

2. Obtain the solutions of the systems of equations using Cramer's Rule

   (a) $2x + y + 3z = 1$, $x + 4y + 6z = 9$, $4x + 3y + 9z = 5$
   (b) $2x - 3y + 2z = 1$, $x + 3y - z = -2$, $x - y + 3z = 3$
   (c) $3x - 4y + 5z = -6$, $x + y - 2z = -1$, $2x + 3y + z = 5$

3. Solve the following systems of equations:

   (a) $3x + 2y = 4$, $2x + y = 3$
   (b) $6x - 3y = -1$, $2x + 2y = -3$
   (c) $2x + 3y + 4z = 8$, $3x + y - z = -2$, $4x - y - 5z = -9$
   (d) $x + 3y - z = 4$, $3x - 2y + 5z = -4$, $5x - y - 4z = -9$

4. Determine which of the following systems of equations will have a unique solution. Also, fmd the solution in such a case.

   (a) $x - 2y = 4$, $-3x + 5y = -7$
   (b) $2x - y + z = 0$, $x + y - 2z = 0$, $3x + 2y - z = 0$

KEY WORDS

- The expression $a_1b_2 - a_2b_1$ is denoted by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.
- With each square matrix, a determinant of the matrix can be associated.
- The *minor* of any element in a determinant is obtained from the given determinant by deleting the row and column in which the element lies.
- The *cofactor* of an element a_{ij} in a determinant is the minor of a_{ij} multiplied by $(-1)^{i+j}$.
- A determinant can be expanded using any row or column. The value of the determinant will be the same.
- A square matrix whose determinant has the value zero, is called a*singular matrix*.
- The value of a determinant remains unchanged, if its rows and columns are interchanged.
- If two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign only.
- If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
- If each element of a row (or column) of a determinant is multiplied by the same constant, then the value of the determinant is multiplied by the constant.
- If any two rows (or columns) of a determinant are proportional, then its value is zero.
- If each element of a row or column from of a determinant is expressed as the sum (or differenence) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants of the same order.

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- The value of a determinant does not change if to each element of a row (or column) be added to (or subtracted from) some multiples of the corresponding elements of one or more rows (or columns).
- The solution of a system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and}$$

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}, \text{ provided } D \neq 0.$$

- The system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

(a) is consistent and has unique solution, when $D \neq 0$.

(b) is inconsistent and has no solution, when $D = 0$ and D_1, D_2, D_3 are not all zero.

(c) is consistent and has infmitely many solutions, when

$$D = D_1 = D_2 = D_3 = 0.$$

SUPPORTED WEBSITES

- http://www.wikipedia.org
- http://mathworld.wolfram.com

PRACTICE EXERCISE

1. Find all the minors and cofactors of $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 2 & 3 & 1 \end{vmatrix}$.

2. Evaluate $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$ by expanding it using the fIrst column.

3. Using Sarrus diagram, evaluate

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 2 & -3 \\ 3 & -1 & -4 \end{vmatrix}$$

4. Solve for x, if

$$\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$$

5. Using properties of determinants, show that

(a) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b)$

(b) $\begin{vmatrix} 1 & x+y & x^2+y^2 \\ 1 & y+z & y^2+z^2 \\ 1 & z+x & z^2+x^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

6. Evaluate :

(a) $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & w^3 & w^5 \\ w^3 & 1 & w^4 \\ w^5 & w^5 & 1 \end{vmatrix}$

w being an imaginary cube-root of unity

7. Using Cramer's rule, solve the following system of linear equations:

(a) $x-2y=3$, $2x+3y=4$ (b) $2x-3y=3$, $3x+2y=11$ (c) $x+y=2$, $4x-3y=3$

8. Using Cramer's rule, solve the following systems of equations:

(a) $2x+y+3z=8$, $3x+2y+3=z$, $x+3z=1+2y$ (b) $x-2y=3+z$, $2x+z=y$, $3-x=2z-3y$

9. Determine which of the following systems of equations will have a unique solution:

(a) $2x-6y+1=0$, $x-3y+2=0$ (b) $2x-3y=5$, $x-2y=6$

(c) $2x+3y+z=1$, $4x-6y+z=3$, $6x-3y+2z=5$ (d) $3x+y+2z=-1$, $x+2y-z=2$, $2x-y+3z=1$

ANSWERS

EXERCISE 5.1

1. (a) 11 (b) 1 (c) 0 (d) $(a^2+b^2)-(c^2+d^2)$

2. (a) and (d)

3.(a) 18 (b) -54 (c) $adf+2bce-ae^2-fb^2-de^2$

EXERCISE 5.2

1. (a) $M_{21}=39$; $C_{21}=-39$
   $M_{22}=3$; $C_{22}=3$
   $M_{23}=-11$; $C_{23}=11$

2. $M_{13}=-5$; $C_{13}=-5$
   $M_{23}=-7$; $C_{23}=7$
   $M_{33}=1$; $C_{33}=1$

3. (a) 19 (b) 0 (c) -131

(d) $(a-b)(b-c)(c-a)$ (e) $4abc$ (f) 0

4. (a) 4 (b) 20 (c) $a^2+b^2+c^2+1$

6. (a) $x=2$ (b) $x=2, 3$ (c) $x=2, -\frac{17}{7}$

EXERCISE 5.3

7. (a) a^3 (b) $2abc(a+b+c)^3$

8. $x=\frac{2}{3}, \frac{11}{3}, \frac{11}{3}$

EXERCISE 5.4

1. (a) $x=\frac{23}{14},\ y=\frac{1}{14}$ (b) $x=-1, y=1$

2. (a) $x=-1, y=2, z=\frac{1}{3}$ (b) $x=-\frac{3}{4},\ y=0,\ z=\frac{5}{4}$

(c) $x=-1, y=2,\ z=1$

3. (a) $x=2, y=-1$ (b) $x=-\frac{11}{8},\ y=-\frac{8}{9}$

(c) $x=1, y=-2, z=3$ (d) $x=-1,\ y=\frac{24}{13},\ z=\frac{7}{13}$

4. (a) Yes; $x=-6, y=-5$ (b) Yes; $x=0, y=0, z=0$

(c) $x=0, y=3, z=2$

PRACTICE EXERCISE

1. $M_{11}=-2, M_{12}=-1, M_{13}=1, M_{21}=-7, M_{22}=-5,$

$M_{23}=-1, M_{31}=-8, M_{32}=-7, M_{33}=-2$

$C_{11} = -2, C_{12} = 1, C_{13} = 1, C_{21} = 7, C_{22} = -5,$

$C_{23} = 1, C_{31} = -8, C_{32} = 7, C_{33} = -2.$

2. 0 3. –31 4. $x = 0, \; x = 1$

6. (a) –8 (b) 0

7. (a) $x = \frac{17}{7}, \; y = -\frac{2}{7}$ (b) $x = 3, y = 1$

(c) $x = \frac{9}{7}, \; y = \frac{5}{7}$

8. (a) $x = 2, y = 3, z = 3$ (b) $x = -\frac{1}{2}, \; y = -\frac{3}{2}, \; z = -\frac{1}{2}$

9. (b) only

Chapter
6

INVERSE OF A MATRIX AND ITS APPLICATION

LEARNING OUTCOMES

After studying this chapter, student will be able to :

- Compute adjoint and inverse of matrix
- Define singular and non singular matrices
- Represent system of linear equation in the matrix form AX = B
- Solve a system of linear equations by matrix method

PREREQUISITES

- Determinant of a matrix, Transport of a matrix.

INTRODUCTION

In this chapter, we will learn to find the inverse of a matrix, if it exists later we will use matrix inverse to solve linear systems.

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LET US CONSIDERAN EXAMPLE

Abhinav spends Rs. 120 in buying 2 pens and 5 note books whereas Shantanu spends Rs. 100 in buying 4 pens and 3 note books. We will try to find the cost of one pen and the cost of one note book using matrices.

Let the cost of 1 pen be Rs. x and the cost of 1 note book be Rs. y. Then the above information can be written in matrix form as:

$$\begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 120 \\ 100 \end{bmatrix}$$

This can be written as $AX = B$

where $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 120 \\ 100 \end{bmatrix}$.

Our aim is to find $X = \begin{bmatrix} x \\ y \end{bmatrix}$

In order to find X, we need to find a matrix A^{-1} so that $X = A^{-1}B$

This matrix A^{-1} is called the inverse of the matrix A.

In this lesson, we will try to find the existence of such matrices.We will also learn to solve a system of linear equations using matrix method.

6.1 SINGULAR AND NON SINGULAR MATRICES

We have already learnt that with each square matrix, a determinant is associated. For any given matrix , say $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$

its determinant will be $\begin{vmatrix} 2 & 5 \\ 4 & 3 \end{vmatrix}$ It is denoted by $|A|$

Similarly, forthe matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 1 & -1 & 7 \end{bmatrix}$ the corresponding determinant is

$$|A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 1 & -1 & 7 \end{vmatrix}$$

A square matrix A is said to be singular if its determinant is zero, i.e. $|A| = 0$.

A square matrix A is said to be non-singular if its determinant is non-zero,

i.e. $|A| \neq 0$

Example 6.1 : Determine whether matrix A is singular or non-singular where

(a) $A = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix}$

Solution: (a) Here, $|A| = \begin{vmatrix} -6 & -3 \\ 4 & 2 \end{vmatrix}$

$= (-6)(2) - (4)(-3)$

$= -12 + 12 = 0$

Therefore, the given matrix A is a singular matrix.

(b) $A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{vmatrix}$

Here $|A| = 1\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix}$

$= -7 + 4 - 3$

$= -6 \neq 0.$

Therefore, the given matrix is non-singular.

Example 6.2 : Find the value of x for which the following matrix is singular:

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$$

Solution: Here,

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ x & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ x & 2 \end{vmatrix}$$

$= 1(-6 - 2) + 2(-3 - x) + 3(2 - 2x)$

$= -8 - 6 - 2x + 6 - 6x.$

$= -8 - 8x.$

Since the matrix A is singular, we have $|A| = 0$.

$|A| = -8 - 8x = 0.$

or $\quad x = -1.$

Thus, the required value of x is -1.

Example 6.3: Given $A = \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix}$ Show that $|A| = |A'|$ where A' denotes the transpose of the matrix.

Solution: Here, $A = \begin{bmatrix} 1 & 6 \\ 3 & 2 \end{bmatrix}$

This gives $A' = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}$

Now, $\quad |A| = \begin{vmatrix} 1 & 6 \\ 3 & 2 \end{vmatrix} = 1 \times 2 - 3 \times 6 = -16 \quad$ ...(1)

and $\quad |A'| = \begin{vmatrix} 1 & 3 \\ 6 & 2 \end{vmatrix} = 1 \times 2 - 6 \times 3 = -16 \quad$ ...(2)

From (1) and (2) we find that $|A| = |A'|$

6.2 MINORS AND COFACTORS OF THE ELEMENTS OF SQUARE MATRIX

Consider a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of *A,* is called the minor of a_{ij} and is denotes by M_{ij}

Cofactor C_{ij} of a_{ij} is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

For example, M_{23} = Minor of a_{23}.

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

and C_{23} = Cofactor of a_{23}

$$= (-1)^{2+3} M_{23}$$

$$= (-1)^5 M_{23}$$

$$= -M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 6.4: Find the minors and the cofactors of the elements of matrix

$$A = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix}$$

Solution: For matrix A, $|A| = \begin{vmatrix} 2 & 5 \\ 6 & 3 \end{vmatrix} = 6 - 30 = -24$

M_{11} (minor of 2) = 3 $C_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = 3.$

M_{12} (minor of 5) = 6 $C_{12} = (-1)^{1+2} M_{12} = (-1)^3 = M_{11} = -6.$

M_{21} (minor of 6) = 5 $C_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21} = -5.$

M_{22} (minor of 3) = 2 $C_{22} = (-1)^{2+2} M_{22} = (-1)^{2+2} M_{22} = 2.$

Example 6.5: Find the minors and the cofactors of the elements of matrix

$$A = \begin{bmatrix} -1 & 3 & 6 \\ 2 & 5 & -2 \\ 4 & 1 & 3 \end{bmatrix}$$

Solution: Here, $M_{11} = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17;\ C_{11} = (-1)^{1+1}\ M_{11} = 17$

$$M_{12} = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6 + 8 = 14;\ C_{12} = (-1)^{1+2}\ M_{12} = -14$$

$$M_{13} = \begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} = 2 - 20 = -18;\ C_{13} = (-1)^{1+3}\ M_{13} = -18$$

$$M_{21} = \begin{vmatrix} 3 & 6 \\ 1 & 3 \end{vmatrix} = 9 - 6 = 3;\ C_{21} = (-1)^{2+1}\ M_{21} = -3$$

$$M_{22} = \begin{vmatrix} -1 & 6 \\ 4 & 3 \end{vmatrix} = (-3 - 24) = -27;\ C_{22} = (-1)^{2+2}\ M_{22} = -27$$

$$M_{23} = \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} = (-1 - 12) = -13;\ C_{23} = (-1)^{2+3}\ M_{23} = 13$$

$$M_{31} = \begin{vmatrix} 3 & 6 \\ 5 & -2 \end{vmatrix} = (-6 - 30) = -36;\ C_{31} = (-1)^{3+1}\ M_{31} = -36$$

$$M_{32} = \begin{vmatrix} -1 & 6 \\ 2 & -2 \end{vmatrix} = (2 - 12) = -10;\ C_{32} = (-1)^{3+2}\ M_{32} = 10$$

and $M_{33} = \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix} = (-5 - 6) = -11;\ C_{33} = (-1)^{3+3}\ M_{33} = -11$

EXERCISE 6.1

1. Find the value of the determinant of following matrices:

(a) $A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \end{bmatrix}$ (b) $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

2. Determine whether the following matrix are singular or non-singular.

(a) $A = \begin{bmatrix} 3 & 2 \\ -9 & 6 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & -1 \end{bmatrix}$

3. Find the minors of the following matrices:

(a) $A = \begin{bmatrix} 3 & -1 \\ 7 & 4 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 6 \\ 2 & 5 \end{bmatrix}$

4.(a) Find the minors of the elements of the 2^{nd} row of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ -2 & -3 & 1 \end{bmatrix}$$

(b) Find the minors of the elements of the 3^{rd} row of matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 4 & 1 \\ -2 & 0 & 3 \end{bmatrix}$$

5. Find the cofactors of the elements of each the following matrices:

(a) $A = \begin{bmatrix} 3 & -2 \\ 9 & 7 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 4 \\ -5 & 6 \end{bmatrix}$

6.(a) Find the cofactors of elements of the 2^{nd} row of matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 0 \\ 4 & -1 & -2 \end{bmatrix}$$

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(b) Find the cofactors of the elements of the 1^{st} row of matrix

$$A = \begin{bmatrix} 2 & -1 & 5 \\ 6 & 4 & -2 \\ -5 & 3 & 0 \end{bmatrix}$$

7. $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $A = \begin{bmatrix} -2 & 3 \\ 7 & 4 \end{bmatrix}$ verify that

(a) $|A| = |A'|$ =°i†Çò $|B| = |B'|$

(b) $|AB| = |A|\ |B| = |BA|$

6.3 ADJOINT OF A SQUARE MATRIX

Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix}$ be a matrix. Then $A = \begin{vmatrix} 2 & 1 \\ 5 & 7 \end{vmatrix}$.

Let M_{ij} and C_{ij} be the minor and cofactor of a .. respectively. Then

$M_{11} = |7| = 7;\ \ C_{11} = (-1)^{1+1}\ .\ |7| = 7.$

$M_{12} = |5| = 5;\ \ C_{12} = (-1)^{1+2}\ |5+ = -5$

$M_{21} = |1| = 1;\ \ C_{21} = (-1)^{2+1}\ |1+ = -1$

$M_{22} = |2| = 2;\ \ C_{22} = (-1)^{2+2}\ |2| = 2$

We replace each element of A by its cofactor and get

$$B = \begin{bmatrix} 7 & -5 \\ -1 & 2 \end{bmatrix} \qquad ...(1)$$

The transpose of the matrix B of co factors obtained in (1) above is

$$B' = \begin{bmatrix} 7 & -1 \\ -5 & 2 \end{bmatrix} \qquad ...(1)$$

The matrix B' obtained above is called the adjoint of matrix A. It is denoted by Adj A.

Thus, adjoint of a given matrix is the transpose of the matrix whose elements are the cofactors of the elements of the given matrix.

Working Rule: To find the Adj A of a matrix A:

(a) replace each element of A by its cofactor and obtain the matrix of cofactors; and

(b) take the transpose of the marix of cofactors, obtained in (a).

Example 6.6: Find the adjoint of

$$A = \begin{bmatrix} -4 & 5 \\ 2 & -3 \end{bmatrix}$$

Solution: Here, $|A| = \begin{vmatrix} -4 & 5 \\ 2 & -3 \end{vmatrix}$ Let A_{ij} be the cofactor of the element a_{ij}

Then $A_{11} = (-1)^{1+1}\,(-3) = -3$ $\qquad$ $A_{21} = (-1)^{2+1}\,(5) = -5$

$A_{12} = (-1)^{1+2}\,(2) = -2$ $\qquad$ $A_{22} = (-1)^{2+2}\,(-4) = -4.$

We replace each element of A by its cofactor to obtain its matrix of cofators as

$$\begin{bmatrix} -3 & -2 \\ -5 & -4 \end{bmatrix} \qquad ...(1)$$

Transpose of matrix in (1) is Adj A.

Thus, $\text{Adj } A = \begin{bmatrix} -3 & -5 \\ -2 & -4 \end{bmatrix}.$

Example 6.7: Find the adjoint of $A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 5 & 2 & -1 \end{bmatrix}$

Solution: Here, $A = \begin{vmatrix} 1 & -1 & 2 \\ -3 & 4 & 1 \\ 5 & 2 & -1 \end{vmatrix}$

Let A_{ij} be the cofactor of the element a_{ij} of $|A|$.

Then $\quad A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix} = -4-2 = -6$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} -3 & 1 \\ 5 & -1 \end{vmatrix} = -(3-5) = 2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} -3 & 4 \\ 5 & 2 \end{vmatrix} = -6-20 = -26$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = -(1-4) = 3.$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} = -1-10 = -11.$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} = -(2+5) = -7.$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = -1-8 = -9.$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix} = -(1+6) = -7.$$

and $A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -3 & 4 \end{vmatrix} = 4-3 = 1.$

Replacing the elements of A by their cofactors, we get the matrix of cofactors as

$$\begin{bmatrix} -6 & 2 & -26 \\ 3 & -11 & -7 \\ -9 & -7 & 1 \end{bmatrix}$$

Thus, $\text{Adj A} = \begin{bmatrix} -6 & 3 & -9 \\ 2 & -11 & -7 \\ -26 & -7 & 1 \end{bmatrix}$

If A is any square matrix of order n, then A.(Adj A) = (Adj A) A = |A| I_n

where I_n is the unit matrix of order n.

Verification:

(1) Consider $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix}$ or $|A| = 2\times 3 - (-1)\times(4) = 10.$

Here $A_{11} = 3$; $A_{12} = 1$; $A_{21} = -4$ and $A_{22} = 2$

Therefore, $\text{Adj } A = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$

Now, $A(\text{Adj } A) = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|.I_2$$

2. Consider, $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Then, $|A| = 3(-6-1) - 5(4-1) + 7(2+3) = -1$

Here $A_{11} = -7$; $A_{12} = -3$; $A_{13} = 5$

$A_{21} = -3$; $A_{22} = -1$; $A_{23} = 2$

$A_{31} = 26$; $A_{32} = 11$; $A_{33} = -19$.

Therefore, $\text{Adj } A = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$

Now $(A)(\text{Adj } A) = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\ I_3.$$

Also, $(\text{Adj } A) . A = \begin{bmatrix} -7 & -3 & 26 \\ -3 & -1 & 11 \\ 5 & 2 & -19 \end{bmatrix}\begin{bmatrix} 3 & 5 & 7 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3.$$

Note: If A is a singular matrix, i.e. $|A| = 0$, then A . (Adj A) = 0

EXERCISE 6.2

1. Find adjoint of the following matrices:

   (a) $\begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (c) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

2. Find adjoint of the following matrices:

   (a) $\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & -i \\ i & i \end{bmatrix}$

   Also verify in each case that A. (Adj A) = (Adj A) A = $|A| I_2$.

3. Verify that

   A. (Adj A) = (Adj A) A = $|A| I_3$, where A is given by

   (a) $\begin{bmatrix} 6 & 8 & -1 \\ 0 & 5 & 4 \\ -3 & 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 7 & 9 \\ 0 & -1 & 2 \\ 3 & -7 & 4 \end{bmatrix}$

   (c) $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & -6 & 1 \\ -1 & -1 & 1 \\ -4 & 11 & -1 \end{bmatrix}$

6.4 INVERSE OF A MATRIX

Consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ We will fmd, if possible, a matrix

$B = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$ such that AB = BA = I i.e., $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

or $\begin{bmatrix} ax+bu & ay+bv \\ cx+du & cy+dv \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

On comparing both sides, we get

$ax + by = 1; \quad ay + bv = 0$

$cx + du = 0; \quad cy + dv = 1$

Solving for x, y, u and v , we get

$$x = \frac{d}{ad-bc},\ y = \frac{-b}{ad-bc},\ u = \frac{-c}{ad-bc},\ v = \frac{a}{ad-dc}$$

provided $ad - bc \neq 0$ i.e., $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$.

Thus, $B = \begin{vmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-dc} \end{vmatrix}$

or $B = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

It may be verified that BA = I

It may be noted from above that, we have been able to find a matrix.

$$B = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{|A|}\text{Adj A} \qquad ...(1)$$

This matrix B, is called the inverse of A and is denoted by A^{-1}.

For a given matrix A, if there exists a matrix B such that AB = BA = I, then B is called the multiplicative inverse of A. We write this as $B = A^{-1}$.

Note : Observe that if $ad - bc = 0$ i.e., $|A| = 0$ the R.H.S. of (1) does not exist and $B = (A^{-1})$ is not defined. This is the reason why we need the matrix A to be non-singular in order that *A* possesses multiplicative inverse. Hence only non-singular matrices possess multiplicative inverse. Also *B* is non-singular and $A = B^{-1}$.

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Example 6.8: Find the inverse of the matrix

$$A = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$$

Solution: $A = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$

Therefore, $|A| = -12 - 10 = -22 \neq 0.$

$\therefore$ A is non-singular. It means A has an inverse. i.e. A^{-1} exists.

Now, $\text{Adj } A = \begin{bmatrix} -3 & -5 \\ -2 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A = \frac{1}{-22}\begin{bmatrix} -3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{22} & \frac{5}{22} \\ \frac{1}{11} & \frac{-2}{11} \end{bmatrix}$$

Note: Verify that $AA^{-1} = A^{-1}A = I.$

Example 6.9: Find the inverse of matrix $A = \begin{bmatrix} 3 & 2 & -2 \\ 1 & -1 & 6 \\ 5 & 4 & -5 \end{bmatrix}$.

Solution: Here, $A = \begin{bmatrix} 3 & 2 & -2 \\ 1 & -1 & 6 \\ 5 & 4 & -5 \end{bmatrix}$

$$\therefore \quad |A| = 3(5 - 24) - 2(-5 - 30) - 2(4 + 5)$$

$$= 3(-19) - 2(-35) - 2(9)$$

$$= -57 + 70 - 18 = -5 \neq 0$$

$\therefore$ A^{-1} exists.

Let A_{ij} be the cofactor of the element a_{ij}.

Then,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 6 \\ 4 & -5 \end{vmatrix} = 5 - 24 = -19,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix}1 & 6\\ 5 & -5\end{vmatrix} - (-5-30) = 35,$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix}1 & -1\\ 5 & 4\end{vmatrix} = 4+5 = 9,$$

$$A_{21} = (-1)^{1+2}\begin{vmatrix}2 & -2\\ 4 & -5\end{vmatrix} = -(-10+8) = 2,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix}3 & -2\\ 5 & -5\end{vmatrix} = -15+10 = -5,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix}3 & 2\\ 5 & 4\end{vmatrix} = -(12-10) = -2$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix}2 & -2\\ -1 & 6\end{vmatrix} = 12-2 = 10,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix}3 & -2\\ 1 & 6\end{vmatrix} = -(18+2) = -20,$$

and $$A_{33} = (-1)^{3+3}\begin{vmatrix}3 & 2\\ 1 & -1\end{vmatrix} = -3-2 = -5$$

$$\text{Matrix of cofactors} = \begin{bmatrix}-19 & 35 & 9\\ 2 & -5 & -2\\ 10 & -20 & -5\end{bmatrix},$$

$$\text{Hence}\quad \text{Adj } A = \begin{bmatrix}-19 & 2 & 10\\ 35 & -5 & -20\\ 9 & -2 & -5\end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|}\text{Adj } A = \frac{1}{-5}\begin{bmatrix}-19 & 2 & 10\\ 35 & -5 & -20\\ 9 & -2 & -5\end{bmatrix} = \begin{bmatrix}\frac{19}{5} & \frac{-2}{5} & -2\\ -7 & 1 & 4\\ \frac{-9}{5} & \frac{2}{5} & 1\end{bmatrix}$$

Note: Verify that $A^{-1}A = AA^{-1} = I$.

Example 6.10: If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$; find

(i) $(AB)^{-1}$ (ii) $B^{-1}A^{-1}$ (iii) Is $(AB)^{-1} = B^{-1}A^{-1}$

Solution:(i) Here, $AB = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} -2+0 & 1+0 \\ -4+0 & 2+1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$$

$$\therefore |AB| = \begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} = -6+4 = -2 \neq 0.$$

Then $(AB)^{-1}$ exists.

Let us denote AB by C_{ij}.

Let C_{ij} be the cofactor of the element c_{ij} of |C|.

Then, $C_{11} = (-1)^{1+1}(3) = 3$ $C_{21} = (-1)^{2+1}(1) = -1$

$C_{12} = (-1)^{1+2}(-4) = 4$ $C_{22} = (-1)^{2+2}(-2) = -2$

Hence, $\text{Adj}(C) = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$

$$C^{-1} = \frac{1}{|C|}\text{Adj}(C) = \frac{1}{-2}\begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

$$C^{-1} = (AB)^{-1} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

(ii) To find $B^{-1}A^{-1}$, find we will find B^{-1}.

$$\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \therefore |B| = \begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix} = 2-0 = 2 \neq 0$$

$\therefore$ B^{-1} exists.

Let B_{ij} be the cofactor of the element b_{ij} of |B|.

then $B_{11} = (-1)^{1+1}(-1) = -1$ $\quad B_{21} = (-1)^{2+1}(1) = -1$

$B_{12} = (-1)^{1+2}(0) = 0$ $\quad B_{22} = (-1)^{2+2}(-2) = -2$

Hence, $\text{Adj } B = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$

$$\therefore\ B^{-1} = \frac{1}{|B|}\text{Adj } B = \frac{1}{2}\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} \\ 0 & -1 \end{bmatrix}$$

Also, $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ Therefore, $|A| = \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -1 - 0 = -1 \neq 0$

Therefore, A^{-1} exists.

Let A_{ij} be the cofactor of the element a_{ij} of $|A|$.

then $A_{11} = (-1)^{1+1}(-1) = -1$ $\quad A_{21} = (-1)^{2+1}(0) = 0$

$A_{12} = (-1)^{1+2}(2) = -2$ and $A_{22} = (-1)^{2+2}(1) = 1$

Hence $\text{Adj } A = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$

$$\Rightarrow\ A^{-1} = \frac{1}{|A|}\text{Adj } A = \frac{1}{-1}\begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

Thus $B^{-1}A^{-1} = \begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{-1}{2} - 1 & 0 + \frac{+1}{2} \\ 0 - 2 & 0 + 1 \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$$

(iii) From (i) and (ii), we find that

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} \frac{-3}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$$

Hence, $(AB)^{-1} = B^{-1}A^{-1}$.

EXERCISE 6.3

1. Find, if possible, the inverse of each of the following matrices:

(a) $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

2. Find, ifpossible, the inverse ofeach of the following matrices :

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 4 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 & 2 \\ 5 & 2 & 4 \\ 1 & -3 & -2 \end{bmatrix}$

Verify that $A^{-1}A = AA^{-1} = I$ for (a) and (b).

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 3 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 3 \\ 3 & 0 & -2 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$

4. Find $(A')^{-1}$ if $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$.

5. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$ show that ABA^{-1} is a diagonal matrix.

6. If $\phi(x) = \begin{vmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{vmatrix}$, show that $[\phi(x)]^{-1} = \phi(-x)$.

7. If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A'A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

8. If $A = \begin{bmatrix} a & b \\ c & \dfrac{1+bc}{a} \end{bmatrix}$, show that $aA^{-1} = (a^2 + bc + 1)\,I - aA$.

9. If $A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = A^2$.

10. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & -4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, show that $A^{-1} = A'$.

6.5 SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

In earlier classes, you have learnt how to solve linear equations in two or three unknowns (simultaneous equations). In solving such systems of equations, you used the process of elimination of variables. When the number of variables invovled is large, such elimination process becomes tedious.

You have already learnt an alternative method, called Cramer's Rule for solving such systems of linear equations.

We will now illustrate another method called the matrix method, which can be used to solve the system of equations in large number of unknowns. For simplicity the illustrations will be for system of equations in two or three unknowns.

6.5.1 MATRIX METHOD

In this method, we first express the given system of equation in the matrix form $AX = B$, where A is called the co-efficient matrix.

For example, if the given system of equation is $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ and we express them in the matrix equation form as :

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Here, $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

If the given system of equations is $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$ then this system is expressed in the matrix equation form as :

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

where, $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Before proceding to fmd the solution, we check whether the coefficient matrix A is non-singular or not.

Note: If A is singular, then $|A| = 0$. Hence, A^{-1} does not exist and so, this method does not work.

Consider equation $AX = B$, where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

When $|A| \neq 0$ when $a_1b_2 - a_2b_1 \neq 0$ we multiply the equation AX = B with A^{-1} on both side and get

$$A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B \qquad (\because A^{-1}A = I)$$

$$\Rightarrow X = A^{-1}B$$

$$A^{-1} = \frac{1}{a_1b_2 - a_2b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}, \quad \text{we get}$$

$$X = \frac{1}{a_1b_2 - a_2b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{a_1b_2 - a_2b_1}\begin{bmatrix} b_2c_1 - b_1c_2 \\ -a_2c_1 + a_1c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \\ \dfrac{-a_2c_1 + a_1c_2}{a_1b_2 - a_2b_1} \end{bmatrix}$$

Hence, $x = \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$ and $y = \dfrac{-a_2c_1 + a_1c_2}{a_1b_2 - a_2b_1}$

Example 6.11 : Using matrix method, solve the given system of linear equations.

$$\left.\begin{aligned} 4x - 3y &= 11 \\ 3x + 7y &= -1 \end{aligned}\right\} \qquad ...(1)$$

Solution: This system can be expressed in the matrix equation form as

$$\begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix} \qquad ...(ii)$$

Here, $A = \begin{bmatrix} 4 & -3 \\ 3 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$

so, (ii) reduces to

$$AX = B \qquad ...(iii)$$

Now, $|A| = \begin{vmatrix} 4 & -3 \\ 3 & 7 \end{vmatrix} = 28 + 9 = 37 \neq 0,$

Since $|A| \neq 0$, A^{-1} exists.

Now, on multiplying the equation AX = B with A^{-1} `on both sides, we get

$A^{-1}(AX) = A^{-1}B$

$(A^{-1}A)X = A^{-1}B$

i.e., $IX = A^{-1}B$.

$X = A^{-1}B$.

Hence, $X = \dfrac{1}{|A|}(\text{Adj } A)B$

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$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 7 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 11 \\ -1 \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 77 & -3 \\ -33 & -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{37}\begin{bmatrix} 74 \\ -37 \end{bmatrix}$$

or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

So, $x = 2, y = -1$ is is unique solution of the system of equations.

Example 6.12: Solve the following system of equations, using matrix method.

$$2x - 3y = 7$$
$$x + 2y = 3$$

Solution: The given system of equations in the matrix equation form, is

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

or $\quad AX = B \qquad$ .... (i)

where $\quad A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

$$|A| = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} = 2\times 2 - 1\times(-3)$$

$$= 4 + 3 = 7 \neq 0$$

A^{-1} exists.

Since, $\quad \text{Adj (A)} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\text{adj(A)} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix} \qquad \text{...(ii)}$$

From (i), We have $\quad X = A^{-1}B$

or, $\quad X = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} \\ \frac{-1}{7} & \frac{2}{7} \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 + \frac{9}{7} \\ \frac{-7}{7} + \frac{6}{7} \end{bmatrix} = \begin{bmatrix} \frac{23}{7} \\ -\frac{1}{7} \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{23}{7} \\ \frac{-1}{7} \end{bmatrix}$.

Thus, $x = \frac{23}{7}, \ y = \frac{-1}{7}$ is the solution of this system of equations,

Example 6.13: Solve the following system of equations, using matrix method.

$$\left.\begin{aligned} x+2y+3z &= 14 \\ x-2y+z &= 0 \\ 2x+3y-z &= 5 \end{aligned}\right\}$$

Solution: The given equations expressed in the matrix equation form as :

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 5 \end{bmatrix} \qquad \text{...(i)}$$

which is in the form AX = B, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 14 \\ 0 \\ 5 \end{bmatrix}$$

$\therefore \ X = A^{-1}B$ ...(ii)

Here, $|A| = 1(2 - 3) - 2(-1 -2) + 3(3 + 4)$

$= 26 \neq 0$

$\therefore \ A^{-1}$ exists.

Also, $\text{Adj } A = \begin{bmatrix} -1 & 11 & 8 \\ 3 & -7 & 2 \\ 7 & 1 & -4 \end{bmatrix}$

Hence, from (ii), we have $X = A^{-1}B = \frac{1}{|A|} \text{Adj} A.B$

$$X = \frac{1}{26} \begin{bmatrix} -1 & 11 & 8 \\ 3 & -7 & 2 \\ 7 & 1 & -4 \end{bmatrix} \begin{bmatrix} 14 \\ 0 \\ 5 \end{bmatrix}$$

$$= \frac{1}{26}\begin{bmatrix}26\\52\\78\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

or, $$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Thus, $x = 1; y = 2$ and $z = 3$ is the solution of the given system of equations.

Example 6.14 : Solve the following system of equations, using matrix method:

$$x + 2y + z = 2$$

$$2x - y + 3z = 3$$

$$x + 3y - z = 0$$

Solution: The given system of equation can be represented in the matrix equation form as :

$$\begin{bmatrix}1 & 2 & 1\\2 & -1 & 3\\1 & 3 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\3\\0\end{bmatrix}$$

i.e., $AX = B$ ...(i)

Now, $A = \begin{bmatrix}1 & 2 & 1\\2 & -1 & 3\\1 & 3 & -1\end{bmatrix}$, $X = \begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B = \begin{bmatrix}2\\3\\0\end{bmatrix}$

Now, $|A| = \begin{bmatrix}1 & 2 & 1\\2 & -1 & 3\\1 & 3 & -1\end{bmatrix} = 1(1-9) - 2(-2-3) + 1(6+1) = 9 \neq 0$

Hence, A^{-1} exists.

Also. $\text{Adj } A = \begin{bmatrix}-8 & 5 & 7\\5 & -2 & -1\\7 & -1 & -5\end{bmatrix}$

$$\therefore\ A^{-1} = \frac{1}{|A|}\text{Adj } A = \frac{1}{9}\begin{bmatrix} -8 & 5 & 7 \\ 5 & -2 & -1 \\ 7 & -1 & -5 \end{bmatrix}$$

Form (i) we have $X = A^{-1}B$.

i.e.,
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -8 & 5 & 7 \\ 5 & -2 & -1 \\ 7 & -1 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} -1 \\ 4 \\ 11 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ \frac{4}{9} \\ \frac{11}{9} \end{bmatrix}$$

so, $x = \frac{-1}{9},\ y = \frac{4}{9},\ z = \frac{11}{9}$ is the solution of the given system.

6.6 CRITERION FOR CONSISTENCY OF A SYSTEM OF EQUATIONS

Let $AX = B$ be a system of two or three linear equations.

Then, we have the following criteria:

1. If $|A| \neq 0$ then the system of equations is consistent and has a unique solution, given by $X = A^{-1}B$.
2. If $|A| = 0$ 0, then the system mayor may not be consistent and if consistent, it does not have a unique solution. If in addition,
   (a) (Adj A)B $\neq$ 0, then the system is inconsistent.
   (b) (Adj A)B = 0 then the system is consistent and has infmitely many solutions.

Note: These criteria are true for a system *of* 'n' equations in 'n' variables as well.

We now, verify these with the help of the examples and fmd their solutions wherever possible.

(a) $5x+7y=1$

$2x-3y=3$

This system is consistent and has a unique solution, because $\frac{5}{2} \neq \frac{7}{-3}$, Here,

the matrix equation is $\begin{bmatrix} 5 & 7 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

i.e., AX = B ...(i)

where, $A = \begin{bmatrix} 5 & 7 \\ 2 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Here, |A| = 5(−3) −2 þ 7 = −15 − 14 = −29 ≠ 0

and $A^{-1} = \frac{1}{|A|}$ Adj $A = \frac{1}{-29}\begin{bmatrix} -3 & -7 \\ -2 & 5 \end{bmatrix}$ ...(ii)

From (i), we have $X = A^{-1}B$

i.e., $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-29}\begin{bmatrix} -3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{24}{29} \\ -\frac{13}{29} \end{bmatrix}$ [From (i) and (ii)]

Thus, $x = \frac{24}{29}$, and $y = \frac{-13}{29}$ .is the unique soolution of the given system of equations.

(b) $3x+2y=7$

$6x+4y=8$

This system is incosisntent i.e. it has no solution because $\frac{3}{6} = \frac{2}{4} \neq \frac{7}{8}$

In the matrix form the system can be written as

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

or, AX = B

where $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

or, $|A| = 3 \times 4 - 6 \times 2 = 12 - 12 = 0$

$$\text{Adj } A = \begin{bmatrix} 4 & -6 \\ -6 & 3 \end{bmatrix}$$

$$\text{Also (Adj A) B} = \begin{bmatrix} 4 & -6 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} -20 \\ -18 \end{bmatrix} \neq 0$$

Thus, the given system of equations is inconsistent.

(c) $\left.\begin{array}{r} 3x - y = 7 \\ 9x - 3y = 21 \end{array}\right\}$ This system is consistent and has infinitely

many solutions, because $\frac{3}{9} = \frac{-1}{3} = \frac{7}{21}$,

In the matrix fonn the system can be written as

$$\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \end{bmatrix}$$

or $AX = B$, where

$$A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ 21 \end{bmatrix}$$

Here, $|A| = \begin{vmatrix} 3 & -1 \\ 9 & -3 \end{vmatrix} = 3X = (-3) - 9 \times (-1)$

$$= -9 + 9 = 0.$$

$$\text{Adj } A = \begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix}$$

$$\text{Also} \quad (\text{Adj } A)B = \begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0.$$

$\therefore$ The given system has an infmite number of solutions.

Till now, we have seen that

(i) If $|A| = 0$ and $(\text{Adj } A)\, B \neq 0$ then the system of equations will have a non-zero, unique solution.

(ii) If $|A| = 0$ and $(\text{Adj } A)\, B = 0$ then the system of equations will have trivial solution $x = y = z = 0$.

(iii) If $|A| = 0$ and $(\text{Adj } A)\, B = 0$ then the system of equations will have infinitely many solutions.

(iv) If $|A| = 0$ and $(\text{Adj } A)\, B \neq 0$ then the system of equations is inconsistent.

Let us now consider another system of linear equations, where $|A| = 0$ and $(\text{Adj } A)\, B \neq 0$ Consider the following system of equations

$$x + 2y + z = 5$$

$$2x + y + 2z = -1$$

$$x - 3y + z = 6$$

In matrix equation form, the above system of equations can be written as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$$

i.e., $AX = B$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -3 & 1 \end{vmatrix} = 0 \quad (\because C_1 = C_3)$

Also, $(\text{Adj A})\,\text{B} = \begin{bmatrix} 7 & -5 & 3 \\ 0 & 0 & 0 \\ -7 & 5 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 6 \end{bmatrix}$ (Verify (Adj A) yourself]

$$= \begin{bmatrix} 58 \\ 0 \\ -58 \end{bmatrix} \neq 0$$

Since $|A| = 0$ and $(\text{Adj A})\text{B} \neq 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|}(\text{Adj A})\text{B}$$

$$= \frac{\begin{bmatrix} 58 \\ 0 \\ -58 \end{bmatrix}}{0} \text{ which is undefined.}$$

The given system of linear equation will have no solution.

Thus, we find that if $|A| = 0$ and $(\text{Adj A})\,\text{B} = 0$ then the system of equations will have no solution.

We can summarise the above finding as:

(i) If $|A| = 0$ and $(\text{Adj A})\,\text{B} \neq 0$ then the system of equations will have a non-zero, unique solution.

(ii) If $|A| = 0$ and $(\text{Adj A})\,\text{B} = 0$ then the system of equations will have trivial solutions.

(iii) If $|A| = 0$ and $(\text{Adj A})\,\text{B} = 0$ then the system of equations will have infinitely many solutions.

(iv) If $|A| = 0$ and $(\text{Adj A})\,\text{B} \neq 0$ then the system of equations will have no solution inconsistent.

Example 6.15: Use matrix inversion method to solve the system of equations:

(i) $6x+4y=2$
$9x+6y=3$

(ii) $2x-y+3z=1$
$x+2y-3z=2$
$5y-5z=3$

Solution: (i) The given system in the matrix equation form is

$$\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

i.e., AX = B

where, $A=\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$ $X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Now, $|A|=\begin{vmatrix} 6 & 4 \\ 9 & 6 \end{vmatrix}=6\times6-9\times4=36-36=0$

$\therefore$ The system has either infmitely solutions or no solution.

Let $x = k$, then $6k + 4y = 2$ gives $y=\dfrac{1-3k}{2}$

Putting these values of x *and* y in the second equation, we have

$$9k+6\left(\frac{1-3k}{2}\right)=3$$

$\Rightarrow$ $18k + 6 - 18k = 6$

$\Rightarrow$ $6 = 6$, which is true.

$\therefore$ The given system has infmitely many solutions. These are

$x = k$, $y=\dfrac{1-3k}{2}$, where k is any arbitrary number.

(ii) The given equations are

$2x - y + 3z = 1$ ... (1)

$x + 2y - z = 1$ ... (2)

$5y - 5z = 3$ ... (3)

In matrix equation form, the given system of equations is

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 0 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

i.e., AX = B

where, $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 0 & 5 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 0 & 5 & -5 \end{vmatrix} = 2 \times \begin{vmatrix} 2 & -1 \\ 5 & -5 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & -1 \\ 0 & -5 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix}$

$= 2(-10 + 5) + 1(-5 - 0) + 3(5 - 0)$

$= -10 - 5 + 15 = 0$

$\therefore$ The system has either infmitely many solutions or no solution

Let $z = k$. Then from (1), we have $2x - y = 1 - 3k$; and from (2), we have $x + 2y = 2 + k$.

Now, we have a system of two equations, namely

$2x - y = 1 - 3k$

$x + 2y = 2 + k$

$$\Rightarrow \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1-3k \\ 2+k \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 1-3k \\ 2+k \end{bmatrix}$

Then $|A| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 4 + 1 = 5 \neq 0$

$\therefore$ A^{-1} exists.

Here, $A^{-1} = \frac{1}{|A|}\text{Adj } A = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$

$\therefore$ The solution is $X = A^{-1}B$.

$$= \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} 1-3k \\ 2+k \end{bmatrix} = \begin{bmatrix} -k+\frac{4}{5} \\ k+\frac{3}{5} \end{bmatrix}$$

$\therefore \quad x = -k+\frac{4}{5},\ y = k+\frac{3}{5},$ where k is any number.

Putting these values of x, y and z in (3), we get

$$5\left(k+\frac{3}{5}\right) - 5k = 3$$

$\Rightarrow 5k + 3 - 5k = 3 \Rightarrow 3 = 3$, which is true.

$\therefore$ The given system of equations has infmitely many solutions, given by

$$x = -k+\frac{4}{5};\ y = k+\frac{3}{5} \text{ and } z = k, \text{ where } k \text{ is any number.}$$

6.7 HOMOGENEOUS SYSTEM OF EQUATIONS

A system of linear equations AX = B with matrix, B = 0, a null matrix, is called homogeneous system of equations.

Following are some systems of homogeneous equations:

(i) $\begin{aligned} x+2y&=0 \\ -2x+3y&=0 \end{aligned}$ (ii) $\begin{aligned} 2x+5y-3z&=0 \\ x-2y+z&=0 \\ 3x-y-6z&=0 \end{aligned}$ (iii) $\begin{aligned} 2x+y-3z&=0 \\ x-2y+z&=0 \\ 3x-y-2z&=0 \end{aligned}$

Let us now solve a system of equations mentioned in (ii).

Given system is

$$2x+5y-3z=0$$
$$x-2y+z=0$$
$$3x-y-6z=0$$

In matrix equation form, the system (ii) can be written as

$$\begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., AX = O

where $A = \begin{bmatrix} 2 & 5 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now $|A| = 2(12 + 1) - 5(-6 -3) -3(-1 + 6)$

$= 26 + 45 - 15$

$= 56 \neq 0$

But $B = 0 \Rightarrow (\text{Adj } A)B = 0.$

Thus, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{|A|}(\text{Adj}A)B = 0$

$\therefore\ x = 0;\ y = 0;\ z = 0.$

i.e., the system of equations will have trivial solution.

Remarks : For a homogeneous system of linear equations, if $|A| \neq 0$ and (Adj A) B = 0.

There will be only trivial solution.

Now, consider the system of equations mentioned in (iii) :

$$2x+y-3z=0$$
$$x-2y+z=0$$
$$3x-y-2z=0$$

In matrix equation form, the above system (iii) can be written as

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., AX = 0

where, $A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 2(4+1)-1(-2-3)-3(-1+6)$

$$= 10 + 5 - 15$$

$$= 0$$

Also, $B = 0 \Rightarrow (\text{Adj } A)B = 0.$

Thus, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|}(\text{Adj}A)B = \frac{1}{0}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ The system of equations will have infmitelymany solutions which will be non-trivial. Considering the first two equations, we get

$$2x + y = 3z$$

$$x - 2y = -z$$

Solving, we get $x = z,\ y = z$ let $z = k,$ where k is any number.

Then $x = k, y = k, z = k$ are the solutions of this system.

Note: For a system of homogenous equations, if |A| = 0 and (Adj A) B = 0, there will be infinitely many solutions.

EXERCISE 6.4

1. Solve the following system of equations, using the matrix inversion method:

(a) $2x+3y=4$
$x-2y=5$

(b) $x+y=7$
$3x-7y=11$

(c) $3x+4y-5=0$
$x-2y+6=0$

(d) $2x-3y+6=0$
$6x+y-8=0$

2. Solve the following system of equations using matrix inversion method:

(a) $x+2y+z=3$
$2x-y+3z=5$
$x+y-z=7$

(b) $2x+3y+z=13$
$3x+2y-2=12$
$x+y+2z=5$

(c) $-x+2y+5z=2$
$2x-3y+z=15$
$-x+y+z=-3$

(d) $2x+y-z=2$
$x+2y-3z=-1$
$5x-y-2z=-1$

3. Solve the following system of equations, using matrix inversion method:

(a) $x+y+z=0$
$2x-y+z=0$
$x-2y+3z=0$

(b) $3x-2y+3z=0$
$2x+y+z=0$
$4x-3y+2z=0$

(c) $x+y+1=0$
$y+z-1=0$
$z+x=0$

4. Determine whether the following system of equations are consistent or not. If consistent, find the solution:

(a) $2x-3y=5$
$x+y=7$

(b) $2x-3y=5$
$4x-6y=10$

(c) $3x+y+2z=3$
$-2y-z=7$
$x+15y+3z=11$

(d) $x+2y-3z=0$
$4x-y+2z=0$
$3x+5y-4z=0$

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KEY WORDS

- A square matrix is said to be non-singular if its corresponding determinant is non-zero.
- The determinant of the matrix A obtained by deleting the i^{th} row and j^{th} column of *A,* is called the minor of a_{ij} It is usually denoted by M_{ij}.
- The cofactor of a_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.
- Adjoint of a matrix *A* is the transpose of the matrix whose elements are the cofactors of the elements of the determinat of given matrix. It is usually denoted by Adj *A*.
- If A, is any square matrix of order *n*, then

  A (Adj A) = (Adj A) A = |A| I_n where I_n is the unit matrix of order *n*.
- For a given non-singular square matrix A, if there exists anon-singular square matrix B such thatAB = BA = 1, then B is called the multiplicative inverse ofA. It is written as B = A^{-1}.
- Only non-singular square matrices have multiplicative inverse.
- If $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ then we can express the system in the matrix equation form as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus, $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ then

$$X = A^{-1}B = \frac{1}{a_1b_2 - a_2b_1}\begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

- A system of equations, given by AX = B , is said to be consistent and has a unique solution, if $|A| \neq 0$.
- A system of equations, given by AX = B is said to be inconsistent, if $|A| = 0$ and $(\text{Adj } A)B \neq 0$.
- A system of equations, given by AX = B is said to be consistent and has infinitely many solutions, if $|A| = 0$ and $(\text{Adj } A)B = 0$.
- A system of equations, given by AX = B is said to be homogenous, if B is the null matrix.
- A homogenous system of linear equations, AX = 0 has only a trivial solution $x_1 = x_2 = ... = x_n = 0$ if $|A| \neq 0$.
- A homogenous system of linear equations, AX = 0 has infinitely many solutions, $|A| = 0$.

SUPPORTIVE WEB SITES

http : //www.wikipedia.org

http:// math world . wolfram.com

PRACTICE EXERCISE

1. Find |A|, if

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 1 & 0 \\ -2 & 5 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} -1 & 3 & 4 \\ 7 & 5 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

2. Find the adjoint of A, if

(a) $A = \begin{bmatrix} -2 & 3 & 7 \\ -1 & 4 & 5 \\ -1 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & 2 \\ -2 & 1 & 3 \end{bmatrix}$

Also, verify that $A(\text{Adj } A) = |A|\, I_3 = (\text{Adj } A)\, A$ for (a) and (b)

3. Find A^{-1}, if exists, when

(a) $\begin{bmatrix} 3 & 6 \\ 7 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$

Also, verify that $(A')^{-1} = (A^{-1})'$ for (a), (b) and (c)

4. Find the inverse of the matrix *A,* if

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

5. Solve, using matrix inversion method, the following systems oflinear equations

(a) $\begin{aligned} x + 2y &= 4 \\ 2x + 5y &= 9 \end{aligned}$ (b) $\begin{aligned} 6x + 4y &= 2 \\ 9x + 6y &= 3 \end{aligned}$

(c) $\begin{aligned} 2x + y + z &= 1 \\ x - 2y - z &= \frac{3}{2} \\ 3y - 5z &= 9 \end{aligned}$ (d) $\begin{aligned} x - y + z &= 4 \\ 2x + y - 3z &= 0 \\ x + y + z &= 2 \end{aligned}$

(e) $\begin{aligned} x + y - 2z &= -1 \\ 3x - 2y + z &= 3 \\ 2x + y - z &= 0 \end{aligned}$

6. Solve, using matrix inversion method

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4; \ \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1; \ \frac{8}{x} + \frac{9}{y} - \frac{20}{z} = 3$$

7. Find non-trivial solution of the following system oflinear equations:

$$\begin{aligned} 3x + 2y + 7z &= 0 \\ 4x - 3y - 2z &= 0 \\ 5x + 9y + 23z &= 0 \end{aligned}$$

8. Solve the following homogeneous equations:

(a) $\begin{aligned} x+y-z&=0 \\ x-2y+z&=0 \\ 3x+6y-5z&=0 \end{aligned}$ (b) $\begin{aligned} x+2y-2z&=0 \\ 2x+y-3z&=0 \\ 5x+4y-9z&=0 \end{aligned}$

9. Find the value of 'p' for which the equations

$$\begin{aligned} x+2y+z&=px \\ 2x+y+z&=py \\ x+y+2z&=pz \end{aligned}$$

have anon-trivial solution

10. Find the value of A for which the following system of equation becomes consistent

$$\begin{aligned} 2x-3y+4&=0 \\ 5x-2y-1&=0 \\ 21x-8y+\lambda&=0 \end{aligned}$$

ANSWERS

EXERCISE 6.1

1. (a) -12 (b) 10

2. (a) singular (b) non-singular

3. (a) $M_{11} = 4$; $M_{12} = 7$; $M_{21} = -1$; $M_{22} = 3$

   (b) $M_{11} = 5$; $M_{12} = 2$; $M_{21} = 6$; $M_{22} = 0$

4. (a) $M_{21} = 11$; $M_{22} = 7$; $M_{23} = 1$

   (b) $M_{31} = -13$; $M_{32} = -13$; $M_{33} = 13$

5. (a) $C_{11} = 7;\ C_{12} = -9;\ C_{21} = 2;\ C_{22} = 3.$

(b) $C_{11} = 6;\ C_{12} = 5;\ C_{21} = -4;\ C_{22} = 0$

6. (a) $C_{21} = 1;\ C_{22} = -8;\ C_{23} = -2$

(b) $C_{11} = -6;\ C_{12} = 10;\ C_{33} = 2$

EXERCISE 6.2

1. (a) $\begin{bmatrix} 6 & 1 \\ -3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} d & b \\ -c & a \end{bmatrix}$ (c) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

2. (a) $\begin{bmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$ (b) $\begin{bmatrix} i & i \\ -i & i \end{bmatrix}$

EXERCISE 6.3

1. (a) $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -4/10 & -2/10 \\ 3/10 & -1/10 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

2. (a) $\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} & \frac{2}{5} \\ -\frac{8}{5} & \frac{6}{5} & -\frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{7}{12} & \frac{1}{3} & \frac{1}{12} \\ \frac{17}{24} & -\frac{1}{3} & -\frac{11}{24} \end{bmatrix}$

4. $(A')^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

EXERCISE 6.4

1. (a) $x = \frac{23}{7}, y = \frac{-6}{7}$ (b) $x = 6, y = 1$

   (c) $x = -\frac{7}{5}, y = \frac{23}{10}$ (d) $x = \frac{27}{30}, y = \frac{13}{5}$

2. (a) $x = \frac{58}{11}, y = -\frac{2}{11}, z = -\frac{21}{11}$ (b) $x = 2, y = 3, z = 0$

   (c) $x = 2, y = -3, z = 2$ (d) $x = 1, y = 2, z = 2$

3. (a) $x = 0, y = 0, z = 0$ (b) $x = 0, y = 0, z = 0$

   (c) $x = 0, y = 0, z = 0$ (d) $x = 0, y = 0, z = 0$

4. (a) Consistent; $x = \frac{26}{5}, y = \frac{9}{5}$

   (b) Consistent; infinitely many solutions

   (c) Inconsistent

   (d) Trivial solution, $x = y = z = 0$

PRACTICE EXERCISE

1. (a) -31 (b) -24

2. (a) $\begin{bmatrix} 4 & -3 & -13 \\ -4 & 5 & 3 \\ 4 & -3 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 8 & -7 \\ -13 & 13 & 13 \\ 5 & 1 & 4 \end{bmatrix}$

3. (a) $\begin{bmatrix} \frac{-1}{18} & \frac{1}{6} \\ \frac{7}{36} & \frac{-1}{12} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{5}{7} & \frac{-1}{7} \\ \frac{-3}{7} & \frac{2}{7} \end{bmatrix}$

   (c) $\begin{bmatrix} \frac{-1}{7} & \frac{-5}{14} \\ \frac{-2}{7} & \frac{-3}{14} \end{bmatrix}$

4. (a) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{3} & 0 \\ 3 & \frac{2}{3} & -1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{3}{2} & -1 & \frac{-1}{2} \\ \frac{-1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{-3}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

5. (a) $x = 2, \ y = 1$

(b) $x = k, \ y = \frac{1}{2} - \frac{3}{2}k$

(c) $x = 1, \ y = \frac{1}{2}, \ z = -\frac{3}{2}$

(d) $x = 2, \ y = -1, \ z = 1$

(e) $x = \frac{1}{2}, \ y = -\frac{1}{2}, \ z = \frac{1}{2}$

6. $x = 2, \ y = 3, \ z = 5$

7. $x = -k, \ y = -2k, \ z = k$

8. (a) $x = k, \ y = 2k, \ z = 3k$

(b) $x = y = z = k$

9. $p = 1, -1, 4$

10. $\lambda = -5.$

Chapter 7

PERMUTATIONS AND COMBINATIONS

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- find out the number of ways in which a given number of objects can be arranged;
- state the Fundamental Principle of Counting;
- define n ! and evaluate it for defferent values of n ;
- state that permutation is an arrangement and write the meaning of ${}^nP_r$.
- state that ${}^nP_r = \dfrac{n!}{(n-r)!}$ and apply this to solve problems;
- show that (i) $(n+1)\,{}^nP_r = (n+1)\,P_n$ (ii) ${}^nP_{r+1} = (n-r)\,{}^nP_r$
- state that a combination is a selection and write the meaning of ${}^nC_r$.
- distinguish between permutations and combinations;
- derive ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ and apply the result to solve problems;
- derive the relation ${}^nP_r = r!\ {}^nC_r$
- verify that ${}^nC_r = {}^nC_{n-r}$ and give its interpretation; and
- derive ${}^nC_r + {}^nC_{r-1} = (n+1)\,C_r$ and apply the result to solve problems.

PREREQUISITES

- Number Systems
- Four Fundamental Operations

INTRODUCTION

We must have come across situations like choosing five questions out of eight questions in a question paper or which items to be chosen from the menu card in a hotel etc. We discuss such situations in this chapter. This chapter 'permutations and combinations' is an important chapter in algebra in view of a number of applications in day - to - day life and in the theory of probability. While learning **'permutations and combinations'**, we should be in a position to clearly see whether the concept of a permutation or the concept of a combination is applicable in a given situation. In general, a combination is only a selection while a permutation involves two steps, namely, selection and arrangement. For example, forming a three digit number using the digits 1, 2, 3, 4, 5 is a **'permutation'**. This involves two steps. In the first step we select three digits, say 2, 4, 5. In the second step, we arrange them to form a three digit number such as 245, 452, 542 etc. Forming a set with three elements using the digits 1, 2, 3, 4, 5 is a**'combination'**. This involves only one process, namely, selection of three elements, say 2, 4, 5. Then the element set formed is {2, 4, 5} which is same as the sets {4, 5, 2} {5, 4, 2} etc. Thus, whenever there is importance to the arrangement or order in which the objects are placed, then it is a '**permutation**' and if there is no importance to the arrangement or order, but only selection is required, then it is a'**combination**'. These notions will help us to arrive at the number of arrangements or combinations without actually counting them.

Before going into formal definitions, we introduce **factorial** notation, which is required to calculate the number of permutations or combinations. If n is a positive integer, we define $n!$ (read as n factorial) by mathematical induction as follows.

$$1! = 1$$

and $$n! = n \, . \, ((n-1)!) \text{ if } n > 1.$$

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For example, $2! = 2\,(1!) = 2$

$3! = 3\,(2!) = 3.2 = 6$

$4! = 4\,(3!) = 4.6 = 24$

$5! = 5\,(4!) = 5.24 = 120$ etc.

By convention , we define $0! = 1$

Throughout this chapter the letters n, r denote nonnegative integers unless otherwise mentioned.

7.1 COUNTING PRINCIPLE

Let us now solve the problem mentioned in the introduction. We will write t_1, t_2 to denote trains from Bangalore to Itarsi and T_1, T_2, T_3, for the trains from Itarsi to Allahabad. Suppose I take t_1 to travel from Bangalore to Itarsi. Then from Itarsi I can take T_1 or T_2 or T_3. So the possibilities are $t_1 T_1$, t_2T_2 and t_3T_3 where $t_1\ T_1$ denotes travel from Bangalore to Itarsi by t_1 and travel from Itarsi to Allahabad by T_1. Similarly, if I take t_2 to travel from Bangalore to Itarsi, then the possibilities are t_2T_1 't_2T_2 and t_2T_3' . Thus, in all there are 6 (2×3) possible ways of travelling from Bangalore to Allahabad.

Here we had a small number of trains and thus could list all possibilities. Had there been 10 trains from Bangalore to Itarsi and 15 trains from Itarsi to Allahabad, the task would have beenvery tedious. Here the Fundamental Principle of Counting or simply the Counting Principle comes In use:

If any event can occur in m ways and after it happens in anyone of these ways, a second event can occur in n ways, then both the events together can occur in $m \times n$ ways.

Example 7.1: How many multiples of 5 are there from 10 to 95 ?

Solution: As you know, multiples of 5 are integers having 0 or 5 in the digit to the extreme right (i.e. the unit's place).

The first digit from the right can be chosen in 2 ways.

The second digit can be anyone of 1,2,3,4,5,6,7,8,9.

i.e., There are 9 choices for the second digit.

Thus, there are $2 \times 9 = 18$ multiples of 5 from 10 to 95.

Example 7.2: In a city, the bus route numbers consist of a natural number less than 100, followed by one of the lettersA,B, C,D,E andF. How many different bus routes are possible?

Solution: The number can be anyone of the natural numbers from 1 to 99.

There are 99 choices for the number.

The letter can be chosen in 6 ways.

$\therefore$ Number of possible bus routes are $99 \times 6 = 594$.

EXERCISE 7.1

1. (a) How many 3 digit numbers are multiples of 5?

   (b) A coin is tossed thrice. How many possible outcomes are there?

   (c) If you have 3 shirts and 4 pairs of trousers and any shirt can be worn with any pair of trousers, in how many ways can you wear your shirts and pairs of trousers?

   (d) A tourist wants to go to another country by ship and return by air. She has a choice of 5 different ships to go by and 4 airlines to return by. In how many ways can she perform the joumey?

2. (a) In how many ways can two vacancies be filled from among 4 men and 12 women if one vacancy is filled by a man and the other by a woman?

   (b) Flooring and painting of the walls of a room needs to be done. The flooring can be done in 3 colours and painting of walls can be done in 12 colours. If any colour combination is allowed, fmd the number of ways of flooring and painting the walls of the room.

So far, we have applied the counting principle for two events. But it can be extended to three or more, as you can see from the following examples:

Example 7.3: There are 3 questions in a question paper. If the questions have 4,3 and 2 solutionsvely, find the total number of solutions .

Solution: Here question 1 has 4 solutions,

question 2 has 3 solutions

and question 3 has 2 solutions.

$\therefore$ By the multiplication (counting) rule,

total number of solutions $= 4 \times 3 \times 2$

$= 24$

Example 7.4: Consider the word ROTOR. Whichever way you read it, from left to right or from right to left, you get the same word. Such a word is known as *palindrome.* Find the maximum possible number of 5-letter palindromes.

Solution: The first letter from the right can be chosen in 26 ways because there are 26 alphabets.

Having chosen this, the second letter can be chosen in 26 ways

$\therefore$ The first two letters can chosen in $26 \times 26 = 676$ ways

Having chosen the first two letters, the third letter can be chosen in 26 ways.

$\therefore$ All the three letters can be chosen in $676 \times 26 = 17576$ ways.

It implies that the maximum possible number of five letter palindromes is 17576 because the fourth letter is the same as the second letter and the fifth letter is the same as the first letter.

Note: *In Example* 7.4 *we found the maximum possible number of five letter palindromes. There cannot be more than* 17576. *But this does not mean that there are* 17576 *palindromes. Because some of the choices like CCCCC may not be meaningful words in the English language.*

Example 7.5: How many 3-digit numbers can be formed with the digits 1,4,7,8 and 9 if the digits are not repeated.

Solution: Three digit number will have unit's, ten's and hundred's place.

Out of 5 given digits anyone can take the unit's place.

This can be done in 5 ways. ... (i)

After filling the unit's place, any of the four remaining digits can take the ten's place.

This can be done in 4 ways. ... (ii)

After filling in ten's place, hundred's place can be filled from any of the three remaining digits.

This can be done in 3 ways. . .. (iii)

$\therefore$ By counting principle, the number of3 digit numbers = 5 x 4 x 3 = 60

Let us now state the General Counting Principle

If there are n events and if the first event can occur inm_2 ways, the second event can occur in m_2 ways after the first event has occured, the third event can occur in m_3 ways after the second event has ocurred, and so on, then all the n events can occur in $m_1 \times m_2 \times \ldots \times m_{n-1} \times m_n$ ways.

Example 7.6: Suppose you can travel from a place A to a place B by 3 buses, from place B to place C by 4 buses, from place C to place D by 2 buses and from place D to place E by 3 buses. In how many ways can you travel from A to E?

Solution: The bus from A to B can be selected in 3 ways.

The bus from B to C can be selected in 4 ways.

The bus from C to D can be selected in 2 ways.

The bus from D to E can be selected in 3 ways.

So, by the General Counting Principle, one can travel from A to E in $3 \times 4 \times 2 \times 3$ ways = 72 ways.

EXERCISE 7.2

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1. (a) What is the maximum number of 6-letter palindromes?

   (b) What is the number of 6-digit palindromic numbers which do not have 0 in the first digit?

2.(a) In a school there are 5 English teachers, 7 Hindi teachers and 3 French teachers. A three member committee is to be formed with one teacher representing each language. In how many ways can this be done?

   (b) b) In a college students union election, 4 students are contesting for the post of President. 5 students are contesting for the post of Vice-president and 3 students are contesting for the post of Secretary. Find the number of possible results.

3.(a) How many three digit numbers greater than 600 can be formed using the digits 1,2,5,6,8 without repeating the digits?

   (b) A person wants to make a time table for 4 periods. He has to fix one period each for English, Mathematics, Economics and Commerce. How many different time tables can he make?

7.2 PERMUTATIONS

Suppose you want to arrange your books on a shelf. If you have only one book, there is onlyone way of arranging it. Suppose you have two books, one of History and one of Geography. You can arrange the Geography and History books in two ways. Geography book first and the History book next, *GH* or History book first and Geography book next; *HG*. In other words, there are two arrangements of the two books.

Now, suppose you want to add a Mathematics book also to the shelf. After arranging History and Geography books in one of the two ways, say *GH,* you can put Mathematics book in one of the following ways: *MGH, GA1H* or *GHM* Similarly, corresponding to *HG*, you have three other ways of arranging the books. So, by

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Notes

the Counting Principle, you can arrange Mathematics, Geography and History books in 3×2 ways = 6 ways.

By permutation we mean an arrangement of objects in a particular order. In the above example, we were discussing the number of permutations of one book or two books.

In general, if you want to find the number of permutations of n obj ects $n \geq 1$, how can you do it? Let us see if we can fmd an answer to this.

Similar to what we saw in the case of books, there is one permutation of 1 object, 2×1 permutations of two objects and $3 \times 2 \times 1$ permutations of 3 objects. It maybe that, there are $n \times (n-1) \times (n-2) \times ... \times 2 \times 1$ permutations of n objects. In fact, it is so, as you will see when we prove the following result.

Theorem 7.1 The total number of permutations of n objects is $n(n-1) ... .2.1$.

Proof: We have to find the number of possible arrangements of n different objects.

The first place in an arrangement can be filled in n different ways. Once it has been done, the second place can be filled by any of the remaining $(n-1)$ objects and so this can be done in $(n-1)$ ways. Similarly, once the first two places have been filled, the third can be filled in $(n-2)$ ways and so on. The last place in the arrangement can be filled only in one way, because in this case we are left with only one object.

Using the counting principle, the total number of arrangements of n different objects is $n(n-1)(n-2)$ .... 2.1. ..... (7.1)

The product $n(n-1) ... 2.1$ occurs so often in Mathematics that it deserves a name and notation. It is usually denoted by $n!$ (or by $\underline{|n}$ read as n factorial).

$$\boldsymbol{n! = n(n-1)(n-2) \quad .... 3.2.1}$$

Here is an example to help you familiarise yourself with this notation.

Example 7.7: Evaluate (a) $3!$ (b) $2! + 4!$ (b) $2! \times 3!$

Solution: (a) $3! = 3 \times 2 \times 1 = 6$

(b) $2! = 2 \times 1 = 2$

$4! = 4 \times 3 \times 2 \times 1 = 24$

Therefore, $2! + 4! = 2 + 24 = 26$.

(c) $2! \times 3! = 2 \times 6 = 12$

Notice that $n!$ satisfies the relation

$$n! = n \times (n-1)! \qquad \text{... (7.2)}$$

This is because, $n(n-1)! = n \,.\, [(n-1)(n-2) \text{...} 3.2.1]$

$= n \,.\, (n-1)(n-2) \text{...} 3.2.1$

$= n!$

Of course, the above relation is valid only for $n \geq 2$ because 0! has not been defined so far. Let us see if we can define 0! to be consistent with the relation. In fact, if we define

$$0! = 1 \qquad \text{...(7.3)}$$

then the relation 7.2 holds for $n = 1$ also.

Example 7.8 : Suppose you want to arrange your English, Hindi, Mathematics, History, Geography and Science books on a shelf. In how many ways can you do it?

Solution: We have to arrange 6 books.

The number of permutations of n objects is $n! = n(n-1)$

Here $n = 6$ and therefore, number of permutations is 6.5.4.3.2.1 = 720.

EXERCISE 7.3

1. (a) Evaluate: (i) $6!$ (ii) $7!$ (iii) $7! + 3!$

   (iv) $6! \times 4!$ (v) $\dfrac{5!}{3!\,2!}$.

   (b) Which of the following statements are true?

   (i) $2! \times 3! = 6!$ (ii) $2! + 4! = 6!$

   (iii) 3! divides 4! (iv) $4! - 2! = 2!$

2. (a) 5 students are staying in a dormitory. In how many ways can you allot 5 beds to them?

(b) In how many ways can the letters of the word 'TRIANGLE' be arranged?

(c) How many four digit numbers can be formed with digits 1, 2, 3 and 4 and with distinct digits?

7.3 PERMUTATION OF *r* OBJECTS OUT OF *n* OBJECTS

Suppose you have five story books and you want to distribute one each to Asha, Akhtar and Jasvinder. In how many ways can you do it? You can give anyone of the five books to Asha and after that you can give anyone of the remaining four books to Akhtar. After that, you can **Algebra** give one of the remaining three books to Jasvinder. So, by the Counting Principle, you can distribute the books in $5 \times 4 \times 3$ ie.60 ways.

More generally, suppose you have to arrange r objects out of n objects. In how many ways can you do it? Let us view this in the following way. Suppose you have n objects and you have to arrange r of these in r boxes, one object in each box.

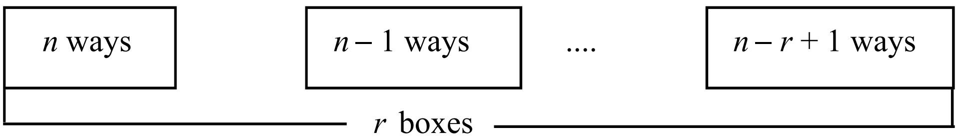

Fig. 7.1

Suppose there is one box. $r = 1$. You can put any of the n objects in it and this can be done in n ways. Suppose there are two boxes. $r = 2$. You can put any of the objects in the first box and after that the second box can be fIlled with any of the remaining $n - 1$ objects. So, by the counting principle, the two boxes can be fIlled in $n(n - 1)$ ways. Similarly, 3 boxes can be filled in $n(n - 1)(n - 2)$ ways.

In general, we have the following theorem.

Theorem 7.2: The number of permutations of r objects out of n objects is

$$n(n-1)\ ...(n-r+1).$$

The number of permutations of r objects out of n objects is usually denoted by ${}^nP_r$.

Thus,

$${}^nP_r = n(n-1)(n-2)\ ....\ (n-r+1) \quad .....\ (7.4)$$

Proof: Suppose we have to arrange r objects out of n different objects. In fact it is equivalent to filling r places, each with one of the objects out of the given n objects.

The first place can be filled in n different ways. Once this has been done, the second place can be filled by anyone of the remaining $(n-1)$ objects, in $(n-1)$ ways. Similarly, the third place can be filled in $(n-2)$ ways and so on. The last place, the r^{th} place can be filled in $[n-(r-1)]$ i.e. $(n-r+1)$ different ways. You may easily see, as to why this is so. Using the Counting Principle, we get the required number of arrangements of r out of n objects is $n\ (n-1)\ (n-2)\ ....\ (n-r+1)$

Example 7.9: Evaluate (a) ${}^4P_2$ (b) ${}^6P_3$ (c) $\frac{{}^4P_3}{{}^3P_2}$

(d) ${}^6P_3 \times {}^5P_2$

Solution: (a) ${}^4P_2 = 4(4-1) = 4 \times 3 = 12$

(b) ${}^6P_3 = 6(6-1)\ (6-2) = 6 \times 5 \times 4 = 120$

(c) $\frac{{}^4P_3}{{}^3P_2} = \frac{4(4-1)\ (4-2)}{3(3-1)} = \frac{4 \times 3 \times 2}{3 \times 2} = 4$

(d) ${}^6P_3 \times {}^5P_2 = 6(6-1)\ (6-2) \times 5\ (5-1)$

$$= 6 \times 5 \times 4 \times 5 \times 4 = 2400$$

Example 7.10 : If you have 6 New Year greeting cards and you want to send them to 4 of your friends, in how many ways can this be done?

Solution: We have to find number of permutations of 4 objects out of 6 objects.

This number is ${}^6P_3 = 6(6-1)\ (6-2)\ (6-3) = 6.5.4.3 = 360$

Therefore, cards can be sent in 360 ways.

Consider the formula for ${}^{n}P_{r}$, namely, $n(n-1)(n-2)....(n-r+1)$. This can be obtained by removing the terms $n-r, n-r-1, ... 2, 1$ from the product for $n!$. The product of these terms is $(n-r)(n-r-1)\ ...\ 2.1.$ i.e., $(n-r)!$

Now, $$\frac{n!}{(n-r)} = \frac{n.(n-1)\ (n-2)\ .(n-r+1)\ (n-r)...\ 3.2.1}{(n-r)\ (n-r-1)\ ...3.2.1}$$

$$= n(n-1)\ (n-2)\ ....\ (n-r+1)$$

$$= {}^{n}P_{r}$$

So, using the factorial notation, this formula can be written as follows:

$${}^{n}P_{r} = \frac{n!}{(n-r)\,!} \qquad ...\ (7.5)$$

Example 7.11: Find the value of ${}^{n}P_{0}$.

Solution: Here $r = 0$ Using relation 7.5 we get

$${}^{n}P_{0} = \frac{n!}{n\,!} = 1$$

Example 7.12: Show that $(n+1)\ .\ {}^{n}P_{r} = {}^{n+1}P_{r+1}$

Solution: $(n+1)\ {}^{n}P_{r} = (n+1)\ .\ \dfrac{n!}{(n-r)\,!} = \dfrac{(n+1).n!}{(n-r)!}$

$$= \frac{(n+1)\,!}{\left((n+1)-(r+1)\right)!} \quad \text{[writingn-ras } [(n+1)-(r+1)]$$

$$= {}^{n+1}P_{r+1}$$

EXERCISE 7.4

1.(a) (i) ${}^{4}P_{2}$ (ii) ${}^{6}P_{3}$

(iii) $\dfrac{{}^{4}P_{3}}{{}^{3}P_{2}}$ (iv) ${}^{6}P_{3} \times {}^{5}P_{2}$ (v) ${}^{n}P_{n}$ q °= ° H›#°Qù#°=ò

(b) Verify each of the following statements:

(i) $6 \times {}^5P_2 = {}^6P_3$ (ii) $4 \times {}^7P_3 = {}^7P_4$

(iii) ${}^4P_2 \times {}^4P_2 = {}^{12}P_4$ (v) ${}^3P_2 \times {}^4P_2 = {}^7P_4$

2.(a) (i) What is the maximum possible number of 3-letterwords in English that do not contain any vowel?

(ii) What is the maximum possible number of 3-letterwords in English which do not have any vowel other than 'a'?

(b) Suppose you have 2 cots and 5 bedspreads in your house. In how many ways can you put the bedspreads on your cots?

(c) You want to send Diwali Greetings to 4 friends and you have 7 greeting cards with you. In how many ways can you do it?

3. Show that ${}^nP_{n-1} = {}^nP_n$ Jx KÇ¶„¬ô=ò.

4. Show that $(n - r)\,{}^nP_n = {}^nP_{n+1}$.

7.4 PERMUTATIONS UNDER SOME CONDITIONS

We will now see examples involving permutations with some extra conditions.

Example 7.13: Suppose 7 students are staying in a hall in a hostel and they are allotted 7 beds. Among them, Parvin does not want a bed next to Anju because Anju snores. Then, in how many ways can you allot the beds?

Solution: Let the beds be numbered 1 to 7.

Case 1: Suppose Anju is allotted bed number 1.

Then, Parvin cannot be allotted bed number 2.

So Parvin can be allotted a bed in 5 ways.

After alloting a bed to Parvin, the remaining 5 students can be allotted beds in 5! ways.

So, in this case the beds can be allotted in $5 \times 5!$ ways = 600 ways.

Case 2: Anju is allotted bed number 7.

Then, Parvin cannot be allotted bed number 6

As in Case 1, the beds can be allotted in 600 ways.

Case 3: Anju is allotted one of the beds numbered 2,3,4,5 or 6.

Parvin cannot be allotted the beds on the right hand side and left hand side ofAnju's bed. For example, if Anju is allotted bed number 2, beds numbered 1 or 3 cannot be allotted to Parvin.

Therefore, Parvin can be allotted a bed in 4 ways in all these cases.

After allotting a bed to Parvin, the other 5 can be allotted a bed in 5! ways.

Therefore, in each of these cases, the beds can be allotted in $4 \times 5! = 480$ ways.

The beds can be allotted in

$(2 \times 600 + 5 \times 480)$ ways = $(1200 + 2400)$ ways = 3600 ways.

Example 7.15: There are 4 books on fairy tales, 5 novels and 3 plays. In how many ways can you arrange these so that books on fairy tales are together, novels are together and plays are together and in the order, books on fairytales, novels and plays.

Solution: There are 4 books on fairy tales and they have to be put together.

They can be arranged in 4! ways.

Similarly, there are 5 novels.

They can be arranged in 5! ways.

And there are 3 plays.

They can be arranged in 3! ways.

So, by the counting principle all of them together can be arranged in $4! \times 5! \times 3!$ ways = 17280 ways.

Example 7.16: Suppose there are 4 books on fairy tales, 5 novels and 3 plays as in Example 7.15. They have to be arranged so that the books on fairy tales are together, novels are together and plays are together, but we no longer require that they should be in a specific order. In how many ways can this be done?

Solution: First, we consider the books on fairy tales, novels and plays as single objects.

These three objects can be arranged in 3 !ways = 6 ways .

Let us fix one of these 6 arrangements.

This may give us a specific order, say, novels $\rightarrow$ fairy tales $\rightarrow$ plays.

Given this order, the books on the same subject can be arranged as follows.

The 4 books on fairy tales can be arranged among themselves in 4! = 24 ways.

The 5 novels can be arranged in 5! = 120 ways.

The 3 plays can be arranged in 3! = 6 ways.

F or a given order, the books can be arranged in $24 \times 120 \times 6 = 1\ 7280$ ways.

Therefore, for all the 6 possible orders the books can be arranged in $6 \times 17280 = 103680$ ways.

Example 7.17: How many ways can 4 girls and 5 boys be arranged in a row so that all the four girls are together?

Solution: Solution: Let 4 girls be one unit and now there are 6 units in all.

They can be arranged in 6! ways.

In each of these arrangements 4 girls can be arranged in 4! ways.

Total number of arrangements in which girls are always together

$= 6! \times 4!$

$= 720 \times 24$

$= 17280$

Example 7.18: How many arrangements of the letters of the word 'BENGALI' can be made

(i) if the vowels are never together.

(ii) if the vowels are to occupy only odd places.

Solution: There are 7 letters in the word 'Bengali; of these 3 are vowels and 4 consonants.

(i) Considering vowels *a, e, i* as one letter, we can arrange 4+ 1 letters in 5! ways in each of which vowels are together. These 3 vowels can be arranged among themselves in 3 ! ways.

$\therefore$ Total number of words = (5 !) × (3 !)

$$= 120 \times 6 = 720.$$

(ii) There are 4 odd places and 3 even places. 3 vowels can occupy 4 odd places in ${}^4P_3$ ways and 4 constants can be arranged in ${}^4P_4$ ways.

$\therefore$ Number of words $= {}^4P_3 \times {}^4P_4 = 24 \times 24$

$$= 576.$$

EXERCISE 7.5

1. Mr. Gupta with Ms. Gupta and their four children is travelling by train. Two lower berths, two middle berths and 2 upper berths have been allotted to them. Mr. Gupta has undergone a knee surgery and needs a lower berth while Ms. Gupta wants to rest during the journey and needs an upper berth. In how many ways can the berths be shared by the family?

2. Consider the word UNBIASED. How many words can be formed with the letters of the word in which no two vowels are together?

3. There are 4 books on Mathematics, 5 books on English and 6 books on Science. In how many ways can you arrange them so that books on the same subject are together and they are arranged in the order Mathematics $\rightarrow$ English $\rightarrow$ Science.

4. There are 3 Physics books, 4 Chemistry books, 5 Botany books and 3 Zoology books. In how many ways can you arrange them so that the books on the same subject are together?

5. 4 boys and 3 girls are to be seated in 7 chairs such that no two boys are together. In how many ways can this be done?

6. Find the number of permutations of the letters of the word 'TENDULKAR', in each of the following cases:

   (i) beginning with T and ending with R.

   (ii) vowels are always together.

   (iii) vowels are never together.

7.5 COMBINATIONS

Let us consider the example of shirts and trousers as stated in the introduction. There you have 4 sets of shirts and trousers and you want to take 2 sets with you while going on a trip. In how many ways can you do it?

Let us denote the sets by S_1, S_2, S_3, S_4 Then you can choose two pairs in the following ways:

1. $\{S_1, S_2\}$	2. $\{S_1, S_3\}$	3. $\{S_1, S_4\}$
4. $\{S_2, S_3\}$	5. $\{S_2, S_4\}$	6. $\{S_3, S_4\}$

[Observe that $\{S_1, S_2\}$ is the same as $\{S_2, S_1\}$]' So, there are 6 ways of choosing the two sets that you want to take with you. Of course, if you had 10 pairs and you wanted to take 7 pairs, it will be much more difficult to work out the number of pairs in this way.

Now as you may want to know the number of ways of wearing 2 out of 4 sets for two days, say Monday and Tuesday, and the order of wearing is also important to you. We know from section 7.3, that it can be done in ${}^4P_2 = 12$ ways.

But note that each choice of 2 sets gives us two ways of wearing 2 sets out of 4 sets as shown below:

1. $\{S_1, S_2\} \rightarrow S_1$ on Monday and S_2 on Tuesday or S_2 on Monday and S_1 on Tuesday
2. $\{S_1, S_3\} \rightarrow S_1$ on Monday and S_3 on Tuesday or S_3 on Monday and S_1 on Tuesday
3. $\{S_1, S_4\} \rightarrow S_1$ on Monday and S_4 on Tuesday or S_4 on Monday and S_1 on Tuesday
4. $\{S_2, S_3\} \rightarrow S_2$ on Monday and S_3 on Tuesday or S_3 on Monday and S_2 on Tuesday
5. $\{S_2, S_4\} \rightarrow S_2$ on Monday and S_4 on Tuesday or S_4 on Monday and S_2 on Tuesday
6. $\{S_3, S_4\} \rightarrow S_3$ on Monday and S_4 on Tuesday or S_4 on Monday and S_3 on Tuesday

$\therefore$ Thus, there are 12 ways of wearing 2 out of 4 pairs.

This argument holds good in general as we can see from the following theorem.

Theorem 7.3: Let $n \geq 1$ be an integer and $r \geq n$. Let us denote the number of ways of choosing r objects out of n objects by ${}^nC_r$. Then

$$ {}^nC_r = \frac{{}^nP_r}{r!} \qquad ...(7.6)$$

Proof : We can choose r objects out *of* n objects in ${}^nC_r$ ways. Each of the r objects chosen can be arranged in $r!$ ways. The number of ways of arranging r objects is $r!$. Thus, by the counting principle, the number of ways of choosing r objects and arranging the r objects chosen can be done in ${}^nC_r\, r!$ ways. But, this is precisely ${}^nP_r$. In other words, we have

$$ {}^nP_r = r!\ {}^nC_r \qquad ....(7.7)$$

Dividing both sides by $r!$, we get the result in the theorem.

Here is an example to help you to familiarise yourself with ${}^nC_r$.

Example 7.19: Evaluate each of the following:

(a) ${}^5C_2$ (b) ${}^5C_3$

(c) ${}^4C_3 + {}^4C_2$ (d) $\dfrac{{}^6C_3}{{}^4C_2}$

Solution: (a) ${}^5C_2 = \dfrac{{}^5P_2}{2\,!} = \dfrac{5.4}{1.2} = 10$ (b) ${}^5C_3 = \dfrac{{}^5P_3}{3\,!} = \dfrac{5.4.3}{1.2.3} = 10$

(c) ${}^4C_3 + {}^4C_2 = \dfrac{{}^4P_3}{3\,!} = \dfrac{{}^4P_2}{2\,!} = \dfrac{4.3.2}{1.2.3} + \dfrac{4.3}{1.2} = 4 + 6 = 10$

(d) ${}^6C_3 = \dfrac{{}^6P_3}{3\,!} = \dfrac{6 \times 5 \times 4}{1 \times 2 \times 3} = 20$

$${}^4C_2 = \frac{4.3}{1.2} = 6$$

$$\therefore \frac{{}^6C_3}{{}^4C_2} = \frac{20}{6} = \frac{10}{3}.$$

Example 7.20: Find the number of subsets of the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} having 4 elements.

Solution: Here the order of choosing the elements doesn't matter and this is a problem in combinations.

We have to find the number ofways of choosing 4 elements of this set which has 11 elements.

By relation (7.6), this can be done in

$${}^{11}C_4 = \frac{11.10.9.8}{1.2.3.4} = 330 \text{ ways.}$$

Example 7.21 : 12 points lie on a circle. How many cyclic quadrilaterals can be drawn by using these points?

Solution: For any set of 4 points we get a cyclic quadrilateral. Number of ways of choosing 4 points out of12 points is ${}^{12}C_4 = 495$. Therefore, we can draw 495 quadrilaterals.

Example 7.22: In a box, there are 5 black pens, 3 white pens and 4 red pens. In how many ways can 2 black pens, 2 white pens and 2 red pens can be chosen?

Solution: Number of ways of choosing 2 black pens from 5 black pens

$$= {}^5C_2 = \frac{{}^5P_2}{2\,!} = \frac{5.4}{1.2} = 10$$

Number of ways of choosing 2 white pens from 3 white pens

$$= {}^3C_2 = \frac{{}^3P_2}{2\,!} = \frac{3.2}{1.2} = 3$$

Number of ways of choosing 2 red pens from 4 red pens

$$= {}^4C_2 = \frac{{}^4P_2}{2\,!} = \frac{4.3}{1.2} = 6$$

$\therefore$ By the Counting Principle, 2 black pens, 2 white pens, and 2 red pens can be chosen in $10 \times 3 \times 6 = 180$

Example 7.23: A question paper consists of 10 questions divided into two parts *A* and *B*. Each part contains five questions. A candidate is required to attempt six questions in all of which at least 2 should be from part *A* and at least 2 from part *B*. In how many ways can the candidate select the questions ifhe can answer all questions equally well?

Solution: The candidate has to select six questions in all of which at least two should be from Part A and two should be from PartB. He can select questions in any of the following ways:

	Part A	**Part B**
(i)	2	4
(ii)	3	3
(iii)	4	2

If the candidate follows choice (i), the number of ways in which he can do so is ${}^5C_2 \times {}^5C_4 = 10 \times 5 = 50$

If the candidate follows choice (ii), the number of ways in which he can do so is ${}^5C_3 \times {}^5C_3 = 10 \times 10 = 100$

Similarly, if the candidate follows choice (iii), then the number of ways in which he can do so is ${}^5C_4 \times {}^5C = 5 \times 10 = 50$

Therefore, the candidate can select the question in

$= 50 + 100 + 50 = 200$ ways.

Example 7.24: committee of 5 persons is to be formed from 6 men and 4 women. In how many ways can this be done when

(i) at least 2 women are included?

(ii) atmost 2 women are included?

Solution: (i) When at least 2 women are included.

The committee may consist of

3 women, 2 men: It can be done in ${}^4C_3 \times {}^6C_2$ ways

or 4 women, 1 man: It can be done in ${}^4C_4 \times {}^6C_1$ ways

or 2 women, 3 men: It can be done in ${}^4C_2 \times {}^6C_3$ ways

$\therefore$ Total number of ways of forming the committee

$= {}^4C_2 . {}^6C_2 + {}^4C_4 . {}^6C_1 + {}^4C_2 . {}^6C_3$

$= 6 \times 20 + 4 \times 15 + 1 \times 6$

$= 120 + 60 + 6 = 186$

(ii) When atmost 2 women are included

The committee may consist of

2 women, 3 men: It can be done in ${}^4C_2 . {}^6C_3$ ways

or 1 woman, 4 men: It can be done in $^4C_4 \,.\, ^6C_1$ ways

or 5 men: It can be done in $^6C_5$ ways

$\therefore$ Total number of ways of forming the committee

$= {}^4C_2 \times {}^6C_3 + {}^4C_1 \times {}^6C_4 + {}^6C_5$

$= 6 \times 20 + 4 \times 15 + 6$

$= 120 + 6 + 6 = 186.$

Example 7.25 : The Indian Cricket team consists of 16 players. It includes 2 wicket keepers and 5 bowlers. In how many ways can a cricket team of eleven be selected if we have to select 1 wicket keeper and atleast 4 bowlers?

Solution: We are to choose 11 players including 1 wicket keeper and 4 bowlers or, 1 wicket keeper and 5 bowlers.

Number of ways of selecting 1 wicket keeper, 4 bowlers and 6 other players

$= {}^2C_1 \,.\, {}^5C_4 \,.\, {}^9C_6$

$$= 2\,\frac{5\times4\times3\times2\times1}{4\times3\times2\times1} \times \frac{9\times8\times7\times6\times5\times4}{6\times5\times4\times3\times2\times1}$$

$$= 2\times5\times\frac{9\times8\times7}{3\times2\times1} = 840$$

Number of ways of selecting 1 wicket keeper, 5 bowlers and 5 other players

$= {}^2C_1 \,.\, {}^5C_5 \,.\, {}^9C_5$

$$= 2\times1\times\frac{9\times8\times7\times6\times5}{5\times4\times3\times2\times1} = 2\times1\times\frac{9\times8\times7\times6}{4\times3\times2\times1} = 252$$

$\therefore$ Total number of ways of selecting the team

$= 840 + 252 = 1092$

EXERCISE 7.6

1. (a) Evaluate (i) ${}^{13}C_3$ (ii) ${}^{a}C_5$ (iii) ${}^{8}C_2 + {}^{8}C_3$

   (iv) $\dfrac{{}^{9}C_3}{{}^{6}C_3}$

   (b) Verify each of the following statement :

   (i) ${}^{5}C_3 = {}^{5}C_3$ (ii) ${}^{4}C_3 \times {}^{3}C_2 = {}^{12}C_6$

   (iii) ${}^{4}C_2 + {}^{4}C_3 = {}^{8}C_5$ (iv) ${}^{10}C_2 + {}^{10}C_3 = {}^{11}C_3$

2. Find the number of subsets of the set $\{1, 3, 5, 7, 9, 11, 13 .... 23\}$ each having 3 elements.

3. There are 14 points lying on a circle. How many pentagons can be drawn using these points?

4. In a fruit basket there are 5 apples, 7 plums and 11 oranges. You have to pick 3 fruits of each type. In how many ways can you make your choice?

5. A question paper consists of 12 questions divided into two parts A and B, containing 5 and 7 questions repectively. A student is required to attempt 6 questions in all, selecting at least 2 from each part. In how many ways can a student select a question?

6. Out of 5 men and 3 women, a committee of 3 persons is to be formed. In how many ways can it be formed selecting (i) exactly 1 woman. (ii) atleast 1 woman.

7. A cricket team consists of 17 players. It includes 2 wicket keepers and 4 bowlers. In how many ways can a playing eleven be selected if we have to select 1 wicket keeper and atleast 3 bowlers?

8. To fill up 5 vacancies, 25 applications were recieved. There were 7 S.C. and 8 O.B.C. candidates among the applicants. If 2 posts were reserved for S.C. and 1 for O.B.C. candidates, find the number of ways in which selection could be made?

7.6 SOME SIMPLE PROPERTIES OF ${}^nC_r$

In this section we will prove some simple properties of ${}^nC_r$ which will make the computations of these coefficients simpler. Let us go back again to Theorem 7.3. Using relation 7.7 we can rewrite the formula for ${}^nC_r$ as follows:

$$ {}^nC_r = \frac{n!}{r!(n-r)!} \qquad ...(7.8)$$

Example 7.26: Find the value of ${}^nC_0$.

Solution: Here $r = 0$, Therefore, ${}^nC_0 = \frac{n!}{0!\,n!} = \frac{1}{0!} = 1$

since we have defined $0! = 1$.

The formula given in Theorem 7.3 was used in the previous section. As we will see shortly, the formula given in Equation 7.8 will be useful for proving certain properties of ${}^nC_r$.

$$ {}^nC_r = {}^nC_{n-r} \qquad ... (7.9)$$

This means just that the number of ways of choosing r objects out of n objects is the same as the number of ways of not choosing $(n - r)$ obj ects out of n obj ects. In the example described in the introduction, it just means that the number of ways of selecting 2 sets of dresses is the same as the number of ways of rejecting $4 - 2 = 2$ dresses. In Example 7.20, this means that the number of ways of choosing subsets with 4 elements is the same as the number of ways of rejecting 8 elements since choosing a particular subset of 4 elements is equivalent to rejecting its complement, which has 8 elements.

Let us now prove this relation using Equation 7.8. The denominator of the right hand side of this equation is $r!\,(n - r)!$. This does not change when we replace r by $n - r$.

$$(n - r)\,!\,.\,[n - (n - r)]\,! = (n - r)\,!\,r\,!$$

The numerator is independent of r. Therefore, replacing r by $n - r$ in Equation 7.8 we get result.

How is the relation 7.9 useful? Using this formula, we get, for example, ${}^{100}C_9$ is the same as ${}^{100}C_2$ The second value is much more easier to calculate than the first one.

Example 7.27 : Evaluate:

(a) ${}^{7}C_5$ (c) ${}^{10}C_9$

(b) ${}^{11}C_9$ (d) ${}^{12}C_9$

Solution: (a) From relation 7.9, we have

$$ {}^{7}C_5 = {}^{7}C_{7-5} = {}^{7}C_2 = \frac{7.6}{1.2} = 21 $$

(b) Similarly ${}^{10}C_9 = {}^{10}C_{10-9} = {}^{10}C_1 = 10$

(c) $$ {}^{11}C_9 = {}^{11}C_{11-9} = {}^{11}C_2 = \frac{11.10}{1.2} = 55 $$

(d) $$ {}^{12}C_{10} = {}^{12}C_{12-10} = {}^{12}C_2 = \frac{12.11}{1.2} = 66 $$

There is another relation satisfied by ${}^{n}C_r$ which is also useful. We have the following relation:

$$ {}^{n-1}C_{r-1} + {}^{n-1}C_r = {}^{n}C_r \qquad ...(7.10) $$

$$ {}^{n-1}C_{r-1} + {}^{n-1}C_r = \frac{(n-1)^n}{(n-r)!\,(r-1)!} + \frac{(n-1)!}{(n-r-1)!r!} $$

$$ = \frac{(n-1)!}{(n-r)\;(n-r-1)!(r-1)!} + \frac{(n-1)!}{r(n-r-1)!(r-1)!} $$

$$ = \frac{(n-1)!}{(n-r-1)!\;(r-1)!}\left[\frac{1}{n-r} + \frac{1}{r}\right] $$

$$ = \frac{(n-1)!}{(n-r-1)!\;(r-1)!}\left[\frac{1}{(n-r)r}\right] $$

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$$= \frac{n(n-1)!}{(n-r)(n-r-1)!\ (r-1)!}$$

$$= \frac{n!}{(n-r)!\ r!} = {}^nC_r$$

Example 7.28 : Evaluate:

(a) ${}^6C_2 + {}^6C_4$ (b) ${}^8C_2 + {}^8C_4$

(a) ${}^5C_3 + {}^5C_2$ (a) ${}^{10}C_2 + {}^{10}C_4$

Solution:

(a) Let us use relation (7.10) with $n = 7, r = 2$. So, ${}^6C_2 + {}^6C_1 = {}^7C_2 = 21$

(b) Here $n = 9, r = 2$ Therefore, ${}^8C_2 + {}^8C_1 = {}^9C_2 = 36$

(c) Here $n = 6, r = 3$ Therefore, ${}^5C_3 + {}^5C_2 = {}^6C_2 = 20$

(d) Here $n = 11, r = 3$ Therefore, ${}^{10}C_2 + {}^{10}C_1 = {}^{11}C_3 = 165$

To understand the relation 7.10 better, let us go back to Example 7.20 In this example let us calculate the number of subsets of the set, {1, 2, 3,4, 5, 6, 7, 8, 9, 10, 11}. We can subdivide them into two kinds, those that contain a particular element, say 1, and those that do not contain 1. The number of subsets of the set having 4 elements and containing 1 is the same as the number of subsets of {2, 3, 4,5,6, 7, 8, 9, 10, 11} having 3 elements. There are ${}^{10}C_3$ such subsets.

The number of subsets of the set having 4 elements and not containing 1 is the same as the number of subsets of the set {2,3,4,5,6,7,8,9,10,11,} having 4 elements. This is ${}^{10}C_4$. So, the number of subsets of {1, 2, 3,4,5,6, 7,8,9, 10, 11} having four elements is ${}^{10}C_3 + {}^{10}C_4$ .But, this is ${}^{11}C_4$ as we have seen in the example. So, ${}^{10}C_3 + {}^{10}C_4 = {}^{11}C_4$. The same argument works for the number of r—element subsets of a set with n elements.

This reletion was noticed by the French Mathematician Blaise Pascal and was used in the so called Pascal Triangle, given below.

$n = 0$				1			
$n = 1$			1		1		
$n = 2$		1	2		1		
$n = 3$		1	3		3	1	
$n = 4$	1	4	6		4	1	
$n = 5$	1	5	10		10	5	1

The first row consists of single element ${}^0C_0 = 1$. The second row consists of ${}^1C_0$ and ${}^1C_1$. From the third row onwards, the rule for writing the entries is as follows. Add consecutive elements in the previous row and write the answer between the two entries. After exhausting all possible pairs, put the number 1 at the begining and the end of the row. For example, the third row is got as follows. Second row contains only two elements and they add up to 2. Now, put 1 before and after 2. For the fourth row, we have $1 + 2 = 3$, $2 + 1 = 3$. Then, we put $1 + 2 = 3$, $2 + 1 = 3$. Then we put 1 at the beginning and the end. Here, we are able to calculate, for example, ${}^3C_1$, ${}^3C_2$, from ${}^2C_0$, ${}^2C_1$, ${}^2C_2$ by using the relation 7.10. The important thing is we are able to do it using addition alone.

The numbers ${}^nC_r$ occur as coefficents in the binomial expansions, and therefore, they are also called binomial coefficents as we will see in lesson 8. In particular, the property 7.10 will be used in the proof of the binomial theorem.

Example 7.29: If ${}^nC_{10} = {}^nC_{12}$ find n.

Solution: Using ${}^nC_r = {}^nC_{n-r}$ we get

$$n - 10 = 12$$

or, $n = 12 + 10 = 22$

EXERCISE 7.7

1. (a) Find the value of ${}^nC_{n-1}$ Is ${}^nC_{n-1} = {}^nC_n$?

   (b) Show that ${}^nC_n = {}^nC_0$

2. Evaluate :

   (a) ${}^9C_5$ (b) ${}^{14}C_{10}$

   (c) ${}^{13}C_9$ (d) ${}^{15}C_{12}$

3. Evaluate :

   (a) ${}^7C_5 + {}^7C_2$ (b) ${}^8C_4 + {}^8C_5$

   (c) ${}^7C_2 + {}^9C_2$ (d) ${}^{12}C_3 + {}^{12}C_2$

4. If ${}^{10}C_r = {}^{10}C_{2r+1}$, find the value of r.

5. If ${}^{18}C_r = {}^{18}C_{r+2}$ find ${}^nC_5$.

PROBLEMS INVOLVING BOTH PERMUTATIONS AND COMBINATIONS

So far, we have studied problems that involve either permutation alone or combination alone. In this section, we will consider some examples that need both of these concepts.

Example 7.30: There are 5 novels and 4 biographies. In how many ways can 4 novels and 2 biographies can be arranged on a shelf ?

Solution: 4 novels can be selected out of 5 in ${}^5C_4$ ways. 2 biographies can be selected out of 4 in ${}^4C_2$ ways.

Number of ways of arranging novels and biographies

$$= {}^5C_4 \times {}^4C_2 = 5 \times 6 = 30$$

After selecting any 6 books (4 novels and 2 biographies) in one of the 30 ways, they can be arranged on the shelf in 6! = 720 ways.

By the Counting Principle, the total number of arrangements

$$= 30 \times 720 = 21600$$

Example 7.31: From 5 consonants and 4 vowels, how many words can be formed using 3 consonants and 2 vowels?

Solution: From 5 consonants, 3 consonants can be selected in ${}^5C_3$ ways.

From 4 vowels, 2 vowels can be selected in ${}^4C_2$ ways.

Now with every selection, number of ways of arranging 5 letters is ${}^5P_5$.

$\therefore$ Total number of words $= {}^5C_3 \times {}^4C_2 \times {}^5P_5$

$$= \frac{5\times4}{2\times1} \times \frac{4\times3}{2\times1} \times 5!$$

$$= 10 \times 6 \times 5 \times 4 \times 3 \times 2\times 1 = 7200$$

EXERCISE 7.8

1. There are 5 Mathematics, 4 Physics and 5 Chemistry books. In how many ways can you arrange 4 Mathematics, 3 Physics and 4 Chemistry books.
   (a) if the books on the same subjects are arranged together, but the order in which the books are arranged within a subject doesn't matter?
   (b) ifbooks on the same subjects are arranged together and the order in which books are arranged within subject matters?
2. There are 9 consonants and 5 vowels. How many words of7letters can be formed using 4 consonents and 3 vowels?
3. In how many ways can you invite at least one of your six friends to a dinner?
4. In an examination, an examinee is required to pass in four different subjects. In how many ways can he fail?

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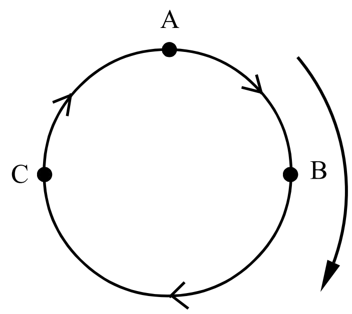
7.9 CIRCULAR PERMUTATIONS

In general, a combination is only a selection permutation involves both selection and arrangement.

7.9.1 Definition: From a set of elements choosing some or all of them are arranging them around a circle is called a "circular permutation".

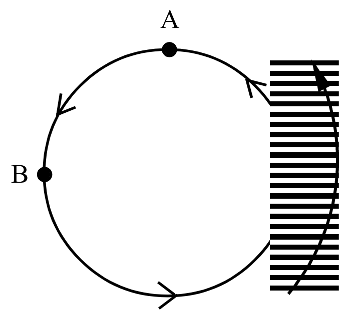
7.9.2 Circular permutations of n dissimilar things "r" at a time is $\frac{{}^{n}p_r}{r}$.

7.9.3 The number of circular permutations of "n" dissimilar things taken all at a time is $\underline{|(n-1)}$ ways.



Clock-wise

Fig. 7.1



anti-clock-wise

Fig. 7.2

Observe circular permutations ABC, ACB. In the first permutation A, B, C are in clock-wise, and in second A, B, C are in anti-clock-wise directions. If we consider the direction (wether it is clock-wise or anti-clock - wise) the permutation is $\underline{|(n-1)}$ ways.

7.9.4 If we do not consider the direction then the permutations is $\frac{1}{2}\underline{|(n-1)}$.

7.9.5 From a set of distinct elements taking some or all the elements and arranging them linearly is called a linear permutation**.**

Imp Note: Linear permutation has a first place and also a last place. Where as a circular permutation has no starting place and no last place in particular. It can be treated as starting from any one of the elements in it. But how the other elements are arranged relative to this starting elements is to be taken into consideration.

7.9.6 Choosing method from A, B, C, D

Linear Method

(i) selecting 2 things from 4 things. (ii) Selecting 2 things from 4 things and their arrangements

Combinations	**Permutations**
a) The number of combinations of 2 elements selected from 4 elements	a) The number of permutations of 2 elements from four elements
$4C_2 = \frac{4.3}{2.1} = 6$ ways	$4P_2 = 4.3 = 12$ ways
AB	AB, BA
AC	AC, CA
AD	AD, DA
BC	BC, CB
BD	BD, DB
CD	CD, DC

(b) (i) No of combinations of 3 elements from 4 elements (ii) No. of permutations of 3 elements from 4 elements

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Combinations	**Permutations**
$4C_3 = 4C_1 = 4$ways $(\because nC_r = nC_{n-r})$ ABC ABD ACD BCD BD CD	$4P_3 = 4.3.2 = 24$ ways ABC ACB BCA BAC CAB CBA BCD BDC CDB CBD DBC DCB $4 \times 6 = 24$ ways

(c) No.of combinations of 4 elements taken all at a time

(c) No.of permutations of 4 things taken 4 at a time

Combinations	**Permutations**
$4C_4 = 4C_0 = 1$ways ABCD	$4P_4 = 4.3.2.1 = 24$ ways D ABC D ACB D BCA D BAC D CAB D CBA Similarly we get $4 \times 6 = 24$ ways

7.9.7 $^{n}\mathrm{P}_{r} = n(n-1)(n-2)...(n-r+1)$

$$^{n}p_{r} = \frac{\lfloor n}{\lfloor (n-r)}$$

$$^{n}p_{n} = \lfloor n$$

$$^{n}p_{0} = 1$$

$$^{n}p_{r} = n(^{n-1}p_{r-1}). \quad (1 \le r \le n)$$

$$^{n}p_{r} = n \cdot {}^{(n-1)}\mathrm{P}_{(r-1)}.$$

Example 7.32: Find the number of ways that 9 boys and 9 girls can set on a round (circlulary) table ?

Sol: Total No.of boys and girls= 18

No.of permutations of sitting circularly in $\lfloor 18-1 = \lfloor 17$ ways.

Example 2: Find the number of ways that 8 charis can be placed beside a round table ? In how many ways can two chairs of specified colors can be placed side by side ?

Sol : First we should treat that two chairs of specified colors as one unit. So now the total no.of chairs 6 + 1 = 7 (8 – 1) can be arranged circularly in $\lfloor (7-1) = \lfloor 6$. ways.

Two specified coloured chairs can be arranged in $\lfloor 2$ ways.

$\therefore$ The no.of circular permitations is $= \lfloor 6 \times \lfloor 2$

$= 6 . 5 . 4 . 3 . 2 .1 \times 2 . 1$

= 1440ways.

Example 7.32 : In how many ways can 6 students sit on a sound table ?

Sol : In number of permutations of sitting is $\lfloor 6-1 = \lfloor 5 = 120$ ways.

If the clock - wise and anti-clockwise directions are not considered, the number pf permutations is $\frac{120}{2} = 60$. ways.

Example 7.33: In how many ways can a gasland be made with different colors of 9 pearls ?

Sol: The arrangements of pearls

$$= \frac{\lfloor 9-1}{2} = \frac{\lfloor 8}{2} = \frac{2016}{2} = 1008$$

Example 7.34: Find the number of ways of arranging 7 men and 5 women around a circular table. In how many of them (i) no two women come cansit together.

Sol : (i) no two women can sit (come) together

First we arrange 7 men around the circular table in $\lfloor 7-1 = \lfloor 6$ ways.

There are 7 places in between them to sit 5 women.

Now we can arrange the 5 women in these 7 places in $7P_5$ ways. Thus, the no.of circular permutations in which no two women (come) can sit together is $\lfloor 6 \times 7P_5$ ways.

(ii) Treat the 5 women as single unit. Then we have 7 men + 1 unit of woment = 8 entities.

They can be arranged around a circular table in $\lfloor 8-1 = \lfloor 7$ ways. Now the 5 women among themselves can be arranged in $\lfloor 5$ ways.

Hence, the required number of arrangements is $\lfloor 5 \; . \; \lfloor 7$ ways.

Example 7.35: Find the number of ways that a garland is made with 14 flowers should be side by side in the garland.

Sol: First we should treat that two specified flowers as one. So now the tatal13 (14 – 1)flowers can be arranged circularly in $\lfloor 13$ ways.

The two specified flowers can be arranged in $\lfloor 2 = 2$ways.

$\therefore$ from the given condition, 14 flowers can be arranged as a garland in $2 \times \lfloor 13$ ways.

But, when preparing a garland, we should not consider the direction. So the flowers in a garland can be arranged in $\frac{2 \times \lfloor 13}{2}$ ways $= \lfloor 13$ ways.

Example 7.36 : Find the number of ways that 6 boys and 6 girls can sit circularly such that a boy should sit between two girls.

Sol: 6 girls can sit circularly in $\lfloor 6-1 = \lfloor 5 = 120$ ways.

As shown in the figure 6 boys can sit in 6 vacant places between 6 girls. After the girls are made to sit.

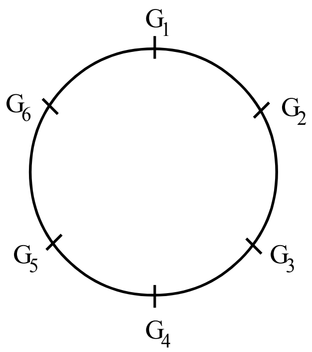

Fig 7.3

we should consider in which place should the first boy sit. If we consider 6 boys should sit in a row, they can sit in 6 vacancies in $\lfloor 6 = 720$ways.

Therefore 6 boys and 6 girls can sit circularly with the given condition in $120 \times 720 = 26,400$ ways.

Permutations & Combinations

- If $p_1, p_2, ... p_k$ are distinct primes and $\alpha_1, \alpha_2, ... \alpha_k$ are positive integers. Then the number of positive divisors of $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} .... p_k^{\alpha_k}$ is $(\alpha_1 + 1)$ $(\alpha_2 + 1) ..... (\alpha_k + 1)$ (This includes 1 and n)

Example 8 : Find the number of positive divisors of 1080.

Solution: $1080 = 2^3 \times 3^3 \times 5^1$

$\therefore$ The number of positive divisors of 1080 = (3 + 1) (3 + 1) (1 + 1)

= 32.

- The sum of all r - digit numbers that can be formed using the given 'n' non-zero digits $(1 \le r \le n \le 9)$ is ${}^{(n-1)}p_{(r-1)}$ × sum of the given digits × 111..1 (r times).

Example 9: Find the sum of all 4-digit numbers that can be formed using the digits 1, 3, 5, 7, 9.

Solution: $r = 4$

$n = 5$

$= {}^{(n-1)}p_{(r-1)} \times$ sum of the given digit $\times 1111$

$= {}^{(5-1)}p_{(4-1)} \times (1+3+5+7+9) \times 1111$

$= {}^{4}p_{3} \times 25 \times 1111$

$= 24 \times 25 \times 1111 = 6{,}66{,}600.$

EXERCISE 7.9

1. 5 boys and 3 girls around a circular table. In how many of them
   (i) no two girls can sit together
   (ii) all the girls can sit together
   (iii) all the girls cannot sit together.
   Find the number of ways of arrangements.
2. Find the number of ways of seating 5 Japanees, 5 Indians at a round table so that no two persons of same country sit together.
3. Find the number of circular permutations of 8 things taken 5 things ?
4. In how many ways can a garland be made with 8 flowers such that 2 specified flowers should be present side by side in the garland ?
5. In how many ways can a necklace be made with different colours of beads.

KEY WORDS

- Fundamental principle of counting states.

  If there are n events and if the first event can occur in m_1 ways, the second event can occur in m_2 ways after the first event has occurred, the third event can occur in m_3 ways after the second event has occurred and so on, then all the n events can occur in

  $m_1 \times m_2 \times m_3 \times ... \times m_{n-1} \times m_n$ ways.
- The number of permutations of n objects taken all at a time is $n!$
- ${}^nP_r = \dfrac{n!}{(n-r)!}$
- ${}^nP_n = n!$
- The number of ways of selecting r objects out of n objects ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$
- ${}^nC_r = {}^nC_{n-r}$
- ${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^nC_r$.
- The number of circular permutations of n dissimilar things is $\lfloor n-1$
- In case of hanging type circular permuataions like garlands of flowers, chains of beds etc., the number of circular permutations of n things is $\frac{1}{2}\lfloor n-1$.

SUPPORTIVE WEB SITES

http : // www.wikipedia.org

http://mathworld.wolfram.com

PRACTICE EXERCISE

1. There are 8 true - false questions in an examination. How many responses are possible?

2. The six faces of a die are numbered 1,2,3,4,5 and 6. Two such dice are thrown simultaneously. In how many ways can they turn up?

3. A restaurant has 3 vegetables, 2 salads and 2 types of bread. If a customer wants 1 vegetable, 1 salad and 1 bread, how many choices does he have?

4. Suppose you want to paper your walls. Wall papers are available in 4 diffrent backgrounds colours with 7 different designs of 5 different colours on them. In how many ways can you select your wall paper?

5. In how many ways can 7 students be seated in a row on 7 seats?

6. Determine the number of 8 letter words that can be formed from the letters of the word ALTRUISM.

7. If you have 5 windows and 8 curtains in your house, in how many ways can you put the curtains on the windows?

8. Determine the maximum number of 3-letter words that can be formed from the letters of the word *POLICY.*

9. There are 10 athletes participating in a race and there are three prizes, 1^{st}, 2^{nd} and 3^{rd} to be awarded. In how many ways can these be awarded?

10. In how many ways can you arrange the letters of the word *ATTAIN* so that the *Ts* are together?

11. A group of 12 friends meet at a party. Each person shake hands once with all others. How many hand shakes will be there ?

12. Suppose that you own a shop which sells televisions. You are selling 5 different kinds of television sets, but your show case has enough space for display of 3 televison sets only. In how many ways can you select the television sets for the display?

13. A contractor needs 4 carpenters. Five equally qualified carpenters apply for the job. In how many ways can the contractor make the selection?

14. In how many ways can a committe of 9 can be selected from a group of 13?

15. In how many ways can a committee of 3 men and 2 women be selected from a group of 15 men and 12 women?

16 In how ways can 6 persons be selected from 4 grade 1 and 7 grade 11 officers, so as to include at least two officers from each category ?

17. Out of 6 boys and 4 girls, a committee of 5 has to be formed. In how many ways can this be done if we take:

    (a) 2 girls.

    (b) at least 2 girls.

18. The English alphabet has 5 vowels and 21 consonants. What is the maximum number of words, that can be formed from the alphabet with 2 different vowels and 2 different consonants?

19. From 5 consonants and 5 vowels, how many words can be formed using 3 consonants and 2 vowels?

20. In a school annual day function a variety programme was organised. !twas planned that there would be 3 short plays, 6 recitals and 4 dance programmes. However, the chief guest invited for the function took much longer time than expected to fmish his speech. To finish in time, it was decided that only 2 short plays, 4 recitals and 3 dance programmes would be perfomed, How many choices were available to them?

    (a) if the programmes can be perfomed in any order?

    (b) if the programmes of the same kind were perfomed at a stretch?

    (c) if the programmes of the same kind were perfomed at a strech and considering the order of performance of the programmes of the same kind?

Answers

EXERCISE 7.1

1. (a) 180 (b) 8 (c) 12 (d) 20
2. (a) 48 (b) 36

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EXERCISE 7.2

1. (a) 17576 (b) 900 2. (a) 105 (b) 60
3. (a) 24 (b) 24

EXERCISE 7.3

1.(a) (i) 720 (ii) 5040 (iii) 5046
(iv) 17280 (v) 10
(b) (i) False (ii) False (iii) True (iv) False
2.(a) 120 (b) 40320 (c) 24

EXERCISE 7.4

1. (a) (i) 12 (ii) 120 (iii) 4 (iv) 7200
(v) $n!$
(b)(i) False (ii) True (iii) False
(iv) False
2. (a)(i) 7980 (ii) 9240
(b) 20 (c) 840

EXERCISE 7.5

1. 96 2. 1152 3. 2073600
4. 2488320
5. 144
6. (i) 5040 (ii) 30240 (iii) 332640

EXERCISE 7.6

1. (a)(i) 286 (ii) 126 (iii) 84 (iv) $\frac{21}{5}$
(b) (i) True (ii) False (iii) False (iv) True

2. 1771   3. 2002   4. 57750

5. 805

6. (i) 30   (ii) 46

7. 3564   8. 7560

EXERCISE 7.7

1. (a) n, No

2. (a) 126   (b) 1001   (c) 715   (d) 455

3. (a) 56   (b) 126   (c) 120   (d) 286

4. 3

5. 56

EXERCISE 7.8

1. (a) 600   (b) 207 3600   2. 6350400   3. 63

4. 15

EXERCISE 7.9

1. (i) $\lfloor\underline{4}\,.\,\lfloor\underline{5}\,P_3 = 1440$

   (ii) $\lfloor\underline{3}\,.\,\lfloor\underline{5} = 720$

   (iii) $\lfloor\underline{7} - \lfloor\underline{3}\,.\,\lfloor\underline{5} = 4320$

2. $\lfloor\underline{9}\ \ \lfloor\underline{4}\,\lfloor\underline{5} = 2880$

3. $\dfrac{8\,P_5}{5} = 1344$

4. 720

5. $\dfrac{\lfloor\underline{8}}{2}$

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PRACTICE EXERCISE

1. 256	2. 36	3. 12	4. 140
5. 5040	6. 40320	7. 6720	8. 120
9. 720	10. 120	11. 66	12. 10
13. 5	14. 715	15. 30030	16. 371
17. (a) 120	(b) 186		
18. 50400			
19. 12000			
20. (a) 65318400	(b) 1080	(c) 311040	

Chapter 8

BINOMIAL THEOREM

LEARNING OUTCOMES

After studying this lesson, you will be able to:

- state the binomial theorem for a positive integral index and prove it using the principle of mathematical induction;
- write the binomial expansion for expressions like $(x + y)^n$ for different values of x and y using binomial theorem;
- write the general term and middle term (s) of a binomial expansion;
- write the binomial expansion for negative as well as for rational indices;
- apply the binomial expansion for fmding approximate values of numbers like $\sqrt[3]{9}$, $\sqrt{2}$, $\sqrt[3]{3}$ etc; and
- app ly the binomial expansion to evaluate algebraic expressions like $\left(3 - \frac{5}{x}\right)^7$ where x is so small that x^2, and higher powers of x can be neglected.

PREREQUISITES

- Number System
- Four fundamental operations on numbers and expressions.
- Algebraic expressions and their simplifications.
- Indices and exponents.

INTRODUCTION

Binomial means two terms connected by either '+' or '–'. We have come across many expansions of Squares, Cubes etc. of a binomial in earlier classes. For example,

$$(x+y)^1 = x^1 + y^1 = x + y$$
$$(x-y)^1 = x^1 - y^1 = x - y$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x-y)^2 = x^2 - 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Each of these is an expansion of a power of the sum or difference of two terms. These are called **binomial expansions.** The coefficients 1, 1 in the expansion of $(x+y)^1$, 1, 2, 1 in the expansion of $(x+y)^2$, 1, 3, 3, 1 in $(x+y)^3$, 1, 4, 6, 4,1 in $(x+y)^4$ etc. are called **binomial coefficients.** From the above examples we observe that the coefficients in these expansions are as follows.

Index	***Coefficients***
1	**1 1**
2	**1 2 1**
3	**1 3 3 1**
4	**1 4 6 4 1**
5	**1 5 10 10 5 1**
6	....

Fig. 8.1

From the above diagram we observe the following pattern of obtaining a row from the previous row from the second row onwards

(i) Each row begins and ends with 1 (one)

(ii) The n^{th} row has $(n+1)$ terms for any $n \in \mathbf{Z}^+$.

(iii) The other numbers (except the first and last) in a row are obtained by adding the two numbers in the previous row on either side of it. This addition is shown by means of the triangle in each row as follows

a ***b***

a+b

The diagram in Fig. 8.1 is called **Pascal triangle** which is named after its inventor, a French mathematician**Blaise Pascal** (1623-1662). But this was mentioned in a different form under a different name **Meru-Prastara** by the renowned Indian scientist **Pingala** in his book **Chanda Shastra** as early as 200 B.C.

The expansion of $(x+y)^n$ using multiplication, as shown at the beginning, becomes difficult as n increases. In this chapter, we derive the expansion of $(x+y)^n$ when n is a positive integer. This result is known as the **Binomial Theorem.** The coefficients of the terms $x^i y^j$ are called **Binomial Coefficients.** We study the properties of these binomial coefficients, give methods to find the middle term(s) and the numerically greatest term(s) in a binomial expansion. Also we outline (without Proof) the binomial theorem for the expansion of $(x+a)^n$ when n is a negative integer or any fraction. We find the coefficient of a particular index (power) of x in the expansion of $(x+a)^n$ (when n is an integer or a rational number). Finally, we find the approximate values of some irrational numbers using the binomial expansions.

8.1 WHAT IS A STATEMENT ?

In your daily interactions, you must have made several assertions in the form of sentences. Of these assertions, the ones that are **either true or false** are called **statement or propositions.**

For instance,

"I am 20 years old" and "If x = 3, then x^2 = 9" are statements, but 'When will you leave?' And 'How wonderful!' are not statements.

Notice that a statement has to be a definite assertion which can be true or false, but not both. F or example, '$x - 5 = 7$' is not a statement, because we don't know what x, is. If $x = 12$, it is true, but if $x = 5$, 'it is not true. Therefore, $x - 5 = 7$' is not accepted by mathematicians as a statement.

But both '$x - 5 = 7 \Rightarrow x = 12$' and $x - 5 = 7$ for any real number x' are statements, the first one true and the second one false.

Example 8.1: Which of the following sentences is a statement?

(i) India has never had a woman President.

(ii) 5 is an even number.

(iii) $x^n > 1$

(iv) $(a + b)^2 = a^2 + b^2 + 2ab$.

Solution: (i) and (ii) are statements, (i) being true and (ii) being false. (iii) is not a statement, since we can not determine whether it is true or false, unless we know the range of values that x andy can take.

Now look at (iv). At first glance, you may say that it is not a statement, for the very same reasons that (iii) is not. But look at (iv) carefully. It is true for any value of a and b. It is an identity. Therefore, in this case, even though we have not specified the range of values for a and b, (iv) is a statement.

Some statements, like the one given below are about natural numbers in general. Let us look at the statement given below:

$$1 + 2 + ... + n = \frac{n(n+1)}{2}$$

This involves a general natural number n. Let us call this statement P(n) [P stands for proposition].

Then P(1) would be $1 = \frac{1(1+1)}{2}$

Similarly, P (2) would be the statement

$$1 + 2 = \frac{2(2+1)}{2} \text{ and so on.}$$

Let us look at some examples to help you get used to this notation.

Example 8.2: If P(n) denotes $2^n > n - 1$, write P(1), P(k) and P(k + 1), where $k \in N$.

Solution: Replacing n by 1, k and k + 1, respectively in P(n), we get

P(1) : $2^1 > 2 - 1$ i.e $2 > 1$

P(k) : $k^2 > k - 1$

P(k+1) : $2^{k+1} > (k + 1) - 1$ i.e. $2^{k+1} > k$

Example 8.3: If P(n) is the statement $1 + 4 + 7 + ... + (3n - 2) = \frac{n(3n-1)}{2}$ write $P(1)$, $P(k)$ and P(k +1).

Solution : To write $P(1)$, the terms on the left hand side (LHS) of $P(n)$ continue till 3 × 1 – 2., 1. So, P (1) will have only one term in its LHS, i.e., the first term.

Also, the nght hand side (RHS) of $P(1) = \frac{1\times(3\times1-1)}{2} = 1$

Therefore, $P(1) = 1$

Replacing n by 2, we get

$$P(2) = 1 + 4 = \frac{2\times(3\times2-1)}{2} \quad \text{i.e., } 5 = 5.$$

Replacing n by k and $k + 1$, respectively, we get

$$P(k) = 1 + 4 + 7 + ... + (3k - 2) = \frac{k(3k-1)}{2}$$

$$P(k+1) = 1 + 4 + 7 + ... + (3k - 2) + [3(k+1) - 2]$$

$$= \frac{(k+1)\ [3(k+1)-1]}{2}$$

i.e., $\quad 1 + 4 + 7 + ... + (3k + 1) = (k + 1)\ \dfrac{(3k+2)}{2}$

EXERCISE 8.1

1. Determine which of the following are statements:

   (a) $1 + 2 + 4 + ... + 2^n > 20$

   (b) $1 + 2 + 3 + ... + 10 = 99$

   (c) Chennai is much nicer than Mumbai

   (d) Where is Timbuktu ?

   (e) $\dfrac{1}{1\times 2} + ...+ \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

2. Given that P(n) : 6 is a factor of $n^3 + 5n$, write $P(1)$, $P(2)$, $P(k)$ and $P(k + 1)$ where k is a natural number.

3. Write $P(1)$, P(k) and P(k+1), if $P(n)$ is

   (a) $2^n \geq n + 1$

   (b) $(1 + x)^n \geq 1 + nx$

   (c) $n(n + 1)(n + 2)$ is divisible by 6.

   (d) $(x^n - y^n)$ is divisible by $(x - y)$.

   (e) $(ab)^n = a^n b^n$

   (f) $\left(\dfrac{n^5}{5} + \dfrac{n^3}{3} + \dfrac{7^n}{15}\right)$ is a natural number.

4. Write $P(1)$, $P(2)$, $P(k)$ and $p(k+1)$, if $P(n)$ is :

a) $\frac{1}{1\times 2}+.....+\frac{1}{n(n+1)}=\frac{n}{n+1}$

b) $1+3+5+....+(2n-1)=n^2$

c) $(1\times 2)+(2\times 3)+....+n(n+1)<n(n+1)^2$

d) $\frac{1}{1\times 3}+\frac{1}{3\times 5}+.....+\frac{1}{(2n-1)\,(2n+1)}=\frac{n}{2n+1}$

Now ,when you are given a statement like the ones given in Examples 2 and 3, how would you check whether it is true or false? One effective method is mathematical induction, which we shall now discuss.

8.2 THE PRINCIPLE OF MATHEMATICAL INDUCTION

In your daily life, you must be using various kinds of reasoning depending on the situation you are faced with. For instance, if you are told that your friendjust had a child, you would know that it is either a girl or a boy. In this case, you would be applying general principles to a particular case. This form of reasoning is an example of deductive logic.

Now let us consider another situation. When you look around, you find students who study regularly, do well in examinations, you may formulate the general rule (rightly or wrongly) that "anyone who studies regularly will do well in examinations". ill this case, you would be formulating a general principle (or rule) based on several particular instances. Such reasoning is inductive, a process of reasoning by which general rules are discovered by the observation and consideration of several individual cases. Such reasoning is used in all the sciences, as well as in Mathematics.

Mathematical induction is a more precise form of this process. This precision is required because a statement is accepted to be true mathematically only if it can be shown to be true for each and every case that it refers to. The following principle allows us to check if this happens.

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The Principle of Mathematical Induction:

Let $P(n)$ be a statement involving a natural number n. If

(i) it is true for $n = 1$, i.e., $P(1)$ is true; and

(ii) assuming $k \geq 1$ and $P(k)$ to be true, it can be proved that $P(k + 1)$ is true; then $P(n)$ must be true for every natural number n.

Note : *that condition (ii) above does not say that P(k) is true. It says that whenever* P(k) *is true, then P(k + 1) is true.*

Let us see, for example, how the principle of mathematical induction allows us to conclude that $P(n)$ is true for $n = 11$.

By (i) $P(1)$ is true. As $P(1)$ is true, we can put $k = 1$ in (ii), So $P(1 + 1)$, i.e., $P(2)$ is true. As $P(2)$ is true, we can put $k = 2$ in (ii) and conclude that $P(2 + 1)$, i.e., $P(3)$ is true. Now put $k = 3$ in (ii), so we get that $P(4)$ is true. lt is now clear that if we continue like this, we shall get that $P(11)$ is true.

lt is also clear that in the above argument, 11 does not play any special role. We can prove that $P(137)$ is true in the same way. Indeed, it is clear that $P(n)$ is true for all $n > 1$.

Let us now see, through examples, how we can apply the principle of mathematical induction to Algebra prove various types of mathematical statements.

Example 8.4 : Prove that

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2} \text{ where } n \text{ is a natural number.}$$

Solution: We have

$$P(n) : 1 + 2 + ... + n = \frac{n(n+1)}{2}$$

Therefore, P(1) is $1 = \frac{1\times(1+1)}{2}$, which is true.

Therefore, P(1) is true.

Let us now see, if $P(k + 1)$ is true whenever $P(k)$ is true.

Let us, therefore, assume that P(k) is true, i.e.,

$$1 + 2 + ... + K = \frac{k}{2}(k+1) \qquad ...(i)$$

Now P(k + 1) : $1 + 2 + \ ... \ k + (k + 1) = \frac{(k+1)(k+2)}{2}$

If will be true, if we can show that LHS = RHS.

The LHS of P(k + 1) = $(1 + 2 + 3 + ... + k) + (k + 1)$

$$= \frac{k}{2}(k + 1) + (k + 1) \quad \text{(From (i))}$$

$$= (k + 1)\left(\frac{k}{2}+1\right)$$

$$= \left(\frac{k+1}{2}\right)(k+2)$$

$$= \text{RHS of } P(k + 1)$$

So, P(k+1) is true, if we assume that P(k) is true.

Since $P(1)$ is also true, both the conditions of the principle of mathematical induction are fulfilled, we conclude that the given statement is true for every natural number n.

As you can see, we have proved the result in three steps - the **basic step** [i.e., checking (i)], the **Induction step** [i.e., checking (ii)], and hence arriving at the end result.

Example 8.5 : Prove that

$$1.2 + 2.2^2 + 3.2^3 + 4.2^4 + ... + n.2^n = (n - 1) \ . \ 2^{n+1} + 2,$$

where n is a natural number.

Solution: Here $P(n)$: $1.2^1 + 2.2^2 + 3.2^3 + ... + n.2^n = (n - 1) \ . \ 2^{n+1} + 2$

Therefore, $P(1)$ is $1.2^1 = (1-1)2^{1+1} + 2$, i.e., $2 = 2$.

So, $P(1)$ is true.

We assume that $P(k)$ is true, i.e., .

$$1.2^1 + 2.2^2 + 3.2^3 + ... + k.2^k = (k - 1) \ 2^{k+1} + 2 \qquad ... (i)$$

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Now will prove that P$(k + 1)$ is true.

Now P$(k + 1)$ is

$$1.2^1 + 2.2^2 + 3.2^3 + ... + k.2^k + (k+1)2^{k+1} = [(k+1) - 1]2^{(k+1+1)} + 2$$

$$= k.2^{k+2} + 2$$

$$\text{LHS of } P(k+1) = (1.2^1 + 2.2^2 + 3.2^3 + ... + k.2^k + (k+1)2^{k+1}$$

$$= 2^{k+1}[(k-1) + (k+1)] + 2$$

$$= 2^{k+1}(2k) + 2 \qquad \text{[Using (i)]}$$

$$= k.2^{k+2} + 2$$

$$= \text{RHS of } P(k+1)$$

Therefore, $P(k + 1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Example 8.6: For ever natural number n, prove that $(x^{2n-1} + y^{2n-1})$ is divisible by $(x + y)$ where $x, y \in N$.

Solution : Let us see if we can apply the principle of induction here. Let us call $P(n)$ the statement $(x^{2n-1} + y^{2n-1})$ is divisible by $(x + y)$.

Then $P(1)$: '$(x^{2-1} + y^{2-1})$ is divisible by $(x + y)$' i.e., '$(x + y)$ is divisible by $(x + y)$' which is true.

Therefore, $P(1)$ is true.

Let us now assume that $P(k)$ is true for some natural number k, i.e., $(x^{2k-1} + y^{2k-1})$ is divisible by $(x + y)$.

This means that for some natural number t, $x^{2k-1}+y^{2k-1}=(x + y)t$

Then $x^{2k-1} = (x + y)t - y^{2k-1}$

We wish to prove that P$(k + 1)$ is true, i.e., $[x^{2(k+1)-1} + y^{2(+1)-1}]$ is divisible by $(x + y)$ is true.

Now,

$$x^{2(k+1)-1} + y^{2(+1)-1}) = x^{2k+1} + y^{2k+1}$$

$$= x^{2k-1+2} + y^{2k+1}$$

$$= x^2\, x^{2k-1} + y^{2k+1}$$

$$= x^2\,[(x+y)t - y^{2k-1}] + y^{2k+1} \qquad \text{[From(1)]}$$

$$= x^2\,(x+y)t - x^2\, y^{2k-1} + y^{2k+1}$$

$$= x^2\,(x+y)t - x^2\, y^{2k-1} + y^2\, y^{2k-1}$$

$$= x^2\,(x+y)t - y^{2k-1}\,(x^2 - y^2)$$

$$= (x+y)\,[x^2\, t - (x-y)\, y^{2k-1}]$$

which is divisible by of $(x + y)$.

Thus, $P(k + 1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Example 8.7 : Prove that $2^n > n$ for every natural number n.

Solution: We have $P(n) : 2^n > n$.

Therefore, $P(1) : 2^1 > 1$ i.e $2 > 1$, which is true.

We assume $P(k)$ to be true, that is,

$$2k > k \qquad \text{... (i)}$$

We wish to prove that $P(k + 1)$ is true, i.e. $2^{k+1} > k + 1$.

Now, multiplying both sides of (i) by 2, we get

$$2 \,.\, 2k > 2k$$

$\Rightarrow$ $2^{k+1} > k + 1$, since $k > 1$

Therefore, $P(k + 1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

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Sometimes, we need to prove a statement for all natural numbers greater than a particular natural number, say a (as in Example 8.8 below). In such a situation, we replace $P(1)$ by $P(a+1)$ in the statement of the principle.

Example 8.8 : Prove that

$n^2 > 2(n+1)$ for all $n \geq 3$, where n is a natural number.

Solution : For all $n \geq 3$, let us call the following statement

$$P(n):\ n^2 > 2(n+1),$$

Since we have to prove the given statement for $n \geq 3$, the first relevant statement is $P(3)$. We, therefore, see whether $P(3)$ is true.

$$P(3):\ 3^2 > 2 \times 4, \quad \text{i.e.,} \quad 9 > 8\ .$$

So, P(3) is true.

Let us assume that $P(k)$ is true, where $k \geq 3$, that is

$$k^2 > 2(k+1) \qquad \text{...(i)}$$

We wish to prove that $P(k+1)$ is true.

$$P(k+1):\ (k+1)^2 > 2(k+2)$$

$$\begin{aligned} \text{LHS of } P(k+1) &= (k+1)^2 \\ &= k^2 + 2k + 1 \\ &> (2k+1) + 2k + 1 \qquad [\text{ By (i)}] \\ &> 3 + 2k + 1, \quad \text{since } 2(k+1) > 3. \\ &= 2\,(k+2) \end{aligned}$$

Thus $(k+1)^2 > 2(k+2)$.

Therefore, $P(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number $n \geq 3$.

Example 8.9 : Using principle of mathematical induction, prove that

$$\left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right)$$ is a natural number for all natural numbers n.

Solution: Let P(n) : $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ be a natural number.

$\therefore$ P(1) : $\left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$ is a natural number.

or, $\frac{1}{5}+\frac{1}{3}+\frac{7}{15}=\frac{3+5+7}{15}=\frac{15}{15}=1$, which is a natural number.

$\therefore$ P(1) is true.

Let P(k) : $\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)$ is a natural number be true ...(i)

Now $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$

$$=\frac{1}{5}[5^5+5k^4+10k^3+10k^2+5k+1]+\frac{1}{3}[k^3+3k^2+3k+1]$$

$$+\left(\frac{7}{15}k+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+(k^4+2k^3+3k^2+2k)+\left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+(k^4+2k^3+3k^2+2k)+1 \quad ...(ii)$$

By (i), $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a natural number.

also $k^4+2k^3+3k^2+2k$ is a natural number and 1 is also a natural number.

(ii) being sum of natural numbers is a natural number.

$\therefore$ P(k + 1) is true, whenever P(k) is true.

$\therefore$ P(n) is true for all natural numbers n.

Hence, $\left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right)$ is a natural number for all natural numbers n.

EXERCISE 8.2

1. Using the principle of mathematical induction, prove that the following statements hold for any natural number n:

   (a) $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n}{6}(n + 1 (2n + 1)$

   (b) $1^3 + 2^3 + 3^3 + ... + n^3 = (1 + 2 + ...+ n)^2$

   (c) $1 + 3 + 5 + ... + (n^3 = (1 + 2 + ... + n)^2$

   (d) $1 + 4 + 7 + ... + (3n - 2) = \frac{n}{2}(3n - 1)$

2. Using principle of mathematical induction, prove the following equalities for any natural number n:

   (a) $\frac{1}{1\times 2}+\frac{1}{2\times 3}+....+\frac{1}{n(n+1)}=\frac{n}{n+1}$

   (b) $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+....+\frac{1}{(n-1)\,(2n+1)}=\frac{n}{2n+1}$

   (c) $(1\times 2)+(2\times 3)+....+n(n+1)\ \frac{n(n+1)(n+2)}{3}$

3. For every natural number n, prove that

   (a) $n^3 + 5n$, is divisible by 6

   (b) $(x^n - 1)$, is divisible by $(x - 1)$

   (c) $n^3 + 2n$, is divisible by 3

   (d) 4 divides $(n^4 + 2n^3 + n^2)$.

4. Prove the following inequalities for any natural number n:

   (a) $3^n \geq 2n + 1$ (b) $4^{2n} > 15n$

   (c) $1 + 2 + ... + n < \frac{1}{8}(2n + 1)^2$

5. Prove the following statements using induction:

   (a) $2^n > n^2$, $n \geq$ N where n is any natural number.

   (b) $\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} > \frac{13}{24}$ for any natural number n greater than 1.

6. Prove that $n(n^2 - 1)$is divisible by 3 for every natural number n greater than 1.

To prove that a statement $P(n)$ is true for every $n \in$ N, both the basic as well as the induction steps must hold.

If even one of these conditions does not hold, then the proof is invalid. For instance, if P(n) is $(a + b)^n \leq a^n + b^n$ for all reals a and b, then $P(1)$ is certainly true. But, $P(k)$ being true does not imply the truth of $P(k + 1)$. So, the statement is not true for every natural number n. (For instance, $(2 + 3)^2\ 2^2 \not\leq 3^2$) .

As another example, take $P(n)$ to be P(n) : $n > \frac{n}{2} + 20$.

In this case, $P(1)$ is not true. But the induction step is true. Since $P(k)$ being true.

$$\Rightarrow \quad k > \frac{k}{2} + 20$$

$$\Rightarrow k + 1 > \frac{k}{2} + 20 + 1 > \frac{k}{2} + 20 + \frac{1}{2} = \frac{k+1}{2} + 20$$

$$\Rightarrow \text{P}(k + 1) \text{ is true.}$$

Or if we want a statement which is false for all n, then take $P(n)$ to be

$$n > \frac{n}{2} + 20.$$

And, as you can see, $P(n)$ is false for large values of n say $n = 100$.

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8.3 THE BINOMIAL THEOREM FOR A NATURAL EXPONENT

You must have multiplied a binomial by itself, or by another binomial. Let us use this knowledge to do some expansions. Consider the binomial $(x + y)$. Now,

$$(x + y)^1 = x + y$$

$$(x + y)^2 = (x + y)\ (x + y) = x^2 + 2xy + y^2$$

$$(x + y)^3 = (x + y)\ (x + y)^2 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = (x + y)\ (x + y)^3 = x^4 + 4x^3y + 6\ x^2\ y^2 + 4x\ y^3 + y^4$$

$$(x + y)^5 = (x + y)\ (x + y)^4 = x^5 + 5x^4y + 10x^3y^2 + 10x^2\ y^3 + 5xy^4 + y^5$$

and so on.

In each of the equations above, the right hand side is called the binomial expansion of the left hand side.

Note that in each of the above expansions, we have written the power of a binomial in the expanded form in such a way that the terms are in descending powers of the first term of the binomial (which is x in the above examples). If you look closely at these expansions, you would also observe the following:

1. The number of terms in the expansion is one more than the exponent of the binomial. For example, in the expansion of $(x + y)^4$, the number of terms is 5.
2. The exponent of x in the first term is the same as the exponent of the binomial, and the exponent decreases by 1 in each successive term of the expansion.
3. The exponent of y in the first tenn is zero (as $y^o = 1$). The exponent of y in the second term is 1, and it increases by 1 in each successive term till it becomes the exponent of the binomial in the last term of the expansion.

4. The sum of the exponents of x and y in each term is equal to the exponent of the binomial. For example, in the expansion of $(x+y)^5$, the sum of the exponents of x andy in each term is 5.

If we use the combinatorial co-efficients, we can write the expansion as

$$(x+y)^3 = {}^3C_0\, x^3 + {}^3C_1\, x^2 y + {}^3C_2\, xy^2 + {}^3C_3\, y^3$$

$$(x+y)^4 = {}^4C_0\, x^4 + {}^4C_1\, x^3 y + {}^4C_2\, x^2y^2 + {}^4C_3\, x y^3 + {}^4C_4\, y^4$$

$$(x+y)^5 = {}^5C_0\, x^5 + {}^5C_1\, x^4 y + {}^5C_2\, x^3y^2 + {}^5C_3\, x^2y^3 + {}^5C_4\, x y^4 + {}^5C_5\, y^5$$

and so on.

More generally, we can write the binomial expansion of $(x+y)^n$, where n is a positive integer, as given in the following theorem. This statement is called the **binomial theorem for a natural** (or **positive integral) exponent.**

Theorem 8.1

$$(x+y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-4} y^1 + {}^nC_2\, x^{n-3}y^2 + .. + {}^nC_{n-3}\, xy^{n-1} + {}^nC_n\, y^n \quad ...(A)$$

where $n \in$ N and $x, y \in$ R.

Proof: Let us try to prove this theorem, using the principle of mathematical induction.

Let statement (A) be denoted by P(n), i.e.,

$$P(n) : (x+y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2}y^2 + .. + {}^nC_{n-1}\, xy^{n-1} + {}^nC_n\, y^n \quad ...(i)$$

Let us examine whether $P(1)$ is true or not.

From (i), we have

$$P(1) : (x+y)^1 = {}^1C_0\, x + {}^1C_1\, y = 1 \times x + xy$$

i.e., $(x+y)^1 = x + y.$

Thus, $P(1)$ holds.

Now, let us assume that P(k) is true, i.e.,

$$P(k) : (x+y)^k = {}^kC_0\, x^k + {}^kC_1\, x^{k-1}\, y + {}^kC_2\, x^{k-2}\, y^2 + \ldots + {}^kC_{k-1}\, x\, y^{k-1} + {}^kC_k\, y^k \qquad \ldots\text{(ii)}$$

AssumingthatP(k) is true, if we prove that $P(k+1)$ is true, thenP(n) holds, for all n. Now,

$$(x+y)^{k+1} = (x+y)\,(x+y)^k$$

$$= (x+y)\,({}^kC_0\, x^k + {}^kC_1\, x^{k-1}\, y + {}^kC_2\, x^{k-2}\, y^2 + \ldots + {}^kC_{k-1}\, x\, y^{k-1} + {}^kC_k\, y^k)$$

$$= {}^kC_0\, x^{k+1} + {}^kC_0\, x^k y + {}^kC_1\, x^k\, y + {}^kC_1\, x^{k-1}\, y^2 + {}^kC_2\, x^{k-2} y^3 + \ldots + {}^kC_{k-1} x^2\, y^{k-1} + {}^kC_{k-1}\, xy^k + {}^kC_k\, xy^k + {}^kC_k\, y^{k+1}$$

i.e., $$(x+y)^{k+1} = {}^kC_0\, x^{k+1} + ({}^kC_0 + {}^kC_1)\, x^4\, y + ({}^kC_1 + {}^kC_2)\, x^{k-1}\, y^2 + \ldots + ({}^kC_{k-1} + {}^kC_k)xy^4 + {}^kC_k\, y^{k+1} \qquad \ldots\text{(iii)}$$

From Lesson 7, you know that ${}^kC_0 = 1 = {}^{k+1}C_0$ ....(iv)

and ${}^kC_k = 1 = {}^{k+1}C_{k+1}$

Also ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$

Therefore,

$$\begin{aligned} {}^kC_0 + {}^kC_1 &= {}^{k+1}C_1 \qquad \ldots\text{(v)} \\ {}^kC_1 + {}^kC_2 &= {}^{k+1}C_2 \\ {}^kC_2 + {}^kC_3 &= {}^{k+1}C_3 \end{aligned}$$

....

.... and so on

Using (iv) and (v) we can write (iii) as

$$(x+y)^{k+1} = {}^{k+1}C_0\, x^{k+1} + {}^{k+1}C_1 x^k\, y + {}^{k+1}C_2\, x^{k-1}\, y^2 + \ldots + {}^{k+1}C_k\, xy^k + {}^{k+1}C_{k+1}\, y^{k+1}$$

which shows that P(k+1) is true.

Thus, we have shown that (a)P(1) is true, and (b) if P(k) is true, then P(k+ 1) is also true.

Therefore, by the principle of mathematical induction, P(n) holds for any value of n. So, we have proved the binomial theorem for any natural exponent.

This result is supported to have been proved first by the famous Arab poet Omar Khayyam, though no one has been able to trace his proof so far.

We will now take some examples to illustrate the theorem.

Example 8.10 : Write the binomial expansion of $(x + 3y)^5$.

Solution: Here the first term in the binomial is X and the second term is $3y$. Using the binomial theorem, we have

$$(x + 3y)^5 = {}^5C_0\, x^5 + {}^5C_1\, x^4\, (3y)^1 + {}^5C_2\, x^3\, (3y)^2 + {}^5C_3\, x^2\, (3y) + {}^5C_4\, x(3y)^4 + {}^5C_5\, (3y)^5$$

$$= 1 \times x^5 + 5x^4 \times 3y + 10x^3 \times (9\,y^2) + 10x^2 \times (27y^3) + 5x \times (81y^4) + 1 \times 243y^5$$

$$= x^5 + 15x^4\, y + 90x^3\, y^2 + 270\, x^2\, y^2 + 405\, xy^4 + 243y^5$$

$$\therefore\ (x + 3y)^5 = x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$$

Example 8.11: Expand $(1 + a)^n$ in terms of powers of a, where a is a real number.

Solution: Taking $x = 1$ and $y = a$ in the statement of the binomial theorem, we have

$$(1 + a)^n = {}^nC_0\, (1)^n + {}^nC_1\, (1)^{n-1} \,.\, a + {}^nC_2\, (1)^{n-2}a^2 + \ldots + {}^nC_{n-1}\, (1)a^{n-1} + {}^nC_n\, a^n$$

i.e., $(1 + a)^n = 1 + {}^nC_1\, a + {}^nC_2\, a^2 + \ldots + {}^nC_{n-1}\, a^{n-1} + a^n$ ....(B)

is another form of the statement of the binomial theorem.

The theorem can also be used in obtaining the expansions of expressions of the type

$$\left(x + \frac{1}{x}\right)^5, \left(\frac{y}{x} + \frac{1}{y}\right)^5, \left(\frac{a}{4} + \frac{2}{9}\right)^5, \left(\frac{2t}{3} - \frac{3}{2t}\right)^6 \text{ etc.,}$$

Let us illustrate it through an example.

Example 8.12: Write the expansion of $\left(\frac{y}{x}+\frac{1}{y}\right)^4$, $x,\ y\neq 0$.

Solution: We have :

$$\left(\frac{y}{x}+\frac{1}{y}\right)^4 = {}^4C_0\left(\frac{y}{x}\right)^4 + {}^4C_1\left(\frac{y}{x}\right)^3\left(\frac{1}{y}\right) + {}^4C_2\left(\frac{y}{x}\right)^2\left(\frac{1}{y}\right)^2$$

$$+ {}^4C_3\left(\frac{y}{x}\right)\left(\frac{1}{y}\right)^3 + {}^4C_4\left(\frac{1}{y}\right)^4$$

$$= 1\times\frac{y^4}{x^4} + 4\times\frac{y^3}{x^3}\times\frac{1}{y} + 6\times\frac{y^2}{x^2}\times\frac{1}{y^2} + 4\times\left(\frac{y}{x}\right)\times\frac{1}{y^3} + 1\times\frac{1}{y^4}$$

$$= \frac{y^4}{x^4} + 4\frac{y^2}{x^3} + \frac{6}{x^2} + \frac{4}{xy^2} + \frac{1}{y^4}.$$

Example 8.13: The population of a city grows at the annual rate of 3%. What percentage increase is expected in 5 years? Give the answer up to 2 decimal places.

Solution: Suppose the population is a at present. After 1 year it will be

$$a + \frac{3}{100}a = a\left(1+\frac{3}{100}\right)$$

After 2 years, it will be $a\left(1+\frac{3}{100}\right)+\frac{3}{100}\left[a\left(1+\frac{3}{100}\right)\right]$

$$= a\left(1+\frac{3}{100}\right)\left(1+\frac{3}{100}\right) = a\left(1+\frac{3}{100}\right)^2$$

Similarly, after 5 years, it will be $a\left(1+\frac{3}{100}\right)^5$

Using the binomial theorem, and ignoring terms involving more than 3 decimal places, we get

$$a\left(1+\frac{3}{100}\right)^5 \approx a[1+5(0.03)+10(0.03)^2] = a\times 1.159$$

So, the increase is $0.159\times 100\% = \frac{159}{1000}\times 100\times\frac{1}{100} = 15.9\%$ in 5 years.

Example 8.14: Using binomial theorem, evaluate

(i) 102^4 (ii) 97^3

Solution:

(i) $102^4 = (100 + 2)^4$

$= {}^4C_0 (100)^4 + {}^4C_1 (100)^3 .2 + {}^4C_2 (100)^2 .2 + {}^4C_3 (100).2^3 + {}^4C_4 .2^4$

$= 100000000 + 8000000 + 240000 + 3200 + 16$

$= 108243216$

(ii) $(97)^3 = (100 - 3)^3$

$= {}^3C_0 (100)^3 - {}^3C_1 (100)^2 .3 + {}^3C_2 (100) .3^2 - {}^3C_3 3^3$

$= 1000000 - 90000 + 2700 - 27$

$= 1002700 - 90027$

$= 912673.$

EXERCISE 8.3

1. Write the expansion of each of the following:

   (a) $(2a + b)^3$ (b) $(x^2 - 3y)^6$ (c) $(4a - 5b)^4$

   (d) $(ax + by)^n$

2. Write the expansions of:

   (a) $(1 - x)^7$ (b) $\left(1 + \frac{x}{y}\right)^7$ (c) $(1 + 2x)^5$

3. Write the expansions of:

   (a) $\left(\frac{a}{3} + \frac{b}{2}\right)^5$ (b) $\left(3x - \frac{5}{x^2}\right)^7$ (c) $\left(x + \frac{1}{x}\right)^4$

   (d) $\left(\frac{x}{y} + \frac{y}{x}\right)^5$

4. Suppose I invest Rs. 1 lakh at 18% per year compound interest. What sum will I get back after 10 years? Give your answer up to 2 decimal places.

5. The population of bacteria increases at the rate of 2% per hour. If the count of bacteria at 9 a.m. is 1.5×10^5, find the number at 1 p.m. on the same day.

6. Using binomial theorem, evaluate each of the following:

   (i) $(101)^4$ (ii) $(99)^4$ (iii) $(1.02)^3$ (iv) $(0.98)^3$

8.4 GENERAL AND MIDDLE TERMS IN A BINOMIAL EXPANSION

Let us examine various terms in the expansion (A) of $(x+y)^n$ i.e., in

$$(x+y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}y^2 + \ldots + {}^nC_{n-1}\, xy^{n-1} + {}^nC_n\, y^n$$

We observe that

the first term is ${}^nC_0\, x^n$, i.e., ${}^nC_{1-1}\, x^n\, y^0$;

the second term is ${}^nC_1\, x^{n-1}\, y$, i.e., ${}^nC_{2-1}\, x^{n-1}\, y^1$;

the third term is ${}^nC_2\, x^{n-2}\, y^2$ i.e., ${}^nC_{3-1}\, x^{n-2}\, y^2$;

and so on.

From the above, we can generalise that

the $(r+1)^{\text{th}}$ term is ${}^nC_{(r+1)-1}\, x^{n-r}\, y^r$; i.e., ${}^nC_r\, x^{n-r}\, y^r$.

If we denote this term by T_{r+1} , we have

$$T_{r+1} = {}^nC_r\, x^{n-r}\, y^n$$

T_{r+1} is generally referred to as the **general term** of the binomial expansion.

Let us now consider some examples and find the general terms of some expansions.

Example 8.15 : Find the $(r + 1)^{th}$ term in the expansion of $\left(x^2 + \frac{1}{x}\right)^n$ where n is a natural number. Verify your answer for the first term of the expansion.

Solution: The general term of the expansion is given by:

$$T_{r+1} = {}^nC_r\, (x^2)^{n-r} \cdot \left(\frac{1}{x}\right)^r$$

$$= {}^nC_r\, x^{2n-2r} \cdot \frac{1}{x^r}$$

$$= {}^nC_r\, x^{2n-3r} \qquad \text{...(i)}$$

Hence, the $(r + 1)^{th}$ term in the expansion is ${}^nC_r\, x^{2n-3r}$.

On expanding $\left(x^2 + \frac{1}{x}\right)^n$, we note that the first term is $(x^2)^n$ or x^{2n}.

Using (i), we find the first term by putting $r = 0$.

Since $T_1 = T_{0+1}$

$\therefore \quad T_1 = {}^nC_0\, x^{2n-0} = x^{2n}$

This verifies that the expression for T_{r+1} is correct for $r + 1 = 1$.

Example 8.16 : Find the fifth term in the expansion of

$$\left(1 - \frac{2}{3}x^3\right)^6$$

Solution: Using here $T_{r+1} = T_5$, which gives $r + 1 = 5$, i.e., $r = 4$

Also $n = 6$ and let $a = -\frac{2}{3}x^3$.

$$\therefore \quad T_5 = {}^6C_4 \left(-\frac{2}{3}x^3\right)^4$$

$$= {}^6C_4 \left(\frac{16}{81}x^{12}\right)$$

$$= \frac{6 \times 5}{1 \times 2} \times \frac{16}{81} \times x^{12} = \frac{80}{27}x^{12}$$

$\therefore$ Thus, the fifth term in the expansion is $\frac{80}{27}x^{12}$

EXERCISE 8.4

1. F or a natural number n, write the $(r+1)^{th}$ term in the expansion of each of the following:

   (a) $(2x+y)^n$ (b) $(2a^2-1)^n$

   (c) $(1-a)^n$ (d) $\left(3+\frac{1}{x^2}\right)^n$

2. Find the specified terms in each of the following expansions:

   (a) $(1+2y)^8$; 6th term (b) $(2x+3)^7$; 4th term

   (c) $(2a-b)^{11}$;7th term (d) $\left(x+\frac{1}{x}\right)^6$; 4th term

   (e) $\left(x^3-\frac{1}{x^2}\right)^7$; 5th term

Now that you are familiar with the general term of an expansion, let us see how we can obtain the **middle term** (or terms) of a binomial expansion. Recall that the number of terms in a binomial expansion is always one more than the exponent of the binomial. This implies that if the exponent is even, the number of terms is odd, and if the exponent is odd, the number of terms is even. Thus, while finding the middle term in a binomial expansion, we come across two cases:

Case 1 : When n is even.

To study such a situation, let us look at a particular value of n, say $n = 6$. Then the number of terms in the expansion will be 7. From Fig. 8.1, you can see that there are three terms on either side of the fourth term.

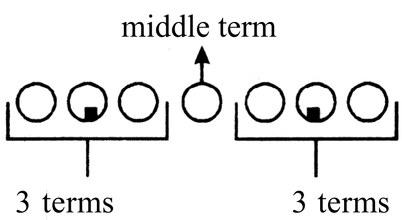

Fig. 8.1

In general, when the exponent n of the binomial is even, there are $\frac{n}{2}$ terms on either side of the $\left(\frac{n}{2}+1\right)$ th term. Therefore, the $\left(\frac{n}{2}+1\right)$th term is the middle term.

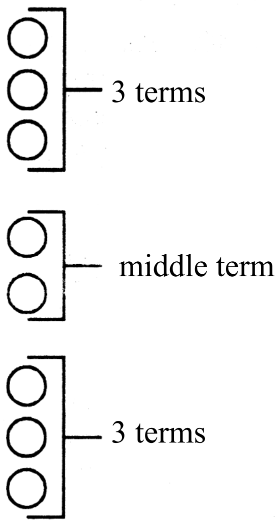

Fig. 8.2

Thus, in this case, there are two middle terms, namely, the fourth,

i.e., $\left(\frac{7+1}{2}\right)$ and the fifth, i.e., $\left(\frac{7+3}{2}\right)$ terms

Similarly, if n = 13, then the $\left(\frac{13+1}{2}\right)$th and the $\left(\frac{13+3}{2}\right)$ th terms, i.e., the 7th and 8th terms are two middle terms, as is evident from Fig. 8.3.

From the above, we conclude that

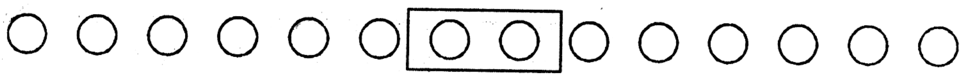

Fig. 8.3

When the exponent n of a binomial is an odd natural number, then the $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms are two middle terms in the corresponding binomial expansion.

Let us now consider some examples.

Example 8.17 : Find the middle term in the expansion of $(x^2 + y^2)^8$.

Solution: Here $n = 8$ (an even number).

Therefore, the $\left(\frac{8}{2} + 1\right)$th i.e., the 5th term is the middle term.

Putting $r = 4$ in the general term $T_{r+1} = {}^8C_r\,(x^2)^{8-r}.\,(y^2)^r$.

$$T_5 = {}^8C_4\,(x^2)^{8-4}\,(y^2)^4 = 70\,x^8\,y^8.$$

Example 8.18: Find the middle term(s) in the expansion of $\left(2x^2 + \frac{1}{x}\right)^9$.

Solution: Here $n = 9$ (an odd number). Therefore, the $\left(\frac{9+1}{2}\right)$th and $\left(\frac{9+3}{2}\right)$th are middle terms i.e., T_5 and T_6 are middle terms.

F or finding T_5 and T_6 putting $r = 4$ and $r = 5$ in the general term

$$T_{r+1} = {}^9C_r\,(2x^2)^{9-r}\,.\left(\frac{1}{x}\right)^6$$

$$T_5 = {}^9C_4\,(2x^2)^{9-4}\left(\frac{1}{x}\right)^4$$

$$= \frac{9\times 8\times 7\times 6}{1\times 2\times 3\times 4}\,.\,32\,x^{10}\,.\left(\frac{1}{x}\right)^4 = 4032\,x^6.$$

and $T_6 = {}^9C_5\,(2x^2)^{9-5}\left(\frac{1}{x}\right)^5 = 2016\,x^3$

Thus, the two middle terms are $4032x^6$ and $2016\,x^3$.

EXERCISE 8.5

1. Find the middle term(s) in the expansion of each of the following:

(a) $(2x + y)^{10}$ (b) $\left(1+\frac{2}{3}x^3\right)^8$

(c) $\left(x + \frac{1}{x}\right)^6$ (d) $(1 - x^2)^{10}$

2. Find the middle term(s) in the expansion of each of the following:

(a) $(a + b)^7$ (b) $(2a - b)^9$

(c) $\left(\frac{3x}{4} - \frac{4y}{3}\right)^7$ (d) $\left(x + \frac{1}{x^2}\right)^{11}$

8.5 BINOMIAL THEOREM FOR RATIONAL EXPONENTS

So far you have applied the binomial theorem only when the binomial has been raised to a power which is a natural number. What happens if the exponent is a negative integer, or if it is a fraction? We will state the result that allows us to still have a binomial expansion, but it will have infinite terms in this case.

The result is a generalised version of the earlier binomial theorem which you have studied.

Theorem 8.2 The Binomial Theorem for a Rational Exponent.

If r is a rational number, and x is a real number such that $|x| < 1$, then

$$(1 + x)^r = 1 + r x + \frac{r(r-1)}{2!} x^2 + \ldots\ldots\ldots\ldots \text{(D)}$$

We will not prove this result here, as it is beyond the scope of this course. In fact, even Sir Issac Newton, who is credited with stating this generalisation, stated it without proof in two letters, written inA.D. 1676. The proofwas developed later, by other mathematicians, in stages. Among those who contributed to the proof of this theorem were English mathematician Colin Maclaurin (A.D. 1698-1746) for rational values of r, Giovanni Francesco, M.M. Salvemini (A.D. 1708-1783) and the German mathematician Abraham G. Kasther (A.D. 1719 - 1800) for integral values of$r$, the Swiss mathematician Leonhard Euler (AD 1707-1783) for fractional exponents and the Norwegian mathematician Neils Henrik Abel (1802-1829) for complex exponents. Let us consider some examples to illustrate the theorem.

Example 8.19 : Write the expansion of $(1 + x)^{-1}$, when $|x| < 1$.

Solution: Here $r = -1$ [[with reference to (D) above].

Therefore,

$$(1 + x)^{-1} = 1 + (-1)\,x + \frac{(-1)\,(-2)}{2!}\,x^2 + \frac{(-1)(-2)\,(-3)}{3\,!}\,x^3 + ...$$

i.e., $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ...$

Similarly, you can write the expansion $(1-x)^{-1} = 1 + x + x^2 + x^3 + ...$

Note the above expansions. In case of $(1+x)^{-1}$ all the terms have positive and negative signs alternate, while in the case of $(1-x)^{-1}$ all the terms have positive sign.

You may have also observed the following points about the binomial expansion (D) in general;

1. If r is a natural number, then (C) and (D) coincide for the case $|x|<1$.
2. Note that ${}^rC_0 = 1$, ${}^rC_1 = r$, ${}^rC_2 = \frac{r(r-1)}{2\,!}$ etc. Thus, the coefficients

   $1, r, \frac{r(r-1)}{2\,!}, ...$ in (D) look like combinatorial coefficients.,

   However, recall that ${}^rC_s$ is defined for natural numbers r and whole numbers only.

   Therefore,

   ${}^rC_0, {}^rC_1, {}^rC_2,$ etc. have no meaning in the present context.
3. The expression (D) will have an infmite number of terms.
4. The sum of the series on the RHS of (D) may not be meaningful if $x > 1$.

   For example, if we put $x = 2$ in Example 1, we have

   $(1 + 2)^{-1} = 1 - 2 + 4 - 8 + 16 - 32 + ...$

   i.e., $\frac{1}{3} = (1 - 2) + (4 - 8) + (16 - 32) + ...$

i.e., $\frac{1}{3} = -1 - 4 - 16 - \ldots,$

which is clearly false.

Therefore, for (D) to hold, it is necessary that $|x| < 1$

Let us look at some more examples of this binomial expansion.

Example 20: Expand $(x + y)^r$, where r is a rational number and $\left|\frac{y}{x}\right| < 1$.

Hence expand $(3 + 5p)^{2/5}$ 5, when $|p| < \frac{3}{5}$

Solution: $(x + y)^r = x^r \left(1 + \frac{y}{x}\right)^r$ ...(1)

Since it is given that $\left|\frac{y}{x}\right| < 1$, we have

$$\left(1 + \frac{y}{x}\right)^r = 1 + r\left(\frac{y}{x}\right) + \frac{r(r-1)}{2\,!}\left(\frac{y}{x}\right)^2 + \frac{r(r-1)\,(r-2)}{3!}\left(\frac{y}{x}\right)^3 + \ldots$$

Therefore, from (1), we have

$$(x+y)^r = x^r\left[1 + r\left(\frac{y}{x}\right) + \frac{r(r-1)}{2\,!}\left(\frac{y}{x}\right)^2 + \frac{r(r-1)\,(r-2)}{3!}\left(\frac{y}{x}\right)^3 + \ldots\right]$$

i.e., $(x + y)^r = x^r + rx^{r-1}y + \frac{r(r-1)}{2\,!}x^{r-2}y^2 + \frac{r(r-1)\,(r-2)}{3!}x^{r-3}y^3 + \ldots(2)$

Now, to solve the second part of the question, note that $|p| < \frac{3}{5}$.

If $\left|\frac{5p}{3}\right| < 1$, then putting $x = 3$, $y = 5p$, $r = \frac{2}{5}$ in (2), we get

$$(3 + 5p)^{2/5} = 3^{2/5} + \frac{2}{5}(3)^{\frac{2}{5}-1}(5p)^1 + \frac{\frac{2}{5}\left(\frac{2}{5}-1\right)}{2\,!}(3)^{\frac{2}{5}-2}(5p)^2 + \ldots$$

$$= 3^{2/5} + (3)^{-3/5}\ (2p) + \frac{\frac{2}{5}\left(-\frac{3}{5}\right)}{2}\ (3)^{-\frac{8}{5}}\ 25p^2\ + ...$$

$$= 3^{2/5} + 3^{-3/5}\ (2p) - 3^{-3/5}\ p^2\ + ...$$

The result we have just obtained in Example 8.20 is another form of the binomial theorem for a rational exponent. Let us restate it formally.

If r is a rational number and $\left|\frac{y}{x}\right| < 1$

$$(x+y)^r = x^r + r\ x^{r-1}\ y + \frac{r(r-1)}{2\,!}\ x^{r-2}\ y^2 + \frac{r(r-1)(r-2)}{3\,!}\ x^{r-3}\ y^3\ + ...\ \text{(E)}$$

Note that you could have expanded $(x + y)^r$ differently if $\left|\frac{y}{x}\right| > 1$ were true. In this case, you would have had $\left|\frac{x}{y}\right| < 1$, and

$$(x + y)^r = y^r\left(1+\frac{x}{y}\right)^r = y^r + r\,y^{r-1}\,x + ...,$$

Consequently, we have the following result:

For a rational number r, an expression like $(ax + by)^r$ can be expanded in two different ways, depending on whether

$$\left|\frac{b\,y}{a\,x}\right| < 1 \ \text{ or } \ \left|\frac{ax}{b\,y}\right| < 1$$

Example 8.21: Expand $(x + y)^{-5}$ then (i) $\left|\frac{y}{x}\right| < 1$ and (ii) $\left|\frac{x}{y}\right| < 1$.

Solution: (i) Since $\left|\frac{y}{x}\right| < 1$, using (E) we have

$$(x+y)^{-5} = x^{-5} + (-5)x^{-5-1}y + \frac{(-5)(-6)}{2!} + x^{-5-2}y^2$$

$$+ \frac{(-5)(-6)(-7)}{3\,!}\ x^{-5-3}y^3 + ...$$

$$= \frac{1}{x^5} - \frac{5y}{x^6} + \frac{15y^2}{x^7} - \frac{35y^3}{x^8} + ...$$

(ii) Since $\left|\frac{x}{y}\right| < 1$ we have to write $(x + y)^{-5}$ in the form $(y + x)^{-5}$.

Using (E), we can write

$$(y + x)^{-5} = y^{-5} + (-5)\, y^{-5-1}\, x$$

$$+ \frac{(-5)\,(-6)}{2!}\, y^{-5-2}x^2 + \frac{(-5)(-6)(-7)}{3!}\, y^{-5-3}x^3 + ...$$

$$= \frac{1}{y^5} - \frac{5x}{y^6} + \frac{15x^2}{y^7} - \frac{35x^3}{y^8} + ...$$

Note that in (i), we have obtained the expansion in ascending powers of y while in (ii), we have obtained the expansion in ascending powers of x.

Binomial Theorem for Rational Index

In the previous section, we have proved that

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_rx^r + ... + {}^nC_nx^n$$

where n is a positive integer and x is any real number. We can write this also in the form.

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{2!}x^2 + ... + \frac{n(n-1)...(n-r+1)}{r!}x^2 + ... \quad ...(i)$$

The RHS of the above equality terminantes automatically after $(n + 1)$ terms, where n is a positive integer since we come across the factor $(n - n)$ from the $(n + 2)^{nd}$ term onwards. But in case n is a negative integer or, more generally, a vational number $\left(\frac{p}{q}\right)$ which is not a positive integer, then (1) contains infinite number of terms in RHS. Still it becomes valid provided we stipulate the condition $|x| < 1$. (The proof of this fact is beyond the scope of this book).

Now, we state without proof, the binomial theorem for rational index.

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Theorem (Binomial Theorem for Rational Index)

If m is a rational number (but not a positive integer) and x is a real number such that $|x| < 1$ (that is, $-1 < x < 1$), then

$$(1+x)^m = 1+\frac{m}{1!}x+\frac{m(m-1)}{2!}x^2+\ldots+\frac{m(m-1)..(m-r+1)}{r!}x^r+\ldots$$

$$= 1+\sum_{r=1}^{\infty}\frac{m(m-1)\ldots(m-r+1)}{r!}x^r$$

Now, we discuss some special cases.

Negative Integral Index

Let m be a negative integer, say $m = -n$ (n is a positive integer) and $|x| < 1$. Then

1. $$(1+x)^{-n} = 1+\frac{(-n)}{1!}x+\frac{(-n)(-n-1)}{2!}x^2+\ldots$$

$$.+\frac{(-n)(-n-1)\ldots(-n-r+1)}{r!}x^r+..$$

$$=1-\frac{n}{1!}x+\frac{n(n+1)}{2!}x^2\ldots.+(-1)^r\frac{n(n+1)\ldots(n+r+1)}{r!}x^r+..$$

$$\therefore\ (1+x)^{-n} = \sum_{r=0}^{\infty}(-1)^r\ {}^{n+r-1}C_r.x^r$$

2. On replacing x by $-x$ in the above, we get

$$(1-x)^{-n} =1-\frac{n}{1!}(-x)+\frac{n(n+1)}{2!}(-x)^2-\frac{n(n+1)(n+2)}{3!}(-x)^3+\ldots$$

$$+(-1)^r.\frac{n(n+1)\ldots.(n+r-1)}{r!}(-x)^r+\ldots$$

$$= 1+\frac{n}{1!}x+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\ldots.$$

$$+\frac{n(n+1)\ldots(n+r-1)}{r!}x^r+\ldots$$

$$\therefore\ (1-x)^{-n} = \sum_{r=0}^{\infty}{}^{(n+r-1)}C_r\ .\ x^r$$

Rational Index

Let m be a positive rational number, say $m = \frac{p}{q}$ where q, p are positive integers ($q \neq 1$) and x is a real number such that $|x| < 1$

3. $$(1+x)^{p/q} = 1 + \frac{\frac{p}{q}}{1!}x + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{2!}x^2 + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)}{3!}x^3 + \ldots$$

$$+\frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\ldots\left(\frac{p}{q}-r+1\right)}{r!}x^r + \ldots$$

$$= 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p-q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p-q)(p-2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots$$

$$+\frac{p(p-q)\ldots(p-(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \ldots$$

Similarly, we can derive that

4. $$(1-x)^{p/q} = 1 - \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p-q)}{2!}\left(\frac{x}{q}\right)^2 - \frac{p(p-q)(p-2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots$$

$$+(-1)^r\,\frac{p(p-q)\ldots.(p-(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \ldots$$

5. $$(1+x)^{-\frac{p}{q}} = 1 - \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 - \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots$$

$$+\ (-1)^r.\ \frac{p(p+q)\ldots\ (p+(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \ldots$$

6. $$(1-x)^{-\frac{p}{q}} = 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \ldots$$

$$+\ \frac{p(p+q)\ldots\ (p+(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \ldots$$

Problems

1. Find the sum of the infinite series

$$1+\frac{2}{3}.\frac{1}{2}+\frac{2.5}{3.6}\left(\frac{1}{2}\right)^2+\frac{2.5.8}{3.6.7}\left(\frac{1}{2}\right)^3+....\infty$$

Solution:

The given series can be written as

$$S = 1+\frac{2}{1}.\frac{1}{6}+\frac{2.5}{1.2}\left(\frac{1}{2}\right)^2+\frac{2.5.8}{1.2.3}\left(\frac{1}{6}\right)^3+....\infty$$

The series is of the form

$$1+\frac{p}{1!}\frac{x}{q}+\frac{p(p+q)}{2!}\left(\frac{x}{2}\right)^2+\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3+....\infty$$

where $p = 2, p + q = 5$ and $\frac{x}{q}=\frac{1}{6}$ or $p = 2, q = 3, x = \frac{1}{2}$.

Hence, by the binomial theorem for rational index.

$$S = (1-x)^{-\frac{p}{q}} = \left(1-\frac{1}{2}\right)^{-\frac{2}{3}} = 2^{2/3} = \sqrt[3]{4}$$

2. Find the sum of the series

$$\frac{3.5}{5.10}+\frac{3.5.7}{5.10.15}+\frac{3.5.7.9}{5.10.15.20}+....\infty$$

Solution:

Write

$$S = \frac{3.5}{5.10}+\frac{3.5.7}{5.10.15}+\frac{3.5.7.9}{5.10.15.20}+....\infty$$

On adding $1+\frac{3}{5}$ both sides, we get

$$1+\frac{3}{5}+S = 1+\frac{3}{5}+\frac{3.5}{5.10}+\frac{3.5.7}{5.10.15}+....\infty$$

$$= 1+\frac{3}{1!}\left(\frac{1}{5}\right)+\frac{3.5}{2!}\left(\frac{1}{5}\right)^2+\frac{3.5.7}{3!}\left(\frac{1}{5}\right)^3+....\infty$$

$$= 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3+....\infty$$

(where $p = 3$, $p + q = 5$ and $\frac{x}{q}=\frac{1}{5}$)

$(1-x)^{-\frac{p}{q}}$ where $p = 3$, $q = 2$ and $x=\frac{2}{5}$

$$=\left(1-\frac{2}{5}\right)^{-\frac{3}{2}}=\left(\frac{3}{5}\right)^{-\frac{3}{2}}=\left(\frac{5}{3}\right)^{\frac{3}{2}}$$

$$\Rightarrow S=\frac{5\sqrt{5}}{3\sqrt{3}}-\frac{8}{5}.$$

3. If $x=\frac{1}{5}+\frac{1.3}{5.10}+\frac{1.3.5}{5.10.15}+...\infty$, then find $3x^2 + 6x$.

Solution:

$$x=\frac{1}{5}+\frac{1.3}{5.10}+\frac{1.3.5}{5.10.15}+...\infty,$$

$$\Rightarrow 1+x=1+\frac{1}{5}+\frac{1.3}{5.10}+\frac{1.3.5}{5.10.15}+...\infty$$

$$= 1+\frac{1}{1!}\left(\frac{1}{5}\right)+\frac{1.3}{2!}\left(\frac{1}{5}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{5}\right)^3+....\infty$$

$$= 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{3!}\left(\frac{3}{q}\right)^3+....\infty$$

where $p = 1$, $p + q = 3$ and $\frac{x}{q}=\frac{1}{5}$

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$$(1-x)^{-\frac{p}{q}} \text{ where } p = 1,\ q = 2 \text{ and } x = \frac{2}{5}$$

$$= \left(1-\frac{2}{5}\right)^{-\frac{1}{2}} = \left(\frac{3}{5}\right)^{-\frac{1}{2}} = \sqrt{\frac{5}{3}}$$

$$\Rightarrow (1+x)^2 = \frac{5}{3} \Rightarrow x^2+2x+1=\frac{5}{3} \Rightarrow 3x^2+6x+3=5$$

$$\therefore\ 3x^2 + 6x = 2.$$

EXERCISE 8.6

1. Expand each of the following:

   (a) $(1 - p)^{-3}$, $|\, p \,| < 1$ (b) $(1 + 3x)^{4/3}$, $|\, x \,| < \frac{1}{3}$

   (c) $(1 - 5z)^{6/5}$, $|\, z \,| < \frac{1}{5}$

2. Expand each of the following:

   (a) $(27-6x)^{\frac{-2}{3}}$, $\left|\frac{2x}{9}\right| < 1$ (b) $(2a+x)^{-3}$, $\left|\frac{x}{2a}\right| < 1$

   (c) $(2+3y)^{1/7}$, $|\,y\,| < \frac{2}{3}$

3. (a) State the condition under which the expansion of $(x + 2y)^{-5}$ will be valid in

   (i) ascending powers of x

   (ii) ascending powers of y.

   Also, write down the expansion in each case.

   (b) Expand, $(3 + 6y)^{-4/3}$stating the range of values of y for which the expansion is valid .

8.6 USE OF BINOMIAL THEOREM IN APPROXIMATIONS

As you have seen, the binomial expansions sometime have infmitely many terms. In such cases, for further calculations; an approximate value involving only the first few terms may be enough for us. Let us illustrate some situations in which we fInd the approximate values.

Example 8.22 : Find the cube root of 1.03 up to three decimal places.

Solution : We want to find $(1.03)^{1/3}$ up to three decimal places.

Now $\quad (1.03)^{1/3} = (1 + 0.03)^{1/3}$

Since $|0.03| < 1$, from (E), we have

$$(1+0.03)^{1/3} = 1 + \frac{1}{3}(0.03) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2\,!}(0.03)^2 + ... \qquad \text{(i)}$$

Now, we need to approximate the value up to three decimal places. Since a non-zero digit in the fourth decimal place may affect the digit in the third place in the process of rounding off, we need to consider those terms in the expansion which produce a non-zero digit in the first, second, third or fourth decimal place.

Therefore, we can take the sum of the fIrst three terms in the Expansion *(i),* and ignore the rest.

$$\therefore \qquad (1.03)^{1/3} \approx 1 + 0.01 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2\,!}(0.0009)$$

$$= 1 + 0.01 - 0.0001$$

$$= 1.0099$$

$$\approx 1.010 \text{ taking the value up to three decimal places.}$$

Now, the digit after the third decimal place is greater than 5, so we have increased the third decimal place by 1.

Thus, the cube root of 1.03, up to three decimal places, is 1.010.

Example 8.23: Assuming y to be so small that y^2 and higher powers of y can be neglected, find the value of $(1-2y)^{\frac{2}{3}}(4+5y)^{\frac{-3}{2}}$.

Solution: Note that y is very small. So, we can assume that $|y|<\frac{1}{2}$ Then, using the binomial theorem, we get

$$(1-2y)^{\frac{2}{3}}=1+\frac{2}{3}(-2y)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2\,!}(-2y)^2+.....$$

and

$$(4+5y)^{\frac{-3}{2}}=4^{\frac{-3}{2}}+\left(-\frac{3}{2}\right)(4)^{\frac{-3}{2}-1}(5y)$$

$$+\frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2\,!}(4)^{\frac{-3}{2}-2}.(5y)^2+....$$

Since we can neglect terms containing y^2 and higher powers of y, we have

$$(1-2y)^{\frac{2}{3}}\approx 1+\frac{2}{3}(-2y)=1-\frac{4}{y}y,$$

$$(4+5y)^{\frac{-3}{2}}\approx(4)^{\frac{-3}{2}}-\frac{3}{2}(4)^{\frac{-5}{2}}(5y)$$

$$=\frac{1}{8}-\frac{15}{64}y$$

Thus, the given product is approximately

$$\left(1-\frac{4}{3}y\right)\left(\frac{1}{8}-\frac{15}{64}\right)=\frac{1}{8}-\frac{1}{6}y-\frac{15}{64}y+\frac{5}{16}y^2\ ....$$

$$\approx\frac{1}{8}-\frac{77}{192}y,$$ again neglecting the term containing y^2.

So, $(1-2y)^{\frac{2}{3}}(4+5y)^{\frac{-3}{2}}$ is $\frac{1}{8}-\frac{77}{192}y$, if we neglect the terms involving y^2 and higher powers of y.

EXERCISE 8.7

1. Find the value of each of the following up to three decimal places:

   (a) $(1.02)^2$ (b) $(1.01)^{-3}$ (c) $(0.97)^4$

   (d) $\sqrt[4]{7.60}$ [Hint : $(7.60)^{1/3} = (8-0.4)^{1/3}$]

   (e) $\sqrt[4]{82}$ [Hint : $(82)^{\frac{1}{4}} = (81+1)^{\frac{1}{4}} = 3\left(1+\frac{1}{81}\right)^{\frac{1}{4}}$]

   (f) $(24)^{\frac{-1}{2}}$ [Hint : $(24)^{\frac{-1}{2}} = (25-1)^{\frac{-1}{2}}$]

2. Assuming z to be so small that z^2 and higher powers of z can be neglected, find the value of

   (a) $(3 + 2z)^{-5}$ (b) $(1 + 3z)^{\frac{2}{3}}(1-5z)^{-2}$

   (c) $\dfrac{\sqrt{1+z}+(1-z)^{2/3}}{(1+z)+\sqrt{1+z}}$ [Hint : LHS $\dfrac{1+\frac{1}{2}z+1-\frac{2}{3}z}{(1+z)+(1+\frac{1}{2}z)}$]

   (d) $\dfrac{(1-z)^{\frac{1}{3}}+(11-5z)^2}{\sqrt[4]{16-z}}$

8.7 PARTIAL FRACTIONS

LEARNING OUTCOMES

After studying this lesson you will be able to :

- To split a fraction in to proper "partial fractions ".
- The partial fraction decomposition is useful in finding the particular integrals of differential equations.

- The partial fraction decomposition is useful in expanding infinite series.

We split a fraction into proper "partial fractions". This splitting is based on the roots of the denominator of a fraction under consideration. This denominator, being a polynomial with real coefficients, can be written as a finite product of linear and/or irreducible quadratic factors.

8.7.1 Definition (proper and improper fractions)

a rational fraction $\frac{f(x)}{g(x)}$ is called a proper fraction if the degree of $f(x)$ is less than the degree of $g(x)$ otherwise it is called an improper fraction.

Definition : (Partial Fraction)

If a proper fraction is expressed as the sum of two or more proper fractions, where in the denominators are powers of irreducible polynomials, then each such proper fraction is called a partial freaction of the given fraction.

8.8 PARTIAL FRACTIONS OF $\frac{f(x)}{g(x)}$, WHEN $g(x)$ CONTAINS NON- REPEATED LINEAR FACTORS

8.8.1 Rule I

Let $\frac{f(x)}{g(x)}$ be a proper freaction. If $(ax+b)$ $a \neq 0$ is a non-repeated linear factor of $g(x)$. Then there will be a partial freaction of the form $\frac{A}{ax+b}$ corresponding to the factor $ax + b$ where A is a non-zero real number to be determined.

Example 1

Resolve $\frac{3x}{(x+3)\,(x-6)}$ in to partial fractions

Solution :

Let $\frac{3x}{(x+3)\,(x-6)} = \frac{A}{x+3} + \frac{B}{x-6}$

$\therefore \quad \frac{3x}{(x+3)\,(x-6)} = \frac{A(x-6) + B\,(x+3)}{(x+3)\,(x-6)}$

$\therefore \quad 3x = A(x - 6) + B(x + 3) \qquad .....(1)$

put $x = 6$ in (1)

$3(6) = B\,(6 + 3) \Rightarrow B = \frac{18}{9} = 2$

put $x = -3$ in (1)

$3(-3) = A(-3 - 6)$

$A = \frac{-9}{-9} = 1$

$\therefore \quad \frac{3x}{(x+3)\,(x-6)} = \frac{1}{x+3} + \frac{2}{x-6}$

Example 2

$\frac{2x-1}{(2x+3)\,(x-1)}$

Solution : Let $\frac{2x-1}{(2x+3)\,(x-1)} = \frac{A}{2x+3} + \frac{B}{x-1}$

$\Rightarrow \frac{2x-1}{(2x+3)\,(x-1)} = \frac{A(x-1) + B(2x+3)}{(2x+3)\,(x-1)}$

$\Rightarrow 2x - 1 = A\,(x - 1) + B\,(2x + 3) \qquad ...(1)$

put $x = 1$ in (1)

$2(1) - 1 = A(0) + B(2 + 3)$

$\Rightarrow 5B = 1 \Rightarrow B = \frac{1}{5}$

put $x = \frac{-3}{2}$ in (1)

$2\left(\frac{-3}{2}\right) - 1 = A\left(\frac{-3}{2} - 1\right) + B(0)$

$$\Rightarrow -4 = A\left(\frac{-5}{2}\right) \Rightarrow A = \frac{8}{5}$$

$$\therefore \quad \frac{2x-1}{(2x+3)\,(x-1)} = \frac{8}{5(2x+3)} + \frac{1}{5(x-1)}$$

8.8.2 Rule 2

Let $\frac{f(x)}{g(x)}$ be a proper fractions. If n is the largest index ($n > 1$) such that $(ax + b)^n$, $a \neq 0$, is a factor of $g(x)$, i.e., $(ax + b)$ is a repeated linear factor of $g(x)$, then there will be n terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_n}{(ax+b)^n}$$

in the partial fraction expansion of $\frac{f(x)}{g(x)}$ of, where $\mathbf{A}_1, A_2, ... A_n$ are real numbers, to be determined.

Example 3

Resolve $\frac{x-1}{(x+1)\,(x+2)^2}$ into partial fractions

Solution : Let $\frac{x-1}{(x+1)\,(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$$\frac{x-1}{(x+1)\,(x+2)^2} = \frac{A(x+2)^2 + B(x+1)(x+2) + C(x+1)}{(x+1)(x+2)^2}$$

$$\Rightarrow x - 1 = A(x + 2)^2 + B(x + 1)(x + 2) + C(x + 1) \quad .... (1)$$

put $x = -2$ in (1)

$-2 - 1 = A(0) + B(0) + C(-2 + 1)$

$\Rightarrow -C = -3 \Rightarrow C = 3$

put $x = -1$ in (1)

$-1 - 1 = A(-1 + 2)^2 + B(0) + C(0)$

$\Rightarrow -2 = A \Rightarrow A = -2$

Now comparing the coefficiants of x^2 in (1) we get.

$$A + B = 0$$

$$\Rightarrow B = -A \Rightarrow B = -(-2) \Rightarrow B = 2$$

$$\therefore \quad \frac{x-1}{(x+1)(x+2)^2} = \frac{2}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^3}$$

Example 4

Resolve $\frac{1}{x^3(2+x)}$ into partial freactions.

Solution : $\frac{1}{x^3(2+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{2+x}$

$$\frac{1}{x^3(2+x)} = \frac{A(x^2)(2+x) + Bx(2+x) + C(2+x) + Dx^3}{x^3(2+x)}$$

$$\therefore \quad Ax^2(2+x) + Bx(2+x) + C(2+x) + Dx^3 = 1 \quad \text{... (1)}$$

put $x = 0$ in (1)

$$A(0) + B(0) + C(2) + D(0) = 1 \Rightarrow C = \frac{1}{2}$$

put $x = -2$ in (1)

$$A(0) + B(0) + C(0) + D(-2)3 = 1 \Rightarrow D = \frac{-1}{8}$$

Now comparing the coefficiants of x^3 in (1)

$$A + D = 0 \Rightarrow A = -D = \frac{1}{8}$$

Now comparing the coefficiants of x^2 in (1)

$$2A + B = 0 \Rightarrow B = -2A = -2\left(\frac{1}{8}\right) = \frac{-1}{4}$$

$$\therefore \quad \frac{1}{x^3(x+2)} = \frac{1}{8x} - \frac{1}{4x^2} + \frac{1}{2x^3} - \frac{1}{8(x+2)}$$

Example 5

Resolve $\frac{x^2+5x+7}{(x-3)^3}$ into partial fractions.

Sol : Let $\frac{x^2+5x+7}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$

$$\therefore \quad \frac{x^2+5x+7}{(x-3)^3} = \frac{A(x-3)^2 + B(x-3) + C}{(x-3)^3}$$

$$\therefore \quad x^2 + 5x + 7 = Ax^2 + (B - 6A)\,x + (9A - 3B + C) \quad ....(1)$$

Now comparing the coefficiants in (1) we get

$$A = 1,\ B - 6A = 5,\ 9A - 3B + C = 7$$

Solving these equations we get

$$A = 1,\ B = 11,\ C = 31$$

$$\therefore \quad \frac{x^2+5x+7}{(x-3)^3} = \frac{1}{x-3} + \frac{11}{(x-3)^3} + \frac{31}{(x-3)^3}$$

Note: The above problem can also be resolved into partial fractions as follows.

Let $x - 3 = y$ then $x = y + 3$

$$\therefore \quad \frac{x^2+5x+7}{(x-3)^3} = \frac{(y+3)^2 + 5(y+3)+7}{y^3}$$

$$= \frac{(y^2 + 11y + 31}{y^3}$$

$$= \frac{1}{y} + \frac{11}{y^2} + \frac{31}{y^3}$$

$$= \frac{1}{x-3} + \frac{11}{(x-3)^2} + \frac{31}{(x-3)^3}$$

EXERCISE 8.8

Resolve the following fractions into partial fractions.

1. $\dfrac{5x+6}{(2+x)(1-x)}$

2. $\dfrac{x+4}{(x^2-4)(x+1)}$

3. $\dfrac{2x+3}{(x-1)^3}$

4. $\dfrac{x^2-x+1}{(x+1)(x-1)^2}$

5. $\dfrac{1}{x^3(x+a)}$

8.8.3 Rule III

Let $\dfrac{f(x)}{g(x)}$ be a proper fraction. It $ax^2 + bx + c,\ a \neq 0$, is a non-repeated irreducible factor of $g(x)$ then corresponding to this factor there will be a partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$ in the expansion of $\dfrac{f(x)}{g(x)}$, where A and B are real numbers, to be determined.

Example 6

Resolve $\dfrac{5x^2+2}{x^3+x}$ into partial fractions.

Solution : Let $\dfrac{5x^2+2}{x^3+x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

Then $\dfrac{5x^2+2}{x(x^2+1)} = \dfrac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$

$\Rightarrow 5x^2 + 2 + A(x^2+1) + (Bx + C)\,x$ .... (1)

put $x = 0$ in (1)

$5(0) + 2 = A \Rightarrow A = 2$

Comparing the coefficiant of x^2 in (1)

$5 = A + B$

$\Rightarrow 5 = 2 + B \Rightarrow B = 3$

Comparing the coefficiant of x in (1)

$0 = C$

$$\therefore \frac{5x^2+2}{x^3+x} = \frac{2}{x} + \frac{3x}{x^2+1}$$

Example 7

Resolve $\dfrac{3x-1}{(1-x+x^2)(x+2)}$ into partial fractions.

Solution: Let $\dfrac{3x-1}{(1-x+x^2)(x+2)} = \dfrac{A}{2+x} + \dfrac{Bx+C}{1-x+x^2}$

$$\frac{3x-1}{(1-x+x^2)(x+2)} = \frac{A(1-x+x^2) + (Bx+C)(2+x)}{(1-x+x^2)(x+2)}$$

$\Rightarrow 3x - 1 = A(1-x + x^2) + (Bx + C)(2 + x)$ ...(1)

put $x = -2$ in (1)

$3(-2) -1 = A(1 + 2 + 4) + [B(-2) + C] (0)$

$\Rightarrow -7 = 7A \Rightarrow A = -1$

comparing the coefficiant of x^2 in (1)

$A + B = 0 \Rightarrow B = -A = -(-1) = 1$

put $x = 0$ in (1)

$-1 = A + 2C$

$\therefore \quad 2C = -1 - A = -1 - (-1) = 0$

$\Rightarrow C = 0$

$$\therefore \quad \frac{3x-1}{(1-x+x^2)\,(2+x)} = \frac{-1}{2+x} + \frac{x}{1-x+x^2}$$

EXERCISE 8.9

I. Resolve the following fractions into partial fractions

1. $\dfrac{x^2+1}{(x^2+4)\,(x-2)}$

2. $\dfrac{x^2+1}{(x^2+x+1)^2}$

3. $\dfrac{x^2-3}{(x+2)\,(x^2+1)}$

Partial Fractions

- Resolve $\dfrac{5x+6}{(2+x)(1-x)}$ into partial fractions

Solution: Let $\dfrac{5a+b}{(2+x)(1-x)} = \dfrac{A}{2+x} + \dfrac{B}{1-x}$

$5x + 6 - A(1 - x) + B(2 + x)$

$x = 1 \Rightarrow 5(1) + 6 = B(2 + 1)$

$11 = 3B \Rightarrow B = \dfrac{11}{3}$

$x = -2 \Rightarrow 5(-2) + 6 = A(-1 - (-2)) + 0$

$-4 = 3A \Rightarrow A = \dfrac{-4}{3}$

MODULE - I
Algebra

Notes

$$\therefore \quad \frac{5x+6}{(2+x)(1-x)} = \frac{-4/3}{2+x} + \frac{11/3}{1-x}$$

$$= \frac{-4}{3(2+x)} + \frac{11}{3(1-x)}$$

- Resolve $\frac{2x+3}{(x-1)^3}$ into partial fraction.

Solution: $\frac{2x+3}{(x-1)^3}$

Let $x - 1 = y \Rightarrow x = y + 1$

$$= \frac{2(y+1)+3}{y^3} = \frac{2y+5}{y^3}$$

$$= \frac{2}{y^2} + \frac{5}{y^3}$$

$$= \frac{2}{(n-1)^2} + \frac{5}{(n-1)^3}$$

$$= (x-1)^2\ (x-1)^3.$$

KEY WORDS

- The statement of the principle of mathematical induction namely.

  P(n), a statement involving a natural number *n,* is true for all $n \geq 1$, where *n* is a fixed natural number, if

  (i) P (1) is true, and

  (ii) Whenever P(k) is true, then P(k + 1) is true for $k \in$ N.

- For a natural number *n,*

  $(x + y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}y^2 +..+ {}^nC_{n-1}\, xy^{n-1} + {}^nC_n\, y^n$

  This is called the **Binomial Theorem for a positive integral (or natural) exponent.**

- Another form of the Binomical Theorem for a positive integral exponent is $(1+a)^n = {}^nC_0 + {}^nC_1\, a + {}^nC_2\, a^2 + ... + {}^nC_{n-1}\, a^{n-1} + {}^nC_n\, a^n$.
- The general term in the expansion of $(x+y)^n$ is ${}^nC_r\, x^{n-r}\, y^r$ and in the expansion of $(1+a)^n$ is ${}^nC_r\, a^r$, where n is a natural number and $0 \le r \le n$.
- If n is an even natural number, there is only one middle term in the expansion of $(x+y)^n$. If n is odd, there are two middle trems in the expansion.
- The formula for the general term can be used for finding the middle term(s) and some other specific terms in an expansion.
- The statement

$$(1+x)^n = 1 + rx + \frac{r(r-1)}{2\,!}\, x + \frac{r(r-1)(r-2)}{3!}\, x^3 + ...$$

where, r is a rational number and $|x| < 1$ is called the **Binomial Theorm for a rational exponent.** In this expansion, the number of terms is infinite if r is not a whole number.

- $$(x+y)^n = rx + r\, x^{r-1}\, y + \frac{r(r-1)}{2\,!}\, x^{r-2}\, y^2 + \frac{r(r-1)(r-2)}{3!}\, x^{r-3}\, y^3 + ...$$

where r is a rational number and $\left|\frac{y}{x}\right| < 1$ is another form of the Binomial Theorem for a rational exponent.

- Expressions like $(ax+by)^n$, where r is a rational number, can be expanded in two different ways, depending on whether $\left|\frac{by}{ax}\right| < 1$ or $\left|\frac{ax}{by}\right| < 1$

The various forms of the partial fractions $\frac{f(x)}{g(x)}$ corresponding to the factors of $g(x)$ are given in the following table.

Sl.No.	Factor of $g(x)$	The form of the partial fractions of $\frac{f(x)}{g(x)}$
1.	$ax + b$	$\frac{A}{ax+b}$
2.	$(ax + b)^n$, $a \neq 0$ $n(>1) \in N$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$
3.	Irreducible $a x^2 + bx + c$, $a \neq 0$	$\frac{A_x + B}{ax^2+bx+c}$
4.	$(ax^2 + bx + c)^n$, $a \neq 0$	$\frac{A_1 x + B_1}{ax^2+bx+c} + \frac{A_2 x + B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_n x + B_n}{(ax^2+bx+c)^n}$

SUPPORTIVE WEBSITES

- http:// www.wikipedia.org
- http:// mathworld.wolfram.com.

PRACTICE EXERCISE

1. Verify each of the following statements, using the principle of mathematical induction:

   (a) The number of subsets of a set with n elements is 2^n.

   (b) $(a + b)^n > a^n + b^n$, $\forall\ n \geq 2$ where a and b are positive real numbers

   (c) $a + ar + ar^2 + \ldots + a\, r^{n-1} = \frac{a(r^n - 1)}{r - 1}$ where $r > 1$ and a is a real number.

(d) $(x^{2n}-1)$is divisible by $(x + 1)$ $\forall\ x \in$ N

(e) $(10^{2n-1} + 1)$ is a multiple of 11.

[Hint : $10^{2k+1} = 10^2\ [(10^{2k-1} + 1) - 99]$

(f) $(4.10^{2n} + 9.10^{2n-1} + 5)$, is a multiple of 99.

(g) $n(n^2 - 1)$, is a multiple of 24, when n is odd.

[Since n is odd, assume that P($2k$+1) is true, as ($2k$ + 3) is always odd. Then try to prove that P($2k$+3) is true.]

(h) $(1 + x)^n > 1 + nx$ where $x > 0$.

(i) If f and g are polynomials in x with real coefficients and $f + g \neq 0$, then $(f + g)$ divides $(f^{\,2n-1} + g^{2n-1}) \forall\ n \in$ N.

2. Write the expansion of each of the following:

(a) $(3x + 2y)^5$ (b) $(P - 2)^8$ (c) $(1 - x)^8$

(d) $\left(1+\frac{2}{3}x\right)^6$ (e) $\left(x+\frac{1}{2x}x\right)^6$ (f) $(3x - y^2)$

(g) $\left(\frac{x^2}{4}+\frac{2}{x}\right)^4$ (h) $\left(x^2-\frac{1}{x^3}\right)^7$ (i) $\left(x^3+\frac{1}{x^2}\right)^5$

(j) $\left(\frac{1}{x^2}-x^3\right)^4$

3. Write the $(r + 1)$)th term in the expansion of each of the following, where $n \in$ N :

(a) $(3x - y^2)^n$ (b) $\left(x^3+\frac{1}{x}\right)^n$

4. Find the specified terms in the expansion of each of the following:

(a) $(1-2x)^7$; 3rd term [Hint: Here $r = 2$]

(b) $\left(x+\frac{1}{2x}\right)^6$; middle term(s)

(c) $(3x - 4y)^6$; 4 th term (d) $\left(y^2 - \frac{1}{y}\right)^{11}$; middle term(s)

(e) $(x^3 - y^3)^{12}$; 4 th term (f) $(1 - 3x^2)^{10}$; middle term(s)

(g) $(-3x - 4y)^6$; 5 th term

(h) Write the rth term in the expansion of $(x - 2y)^6$.

(i) Write the $(r - 1)^{\text{th}}$ term in the expansion of $(1 + 2x)^8$.

5. If T_r, denotes the rth term in the expansion of $(1 + x)^n$ in ascending powers of x (n being a natural number) , prove that

$r(r + 1)\, T_{r+2} = (n - r + 1)(n - r)\, x^2\, T_r$

[Hint : $T_r = {}^nC_{r-1} \,.\, x^{r-1}$ and $T_{r+2} = {}^nC_{r+1}\, x^{r+1}$]

6. k_r is the coefficient of x^{r-1} in the expansion of $(1 + 2x)^{10}$ in ascending powers of x and $k_{r+2} = 4k_r$. Find the value of r.

[**Hint :** $k_r = {}^{10}C_{r-1}\, 2^{r-1}$ and $k_{r+2} = {}^{10}C_{r+1}\, 2^{r+1}$]

7. The coefficients of the 5th, 6th and 7th terms in the expansion of $(1 + a)^n$ (n being a natural number) are inA.P. Find n.
[**Hint :** ${}^nC_5 - {}^nC_4 = {}^nC_6 - {}^nC_5$]

8. Expand $(1 + y + y^2)^4$ [**Hint :** $(1 + y + y^2)^4 = \{(1 + y) + y^2\}^4$

9. Write the expansion of each of the following:

(a) $(1 - x)^{-4}$, $|x| < 1$ (b) $\frac{1}{(1+x)^3}$, $|x| < 1$

(c) $(3 - z)^{-4}$, $|z| < 3$ (d) $\frac{1}{(1+3x)^{3/2}}$, $|x| < \frac{1}{3}$

10. State the condition under which the expansion $(x - 2y)^{-3}$ will be valid in ascending powers of y. Also write the expansion.

11. State the condition under which the expansion of $(x - 3y)^{-1/2}$ will be valid in ascending powers of y. Also write the expansion.

12. Expand the following, stating the condition of y under which the expansion will be valid:

(a) $\frac{1}{(2+y)^4}$ (b) $(3 - y)^{-2/3}$

13. Find the value of each of the following up to three decimal places, using the necessary number of terms in the expansion:

(a) $(0.99)^{-4}$ (b) $(1.03)^{-3}$

(c) $\sqrt[3]{26}$ [Hint : $(26)^{\frac{1}{3}} = (27-1)^{\frac{1}{3}}$]

(d) $\sqrt[7]{13}$ [Hint : $(127)^{\frac{1}{7}} = (128-1)^{\frac{1}{7}}$]

(e) $\sqrt[5]{35}$ [Hint : $(35)^{\frac{1}{5}} = \{32+3)^{\frac{1}{5}}$]

(f) $\sqrt[5]{31}$ [Hint : $(31)^{\frac{1}{5}} = \{32-1)^{\frac{1}{5}}$]

(g) $\sqrt[3]{1001}$ [Hint : $(1001)^{\frac{1}{3}} = (1000+1)^{\frac{1}{3}}$]

14. Assumingy to be so small that y^2 and higher powers of y can be neglected, find the value of each of the following:

(a) $(1+5y)^{-2}(1+2y)^{\frac{-3}{2}}$ (b) $\frac{(1-4y)^{-3}(1-2y^2)^{\frac{1}{2}}}{(4-y)^{3/2}}$

(c) $\frac{\sqrt{1-3y}+(1-y)^{5/3}}{\sqrt{4-y}}$

15. A student puts $n = 0$ in $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + ... + x^n$ and obtains $(1 + x)^0 = 1 + 0 + 0 + ... + x^0$. i.e $1 = 1 + 1$ Can you detect the error in this solution?

16. Assumingthattheexpansions are possible, fmd the coefficientof y^3 in $(1 - 4y)^2 (1 - 2y)^{1/2}$.

17. Prove that

$(1 + x + x^2 + x^3 +...)(1 - x + x^2 - x^3 + ...) = 1 + x^2 + x^4 + x^6 + ...$

[Hint : LHS = $(1 - x)^{-1}(1 + x)^{-1} = (1 - x^2)^{-1}$].

ANSWERS

EXERCISE 8.1

1. (b), (e) and (f) are statements; (a) is not, since we have not given the range of values of n, and therefore we are not in a position to decide, if it is true or not. (c) is subjective and hence not a mathematical statement. (d) is a question, not a statement.

   Note that (f)is universally false.

2. P(1) : 6 is a factor of $1^3 + 5.1$

   P(2) : 6 is a factor of $2^3 + 5.2$

   P(k) : 6 is a factor of $k^3 + 5k$

   P(k + 1) : 6 is a factor of $(k + 1)^3 + 5(k + 1)$

3. (a) P(1) : $2 \geq 2$

   P(k) : $2^k \geq k + 1$

   P(k + 1) : $2^{k+1} \geq k + 2$

   (b) P(1) : $1 + x \geq 1 + x$

   P(k) : $(1 + x)^k \geq 1 + kx$

   P(k + 1) : $(1 + x)^{k+1} \geq 1 + (k + 1)x$

   (c) P(1) : 6 is divisible by 6.

   P(k) : $k(k + 1)(k + 2)$ is divisible by 6.

   P(k + 1): $(k + 1)(k + 2)(k + 3)$ is divisible by 6

   (d) P(1) : $(x - y)$ is divisible by $(x - y)$

   P(k) : $(x^k - y^k)$ is divisible by $(x - y)$

   P(k + 1): $(x^{k+1} - y^{k+1})$ is divisible by $(x - y)$

   (e) P(1) : $ab = ab$

   $P(k)$: $(ab)^k = a^k b^k$

   P(k + 1) : $(ab)^{k+1} = a^{k+1} . b^{k+1}$

(f) P(1) : $\frac{1}{5}+\frac{1}{3}+\frac{7}{15}$ is a natural number.

P(k) : $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a natural number.

P(k + 1): $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$ is a natural number.

4. (a) P(1) : $\frac{1}{1\times 2}=\frac{1}{2}$

P(2) : $\frac{1}{1\times 2}+\frac{1}{2\times 3}=\frac{2}{3}$

P(k) : $\frac{1}{1\times 2}+\frac{1}{2\times 3}+....+\frac{1}{k(k+1)}=\frac{k}{k+1}$

P(k+1) : $\frac{1}{1\times 2}+....+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$

(b) P(1) : $1 = 1^2$

P(2) : $1 + 3 = 2^2$

P(k) : $1 + 3 + 5 + ... + (2k - 1) = k^2$

P(k + 1) : $1 + 3 + 5 +.... + (2k - 1) + [2(k + 1)-1] = (k + 1)^2$

(c) P(1) : $1 \times 2 < 1(2)^2$

P(2) : $(1\times 2) + (2\times 3) < 2(3)^2$

P(k) : $(1\times 2) + (2\times 3) + ... + k(k + 1) < k(k + 1)^2$

P(k + 1) : $(1\times 2) + (2\times 3) +....+ (k + 1)(k + 2) < (k + 1)(k + 2)^2$

(d) P(1) : $\frac{1}{1\times 3}=\frac{1}{3}$

P(2) : $\frac{1}{1\times 3}+\frac{1}{3\times 5}=\frac{2}{5}$

$$P(k):\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+.....+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

$$P(k+1):\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+.....+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

EXERCISE 8.3

1. (a) $8a^3 + 12a^2b + 6ab^2 + b^3$

   (b) $x^{12} - 18x^{10}y + 135x^8y^2 - 540x^6y^3 + 1215x^4y^4 - 1458x^2y^5 + 729y^6$

   (c) $256a^4 - 1280a^3b + 2400a^2b^2 - 2000ab^3 + 625b^4$

   (d) $a^n x^n + na^{n-1}x^{n-1}by + \frac{n(n-1)}{2!}a^{n-2}x^{n-2}b^2y^2 + ... + b^n y^n$

2. (a) $1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$

   (b) $1+\frac{7x}{y}+\frac{21x^2}{y^2}+\frac{35x^3}{y^3}+\frac{35x^4}{y^4}+\frac{21x^5}{y^5}+\frac{7x^6}{y^6}+\frac{x^7}{y^7}$

   (c) $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

3. (a) $\frac{a^5}{243}+\frac{5a^4b}{162}+\frac{5a^3b^2}{54}+\frac{5a^2b^3}{36}+\frac{5ab^4}{48}+\frac{b^5}{32}$

   (b) $2187x^7 - 25515x^4 + 127575x - \frac{354375}{x^2} + \frac{590625}{x^5} - \frac{590625}{x^8} + \frac{328125}{x^{11}} - \frac{78125}{x^{14}}$

   (c) $x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$

   (d) $\frac{x^5}{y^5} + 5\frac{x^3}{y^3} + 10\frac{x}{y} + 10\frac{y}{x} + 5\frac{y^3}{x^3} + \frac{y^5}{x^5}$

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6. (i) 104060401 (ii) 96059601

   (iii) 1.061208 (iv) 0.941192

EXERCISE 8.4

1. (a) ${}^nC_r 2^{n-r} x^{n-r} y^r$

   (b) ${}^nC_r 2^{n-r} a^{2n-2r} (-1)^r$

   (c) ${}^nC_r (-1)^r a^r$

   (d) ${}^nC_r 3^{n-r} x^{-2r}$

2. (a) $1792y^5$

   (b) $15120x^4$

   (c) $14784a^5b^6$

   (d) 20

   (e) $35x$

EXERCISE 8.5

1. (a) $8064\ x^5y^5$

   (b) $\frac{1120}{81}x^{12}$

   (c) 20

   (d) $-252x^{10}$

2. (a) $35a^4b^3,\ 35a^3b^4$

   (b) $4032a^5b^4,\ -2016a^4b^5$

   (c) $\frac{-105}{4}x^4y^3,\ \frac{140}{3}x^3y^4$

   (d) $\frac{462}{x^4},\ \frac{462}{x^7}$

EXERCISE 8.6

1. (a) $1 + 3p + 6p^2 + 10p^3 + ...$ (b) $1 + 4x + 2x^2 + ...$

   (c) $1 - 6z + 3z^2 + 4z^3 + ...$

2. (a) $\frac{1}{9}+\frac{4x}{243}+\frac{20}{6561}x^2+....$ (b) $\frac{1}{8a^3}-\frac{3x}{16a^4}+\frac{3x^2}{16a^5}-....$

   (c) $3^{\frac{1}{7}}y^{\frac{1}{7}}\left[1+\frac{2}{21y}-\frac{4}{147y^2}+....\right]$

3. (a) (i) $\left|\frac{x}{2y}\right|<1:\frac{1}{32y^5}-\frac{5x}{64y^6}+\frac{15x^2}{128y^7}-......$

   (ii) $\left|\frac{2y}{x}\right|<1:\frac{1}{x^5}-\frac{10y}{x^6}+\frac{60y^2}{x^7}-......$

   (b) $|y|<\frac{1}{2}:3^{-4/3}-8(3)^{-7/3}y+56(3)^{-10/3}y^2+......$

EXERCISE 8.7

1. (a) 1.041 (b) 0.971

   (c) 1.130 (d) 1.968

   (e) 3.009 (f) 0.204

2. (a) $\frac{1}{243}-\frac{10z}{729}$ (b) $1 + 12z$

   (c) $1-\frac{5z}{6}$ (d) $61-\frac{10409}{192}z$

EXERCISE 8.8

1. $\frac{11}{3(1-x)}-\frac{4}{3(2+x)}$

2. $\frac{1}{2(x-2)}+\frac{1}{2(x+2)}-\frac{1}{x+1}$

3. $\dfrac{2}{(x-1)^2}+\dfrac{5}{(x-1)^3}$

4. $\dfrac{3}{4(x+1)}+\dfrac{1}{4(x-1)}+\dfrac{1}{2(x-1)^2}$

5. $\dfrac{1}{a^3x}-\dfrac{1}{a^2x^2}+\dfrac{1}{ax^3}-\dfrac{1}{a^3(x+a)}$

EXERCISE 8.9

1. $\dfrac{3x+6}{8(x^2+4)}+\dfrac{5}{8(x-2)}$

2. $\dfrac{1}{x^2+x+1}-\dfrac{x}{(x^2+x+1)^2}$

3. $\dfrac{1}{5(x+2)}+\dfrac{4x-8}{5(x^2+1)}$

PRACTICE EXERCISE

2. (a) $243x^5 + 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240xy^4 + 32y^5$

(b) $p^8-8p^7q+28p^6q^2-56p^5q^3+70p^4q^4-56p^3q^5+28p^2q^6-8pq^7+q^8$

(c) $1-8x+28x^2-56x^3+70x^4-56x^5+28x^6-8x^7+x^8$

(d) $1+4x+\dfrac{20}{3}x^2+\dfrac{160}{27}x^3+\dfrac{80}{27}x^4+\dfrac{64}{81}x^5+\dfrac{64}{729}x^6$

(e) $x^6+3x^4+\dfrac{15}{4}x^2+\dfrac{5}{2}+\dfrac{15}{16x^2}+\dfrac{3}{16x^4}+\dfrac{1}{64x^6}$

(f) $243x^5-405x^4y+270x^3y^4-90x^2y^6+15xy^8-y^{10}$

(g) $\dfrac{x^8}{256}+\dfrac{x^5}{8}+\dfrac{3}{2}x^2+\dfrac{8}{x}+\dfrac{16}{x^4}$

(h) $x^{14}-7x^9+21x^4-\dfrac{35}{x}+\dfrac{35}{x^6}-\dfrac{21}{x^{11}}+\dfrac{7}{x^{16}}-\dfrac{1}{x^{21}}$

(i) $x^{14}-7x^9+21x^4-\frac{35}{x}+\frac{35}{x^6}-\frac{21}{x^{11}}+\frac{7}{x^{16}}+\frac{1}{x^{21}}$

(j) $\frac{1}{x^8}-\frac{4}{x^3}+6x^2-4x^7+x^{12}$

3. (a) $(-1)^r\,{}^nC_r 3^{n-r}\,x^{n-r}\,y^{2r}$ (b) ${}^nC_r\,x^{3n-4r}$

4. (a) $84x^2$ (b) $\frac{5}{2}$

(c) $-34560\,x^3y^3$ (d) $-462y^7, 462\,y^4$

(e) $-220x^{27}y^9$ (f) $-61236x^{10}$

(g) $34560x^2y^4$ (h) $(-2)^{r-1}\;{}^6C_{r-1}\,x^{7-r}\,y^{r-1}$

(i) $-2^{r-2}\;{}^8C_{r-2}\,x^{r-2}$

6. 5 7. 7, 14

8. $1+4y+10y^2+16y^3+19y^4+16y^5+10y^6+4y^7+y^8$

9. (a) $1+4x+10x^2+\ldots.$ (b) $1-3x+6x^2-10x^3+\ldots..$

(c) $\frac{1}{81}+\frac{4}{243}z+\frac{10}{729}z^2+\ldots..$ (d) $1-\frac{9}{2}x+\frac{135}{8}x^2-\frac{945}{16}x^3+\ldots..$

10. $\left|\frac{2y}{x}\right|<1:\frac{1}{x^3}+\frac{6y}{x^4}+\frac{24y^2}{x^5}+\ldots..$

11. $\left|\frac{x}{3y}\right|<1:\frac{1}{\sqrt{-3y}}+\frac{x}{6y\sqrt{-3y}}+\frac{x^2}{24y^2\sqrt{-3y}}+\ldots..$

12. (a) $\frac{1}{16}-\frac{y}{8}+\frac{5y^2}{32}-\frac{5y^3}{32}+\ldots|\,y\,|<2$

(b) $\frac{1}{3^{2/3}}+\frac{2x}{9\times3^{2/3}}+\frac{5x^2}{81\times3^{2/3}}+\ldots..,\;|\,y\,|<3$

Chapter

9

CARTESIAN OF SYSTEM COORDINATES

LEARNING OUTCOMES

After studying this chapter, student will be able to:

- define Cartesian System of Coordinates including the origin, coordinate axes, quadrants, etc;
- derive distance formula and section formula;
- derive the formula for area of a triangle with given vertices;
- verify the collinearityofthree given points;
- state the meaning of the terms: inclination and slope of a line;
- find the formula for the slope of a line through two given points;
- state the condition for parallelism and perpendicularity of lines with given slopes;
- find the intercepts made by a line on coordinate axes;
- define locus as the path of a point moving in a plane under certain conditions; and
- find the equation of locus under given conditions.

PREREQUISITES

- Number system.

- Plotting of points in a coordinate plane
- Drawing graphs oflinear equations.
- Solving systems of linear equations.

INTRODUCTION

The study of geometry began in the third century B.C. the Greek mathematician Menaechmus (Ca 380-320 B.C) used a method which had a similarity with the present methods of Coordinate geometry. Apollonius (262 B.C - 190 B.C) came close to inventing analytic geometry, but could not do so, as he did not take into account the negative magnitudes. However, the deceive step in developing coordinate geometry as a subject was taken later by Descartes and Fermat. Abraham de Moivre (1667-1754) also made contribution to the development of coordinate geometry.

The study of the branch of mathematics which deals with the interrelationship between geometrical and algebraic concepts is called coordinate geometry (or) Cartesian geometry in honour of famous French mathematician Rene Descartes.

In this chapter we shall study the basic conceptys in geometry and its algebraic representation.

9.1 RECTANGULAR COORDINATE AXES

Recall that in previous classes, you have learnt to fix the position of a point in a plane by drawing two mutually perpendicular lines. The fixed point O, where these lines intersect each other is called the origin 0 as shown in Fig. 9.1 These mutually perpencular lines are called the co-ordinate axes. The horizontal line XOX' is the x-axis or axis of x and the vertical line YOY' is the y- axis or axis of y.

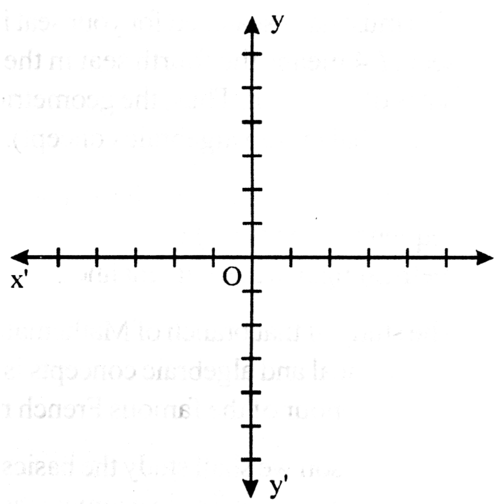

Fig. 9.1

9.1.1 Cartesian coordinates of a point

To find the coordinates of a point we proceed as follows. Take X'OX and YOY' as coordinate axes. Let P be any point in this plane. From point P draw PA $\perp$ XOX' and PB $\perp$ XOY'. Then the distance OA $= x$ measured along x-axis and the distance OB $= y$ measured along y-axis determine the position of the point P with reference to these axes. The distance OA measured along the axis of x is called the abscissa or x-coordinate and the distance OB (=PA) measured along y-axis is called the ordinate or y-coordinate of the point P. The abscissa and the ordinate taken together are called the coordinates of the point P. Thus, the coordinates of the point P are (x and y) which represent the position of the point P point in a plane. These two numbers are to form an ordered pair beacuse the order in which we write these numbers is important. In Fig. 9.3 you may note that the position of the ordered pair (3, 2) is different from that of (2, 3). Thus, we can say that (x, y) and (y, x) are two different ordered pairs representing two different points in a plane.

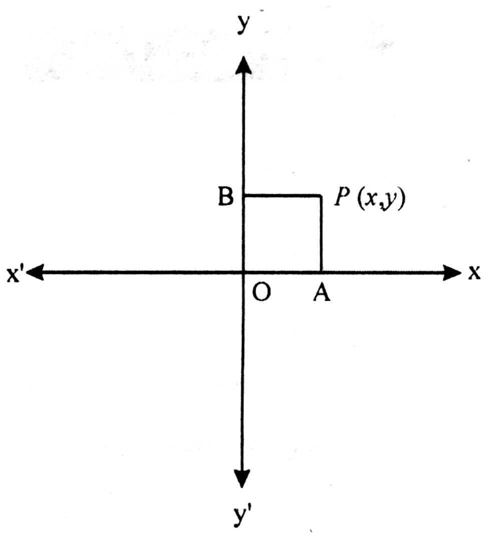

Fig. 9.2

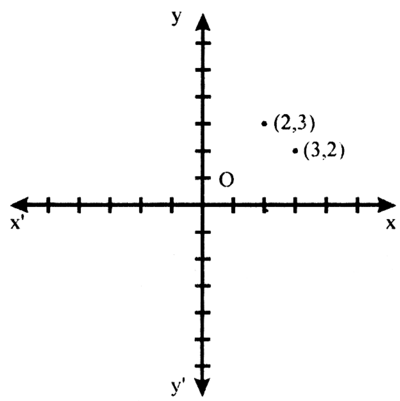

Fig. 9.3

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9.1.2 Quardrants

We know that coordinate axes XOX' and YOY' divide the region of the plane into four regions. These regions are called the quardrants as shown in Fig. 9.4. In accordance with the convention of signs, for a point P (x,y) in different quadrants, we have

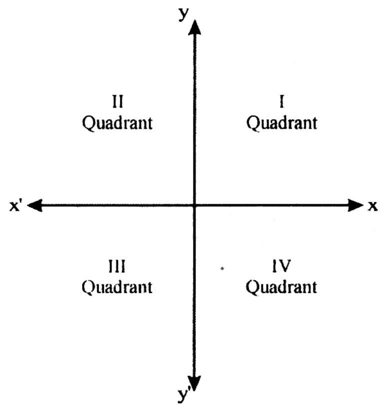

Fig. 9.4

I quadrant: $x > 0, y > 0$

II quadrant: $x < 0, y > 0$

III quadrant: $x < 0, y < 0$

IV quadrant: $x > 0, y < 0$

9.2 DISTANCE BETWEEN TWO POINTS

Recall that you have derived the distance formula between two points P (x_1, y_1) and Q (x_2,y_2) in the following manner:

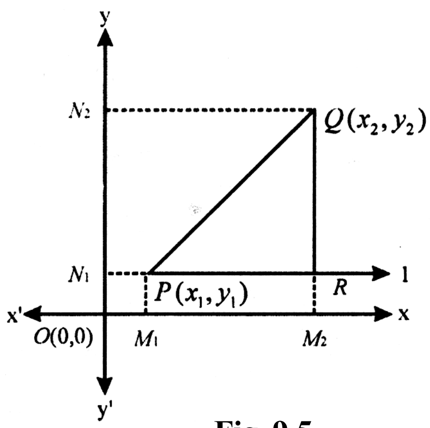

Fig. 9.5

Let us draw a line l II .XX' through P. Let R be the point of intersection of the perpendicular from Q to the line l. Then Δ PQR is a right-angled triangle.

Also PR $= M_1M_2$

$= OM_2 - OM_1$

$= x_2 - x_1$

and QR $= QM_2 - RM_2$

$= QM_2 - PM_1$

$= ON_2 - ON_1$

$= y_2 - y_1$

Now $PQ^2 = PR^2 + QR^2$ (Pythagoras theorem)

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note: This formula holds for points in all quadrants

Also the distance of a point $P(x,y)$ from the origin $0(0,0)$

is $OP = \sqrt{x^2 + y^2}$

Let us illustrate the use of these formulae with some examples.

Example 9.1: Find the distance between the following pairs of points:

i) A(14, 3) and B(10, 6) ii) M(−1, 2) and N(0, −6)

Solution:

i) Distance between two points = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Here $x_1 = 14$, $y_1 = 3$, $x_2 = 10$, $y_2 = 6$

$\therefore$ Distance between A and B = $\sqrt{(10-14)^2 + (6-3)^2}$

$$= \sqrt{(-4)^2 + (3)^2}$$

$$= \sqrt{16+9}$$

$$= \sqrt{25}$$

$$= 5$$

Distance between A and B is 5 units.

ii) Here $x_1 = -1$, $y_1 = 2$, $x_2 = 0$ and $y_2 = -6$

Distance between A and B $= \sqrt{(0-(-1))^2 + (-6-2)^2}$

$$= \sqrt{1+(-8)^2} = \sqrt{1+64} = \sqrt{65}$$

Distance between M and N = $\sqrt{65}$ units.

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Example 9.2 : Show that the points P(−1, 1), Q(2, 3) and R(−2, 6)are the vertices of a right-angled triangle.

$$PQ^2 = (2+1)^2 + (3+1)^2 = 3^2 + 4^2 = 9 + 6 = 25$$

$$QR^2 = (-4)^2 + (3)^2 = 16 + 9 = 25$$

and $$RP^2 = 1^2 + (-7)^2 = 1 + 49 = 50$$

$$PQ^2 + QR^2 = 25 + 25 = 50 = RP^2$$

$\therefore$ ΔPQR is a right-angled triangle (by converse of Pythagoras Theorem)

Example 9.3 : Show that the points A(1, 2), B(4, 5) and C(−1, 0) lie on a straight line.

Solution: Here, $AB = \sqrt{(4-1)^2 + (5-2)^2}$ units

$= \sqrt{18}$ units

$AB = 3\sqrt{2}$ units

$BC = \sqrt{(-1-4)^2 + (0-5)^2}$ units

$= \sqrt{50}$ units

$= 5\sqrt{2}$ units

and AC $= \sqrt{(-1-1)^2 + (0-2)^2}$ units

$= \sqrt{4+4}$ units

$= 2\sqrt{2}$ units

Now AB + AC $= \left(3\sqrt{2} + 2\sqrt{2}\right)$ units

$= 5\sqrt{2}$ units

$=$ BC

i.e., AB + AC = BC

Hence A, B, C lie on a straight line. In other words, A, B, C are collinear.

Example 9.4 : Prove that the points $(2a, 4a)$, $(2a, 6a)$ and $(2a+\sqrt{3}a, 5a)$ are the vertices of an equilateral triangle whose side is $2a$.

Solution: Let the points be A (2a, 4a), B (2a, 6a) and C $(2a+\sqrt{3}a, 5a)$

$$AB = \sqrt{0+(2a)^2} = 2a \text{ units}$$

$$BC = \sqrt{(\sqrt{3}a)^2+(-a)^2} \text{ units}$$

$$= \sqrt{3a^2+a^2} = 2a \text{ units}$$

and $AC = \sqrt{(\sqrt{3}a)^2+(a)^2} = 2a$ units

$\Rightarrow$ AB + BC > AC

BC + AC > AB and

AB + AC > BC and AB = BC = AC = $2a$

$\Rightarrow$ A, B, C form the vertices of an equilateral triangle of side $2a$.

EXERCISE 9.1

1. Find the distance between the following pairs of points.
   (a) (5, 4) and (2, –3) (b) $(a, -a)$ and (b, b)
2. Prove that each of the following sets of points are the vertices of a right angled-trangle.
   (a) (4, 4) (3, 5) (–1, –1) (b) (2, 1), (0, 3), (–2, 1)
3. Show that the following sets of points form the vertices of a triangle:
   (a) (3, 3) (–6, 3) and (0, 0)
   (b) $(0, a)$ (a, b) and (0, 0) (if $ab = 0$)
4. Show that the following sets of points are collinear:
   (a) (3, –6) (2, 4) and (–4, 8) (b) (0, 3) (0, –4) and (0, 6)
5. (a) Show that the points (0,–1) (–2, 3) (6, 7) and (8, 3) are the vertices of a rectangle.

   (b) Show that the points (3,–2) (6, 1) (3, 4) and (0, 1) are the vertices of a square.

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9.3 SECTION FORMULA

9.3.1 Internal Division

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points on a line I and $R(x, y)$ divide PQ internally in the ratio $m_1 : m_2$

To find: The coordinates x and y of point R.

Construction: Draw PL, QN and RM perpendiculars to XX' from P, Q and R respectively and L, M and N lie on XX'. Also draw $RT \perp QN$ and $PV \perp QN$.

Method: R divides PQ internally in the ratio $m_1 : m_2$.

$\Rightarrow$ R lies on PQ and $\dfrac{PR}{RQ} = \dfrac{m_1}{m_2}$

Also, in triangles, RPS and QRT,

$\angle RPS = \angle QRT$ ((Corresponding angles as PS||RT)

and $\angle RSP = \angle QTR = 90^0$

$\therefore \Delta RPS \sim \Delta QRT$ (AAA similarity)

$\Rightarrow \dfrac{PR}{RQ} = \dfrac{RS}{QT} = \dfrac{PS}{RT}$ (i)

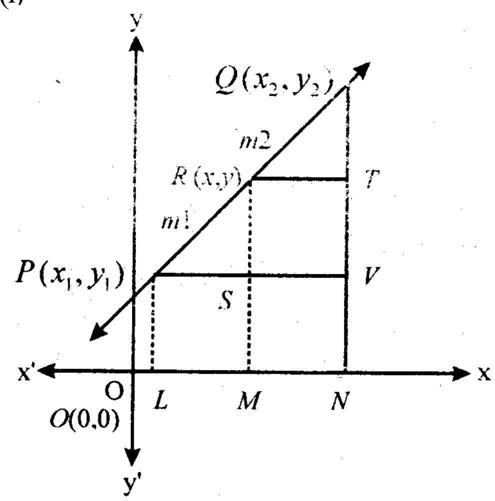

Fig. 9.6

Also PS = LM

$= OM - OL$

$= x - x_1.$

$RT = MN = ON - OM$

$= x_2 - x.$

$RS = RM - SM$

$= y - y_1$

$QT = QN - TN$

$= y_2 - y.$

From (i) we have

$$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

$\Rightarrow \quad m_1(x_2 - x) = m_2(x - x_1)$

and $\; m_1(y_2 - y) = m_2(y - y_1)$

$$\Rightarrow x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \text{ and } y = \frac{m_1y_2 + m_2y_1}{m+n}$$

Thus, the coordinates of R are:

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m+n}\right)$$

Coordinates of the mid-point of a line segment

If R is the mid point of PQ then,

$m_1 = m_2 = 1$ (as R divides PQ in the ratio 1:1)

$\therefore$ Coordinates of the mid point are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

9.3.2 External Division

Let R divide PQ externally in the ratio $m_1 : m_2$

To find: The coordinates of R.

Construction: Draw PL, QN and RM perpendiculars to XX' from P, Q and R respectively and PS$\perp$RM and QT $\perp$ RM.

Clearly, $\Delta RPS \sim \Delta RQT$

$$\frac{RP}{RQ} = \frac{PS}{QT} = \frac{RS}{RT}$$

or $$\frac{m_1}{m_2} = \frac{x - x_1}{x - x_2} = \frac{y - y_1}{y - y_2}$$

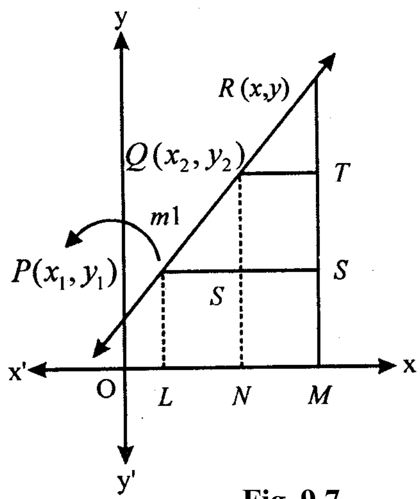

Fig. 9.7

$\Rightarrow \quad m_1(x - x_2) = m_2(x - x_1)$

and $\quad m_1(y - y_2) = m_2(y - y_1)$

These give: $$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} \text{ and } y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

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Hence, the coordinates of the point of external division are

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$$

Let us now take some examples.

Example 9.5 : Find the coordinates of the point which divides the line segment joining the points (4,–2) and (–3, 5) internally and externally in the ratio 2:3.

Solution: Let P(x, y) be the point of internal division.

$$\therefore \quad x = \frac{2(-3)+3(4)}{2+3} = \frac{6}{5} \quad \text{and} \quad y = \frac{2(5)+3(-2)}{2+3} = \frac{4}{5}$$

P has coordinates $\left(\frac{6}{5}, \frac{4}{5}\right)$

If Q(x', y') is the point of external division, then

$$x' = \frac{2(-3)-3(4)}{2-3} = 18 \text{ and } \quad y' = \frac{2(5)-3(-2)}{2-3} = -16$$

Thus, the coordinates of the point of external division are (18, –16).

Example 9.6 : In what ratio does the point (3, –2) divide the line segment joining the points (1,4) and (–3, 16)?

Solution: Let the point P(3, –2) divide the line segement in the ratio k : 1.

Then the coordinates of P are $\left(\frac{-3k+1}{k+1}, \frac{16k+4}{k+1}\right)$

But the given coordinates of P are (3, –2)

$$\therefore \frac{-3k+1}{k+1} = 3$$

$$\Rightarrow -3k+1 = 3k+3$$

$$\Rightarrow k = -\frac{1}{3}$$

$\Rightarrow$ P divides the line segement externally in the ratio 1:3 .

Example 9.7 : The vertices of a quadrilateral ABCD are respectively (1, 4), (−2,1), (0, −1) and (3,2). If E, F, G, H are respectively the midpoints of AB, BC, CD and DA, prove that the quadrilateral EFGH is a parallelogram.

Solution : Since E, F, G, and H, are the midpoints of the sides AB, BC, CD and DA, therefore, the coordinates of E, F, G, and H respectively are:

$$\left(\frac{1-2}{2}, \frac{4+1}{2}\right), \left(\frac{-2+0}{2}, \frac{1-1}{2}\right)\left(\frac{0+3}{2}, \frac{-1+2}{2}\right) \text{and} \left(\frac{1+3}{2}, \frac{4+2}{2}\right)$$

$\Rightarrow$ E$\left(\frac{-1}{2}, \frac{5}{2}\right)$, F(−1, 0), G$\left(\frac{3}{2}, \frac{1}{2}\right)$ and H(2, 3) are the required points.

Also, the mid point of diagonal EG has coordinates

$$\left[\frac{-\frac{1}{2}+\frac{3}{2}}{2}, \frac{\frac{5}{2}+\frac{1}{2}}{2}\right] = \left[\frac{1}{2}, \frac{3}{2}\right]$$

Coordinates of midpoint of FH are

$$\left(\frac{-1+2}{2}, \frac{0+3}{2}\right) = \left(\frac{1}{2}, \frac{3}{2}\right)$$

Since, the midpoints of the diagonals are the same, therefore, the diagonals bisect each other.

Hence EFGH is a parallelogram.

EXERCISE 9.2

1. Find the midpoint of each of the line segements whose end points are given below:
   (a) (−2, 3) and (3, 5) (b)(6, 0) and (−2, 10)
2. Find the coordinates of the point dividing the line segment joining (−5, −2) and (3, 6) intemallyin the ratio 3:1.
3. (a) Three vertices ofa parallelogram are (0,3), (0,6) and (2,9). Find the fourth vertex.

(b) (4,0), (–4,0), (0,–4) and (0,4) are the vertices of a square. Show that the quadrilateral formed by joining the midpoints of the sides is also a square.

4. The line segementjoining (2,3) and (5, –1) is trisected. Find the points of trisection.

5. Show that the figure formed by joining the midpoints of the sides of a rectangle is a rhombus.

9.4 AREA OF A TRIANGLE

Let us find the area of a triangle whose vertices are A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) Draw AL, BM and CN perpendiculars to XX' Area of ΔABC.

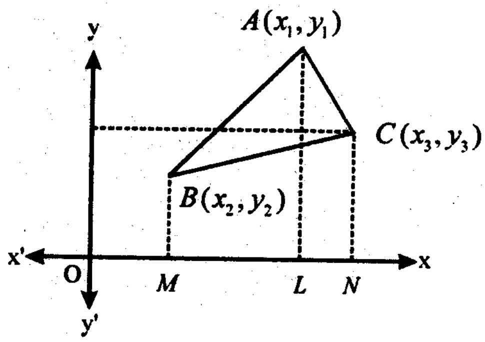

Fig. 9.8

= Area of trapzium. BMLA + Area oftrapzium. ALNC –

Area oftrapzium. BMNC

$$= \frac{1}{2}(\text{BM+AL})\text{ML} + \frac{1}{2}(\text{AL+CN})\text{LN} - \frac{1}{2}(\text{BM+CN})\text{MN}$$

$$= \frac{1}{2}\left(y_2+y_1\right)(x_1-x_2)+\frac{1}{2}(y_1+y_3)(x_3-x_1)-\frac{1}{2}(y_2+y_3)(x_3-x_2)$$

$$= \frac{1}{2}\left[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_3y_3)\right]$$

$$= \frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$$

This can be stated in the determinant form as follows:

$$\text{Area of } \Delta\text{ABC} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Example 9.8 : Find the area of the triangle whose vertices are A(3, 4), B(6, –2) and C(–4, –5).

Solution: The area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 3 & 4 & 1 \\ 6 & -2 & 1 \\ -4 & -5 & 1 \end{vmatrix}$

$$= \frac{1}{2}\left[3(-2+5)-4(6+4)+1(-30-8)\right]$$

$$= \frac{1}{2}\left[9-40-38\right] = -\frac{69}{2}$$

As the area is to be positive

$\therefore$ Area of $\Delta ABC = \frac{69}{2}$ square units.

Example 9.9: If the vertices ofa triangle are (1, *k),* (4, –3) and (–9,7) and its area is 15 square units, find the value(s) of *k*.

Solution: Area of triangle $= \frac{1}{2}\begin{vmatrix} 1 & k & 1 \\ 4 & -3 & 1 \\ -9 & 7 & 1 \end{vmatrix}$

$$= \frac{1}{2}\left[-3-7-k(4+9)+1(28-27)\right]$$

$$= \frac{1}{2}\left[-10-13k+1\right]$$

$$= \frac{1}{2}\left[-9-13k\right]$$

Since the area of the triangle is given to be l5,

$\therefore \quad \frac{-9-13k}{2} = 15$

or, $-9 - 13k = 30$

$- 13k = 39$

or, $k = -3$.

EXERCISE 9.3

1. Find the area of each of the following triangles whose vertices are given below:

   (a) (0, 5) (5, –5) and (0, 0)

   (b) (2, 3), (–2, –3) and (–2, 3)

   (c) $(a, 0)$ $(0, -a)$ and (0, 0)

2. The area of a triangl e ABC, whose verti ces are A(2,–3), B(3, –2) and $C\left(\frac{5}{2}, k\right)$ is $\frac{3}{2}$ sq unit. Find the value of k.

3. Find the area of a rectangle whose vertices are (5, 4) (5, –4) (–5, 4) and (–5, –4).

4. Find the area of a quadrilateral whose vertices are (5, –2), (4, –7) (1, 1) and (3, 4).

9.5 CONDITION FOR COLLINEARITY OF THREE POINTS

The three points $A(x, y)$, $B(x_2, y_2)$ and $C(x_3, y_3)$are collinear if and only if the area of the triangle ABC becomes zero.

i.e., $$\frac{1}{2}\left|x_1y_2 - x_1y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\right| = 0$$

i.e., $$x_1y_2 - x_1y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 = 0$$

In short, we can write this result as

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Let us illustrate this with the help of examples:

Example 9.10: Show that the points A(a, $b+c$), B(b, $c+a$) and C(c, $a+b$) are collinear.

Solution: Area of triangle ABC $= \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}(a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} = 0$$

Hence the points are collinear.

Example 9.11 : For what value of k, are the points (1, 5) , (k, 1) and (4, 11) collinear?

Solution: Area of the triangle formed by the given points is

$$= \frac{1}{2}\begin{vmatrix} 1 & 5 & 1 \\ k & 1 & 1 \\ 4 & 11 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[-10-5k+20+11k-4]$$

$$= \frac{1}{2}[6k+6] = 3k+3$$

Since the given points are collinear, therefore

$$\Leftrightarrow 3k + 3 = 0 \Rightarrow k = -1.$$

Hence, for $k = -1$, the given points are collinear.

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EXERCISE 9.4

1. Show that the points (–1, –1) (5, 7) and (8, 11) are collinear.
2. Show that the points (3, 1), (5, 3) and (6, 4) are collinear.
3. Prove that the points $(a, 0)$, $(0, b)$ and (1, 1) are collinear if $\frac{1}{a}+\frac{1}{b}=1$.
4. If the points (a, b), (a_1, b_1) and $(a - a_1, b - b_1)$ are collinear, show that $a_1 b = a b_1$.
5. Find the value ofk for which the points (5, 7), $(k, 5)$ and (0, 2) are collinear.
6. Find the values ofkforwhich the point $(k, 2-2k)$ $(-k + 1, 2k)$ and $(-4 - k, 6-2k)$ are collinear.

9.6 INCLINATION AND SLOPE OF A LINE

Look at the Fig. 9.9. The line AB makes an angle or $\pi + \alpha$ with the x-axis (measured in anticlockwise direction).

The *inclination* of the given line is represented by the measure of angle made by the line with the positive direction of x -axis (measured in anticlockwise direction)

In a special case when the line is parallel to x-axis or it coincides with the *x-axis,* the inclination of the line is defined to be 0°.

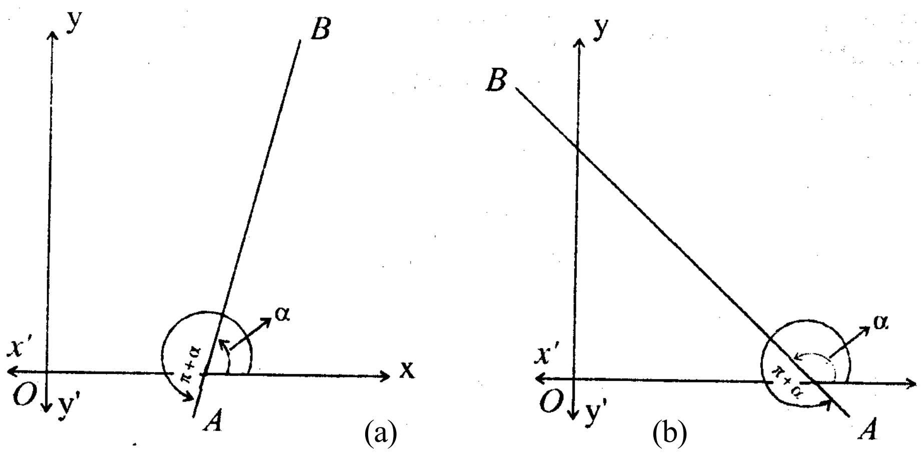
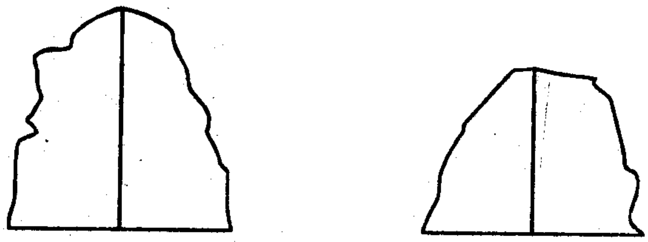


Fig. 9.9

Again look at the pictures of two mountains given below. Here we notice that the mountain in Fig. 9.10 (a) is more steep compared to mountain in Fig. 9.10 (b).



(a) (b)

Fig. 9.10

How can we quantify this steepness? Here we say that the angle of inclination of mountain (a) is more than the angle of inclination of mountain (b) with the ground.

Try to see the difference between the ratios of the maximum height from the ground to the base in each case.

Naturally, you will find that the ratio in case (a) is more as compaired to the ratio in case (b). That means we are concerned with height and base and their ratio is linked with tangent of an angle, so mathematically this ratio or the tangent of the inclination is termed as *slope.* We define the slope as tangent of an angle.

The *slope* of a line is the the tangent of the angle 8 (say) which the line makes with the positive direction of x -axis. Generally, it is denoted by m (= tan θ)

Note: If a line makes an angle of 90^0 or 270^0 with the x-axis, the slope of the line can not be defined.

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Example 9.12: In Fig. 9.9 find the slope oflinesAB and BA.

Solution: Slope of line AB = tan α

Slope of line BA = tan $(\pi + \alpha)$ = tan α.

Note: From this example, we can observe that "slope is independent of the direction of the line segement".

Example 9.13 : Find the slope of a line which makes an angle of 30° with the negative direction of x-axis.

Solution:

Here $\theta = 180^0 - 30^0 = 150^0$

$\therefore \quad m =$ slope of the line

$= \tan(180^0 - 30^0)$

$= -\tan 30^0 = \dfrac{-1}{\sqrt{3}}$

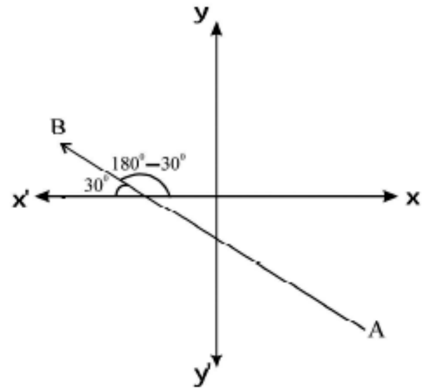

Fig. 9.11

Example 9.14 : Find the slope of a line which makes an angle of 60° with the positive direction of y-axis.

Solution : Here $\theta = 90^0 + 60^0$

$\therefore \quad m =$ Solpe of the line

$= \tan(90^0 + 60^0)$

$= -\cot 60^0$

$= -\tan 30^0$

$= \dfrac{-1}{\sqrt{3}}$

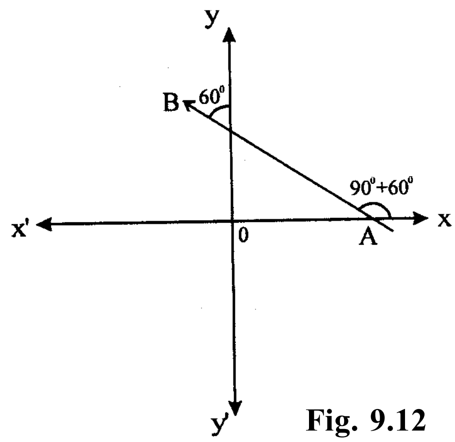

Fig. 9.12

Example 9.15 : If a line is equally inclined to the axes, show that its slope is +1.

Solution: Let a line AB be equally inclined to the axes and meeting axes at points A and B as shown in the Fig. 9.13.

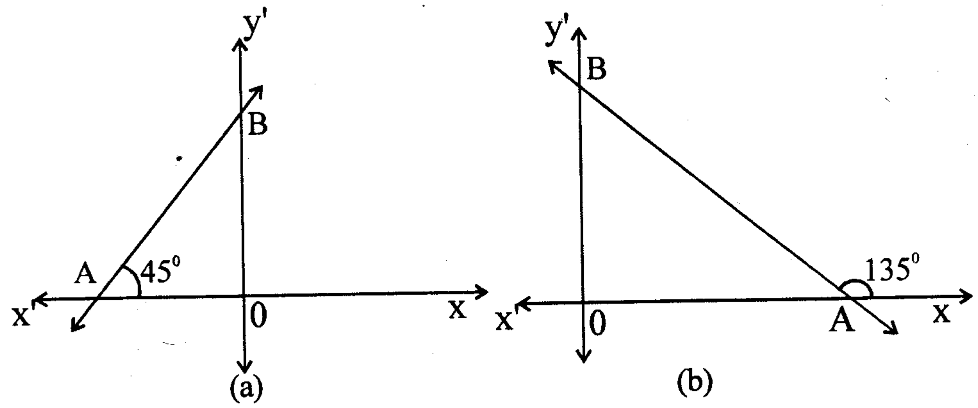

Fig. 9.13

In Fig 9.13(a), inclination of line AB = $\angle XAB = 45^0$

Slope of the line AB = $\tan 45^0 = 1$

In Fig. 9.13 (b) inclination of line AB = $\angle XAB = 180^0 - 45^0$

$\therefore$ Slope of the line $AB = \tan 135°$

$= \tan (180^0 - 45^0) = -\tan 45^0 = -1$

Thus, if a line is equally inclined to the axes, then the slope of the line will be ± 1.

EXERCISE 9.5

1. Find the Slope of a line which makes an angle of i) 60^0 ii) 150^0 with the positive direction of x-axis.
2. Find the slope of a line which makes an angle of 30^0 with the positive direction of y-axis.
3. Find the slope of a line which makes an angle of 60^0 with the negative direction of x-axis.

9.7 SLOPE OF A LINE JOINING TWO DISTINCT POINTS

Let A(x_1, y_1) and B(x_2, y_2) be two distinct points. Draw a line through A andB and let the inclination of this line be 8. Let the point of intersection of a horizontal line through A and a vertical line through B be M, then the coordinates of M are as shown in the Fig. 9.14.

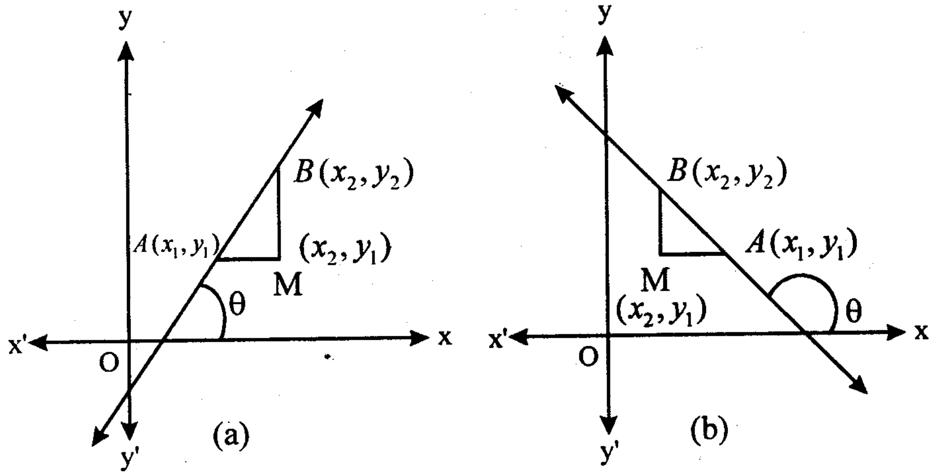

Fig. 9.14

(A) In Fig 9.14 (a), angle of inclination MAB is equal to θ (acute). Consequently.

$$\tan\theta = \tan(\angle MAB) = \frac{MB}{AM} = \frac{y_2 - y_1}{x_2 - x_1}$$

(B) In Fig. 9.14 (b), angle of inclinationθ is obtuse, and since θ and ∠MAB are supplementary, consequently,

$$\tan\theta = -\tan(\angle MAB) = \frac{-MB}{MA} = -\frac{(y_2 - y_1)}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

Hence in both the cases, the slope m of a line through A(x_1, y_1) and B(x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Note : If $x_1 = x_2$ then m is not defined. In that case the line is parallel to y-axis.

Is there a line whose slope is 1? Yes, when a line is inclined at 45° with the positive direction of X-axis.

Is there a line whose slope is $\sqrt{3}$? Yes, when a line is inclined at 60° with the positive direction of x-axis.

From the answers to these questions, you must have realised that given any real number *m*, there will be a line whose slope is *m* (because we can always find an angle α such that $\tan\alpha = m$).

Example 9.16 : Find the slope of the line joining the points A(6, 3) and B(4, 10).

Solution: The slope of the line passing through the points

$$(x_1, y_1) \text{ and } (x_2, y_2) = \frac{y_2 - y_1}{x_2 - x_1}$$

Here, $x_1 = 6;\ y_1 = 3;\ x_2 = 4,\ y_2 = 10$

Now substituting these values, we have slope $= \dfrac{10-3}{4-6} = -\dfrac{7}{2}$.

Example 9.17 : Determine *x*, so that the slope of the line passing through the points (3, 6) and (*x*, 4) is 2.

Solution: Slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4-6}{x-3} = \dfrac{-2}{x-3}$

$$\therefore \frac{-2}{x-3} = 2 \qquad \text{...(Given)}$$

$$\therefore 2x - 6 = -2 \text{ or } x = 2$$

EXERCISE 9.6

1. What is the slope of the line joining the points A(6, 8) and B(4, 14)?
2. Determine *x* so that 4 is the slope of the line through the points A(6, 12) and B(*x*, 8).

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3. Determine y, if the slope of the line joining the points A(−8, 11) and B(2, y) is $-\frac{4}{3}$.

4. A(2, 3), B(0, 4) and (−5, 0) are the vertices of a triangle ABC. Find the slope of the line passing through the point B and the mid point of AC

5. A(−2, 7), B(1, 0), C(4, 3) and D(1, 2) are the vertices of a quadrilateral ABCD. Show that

   (i) slope of AB = slope of CD

   (ii) slope of BC = slope of AD

9.8 CONDITIONS FOR PARALLELISM AND PERPENDICULARITY OF LINES

9.8.1 Slope of Parallel Lines

Let l_1, l_2 be two (non-vertical) lines with their slopes m_1 and m_2 respectively. Let θ_1, θ_2 be angles of inclination of these resepectively.

Case 1 : Let the lines l_1 and l_2 be parallel

Then $\theta_1 = \theta_2$

$\Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow m_1 = m_2$

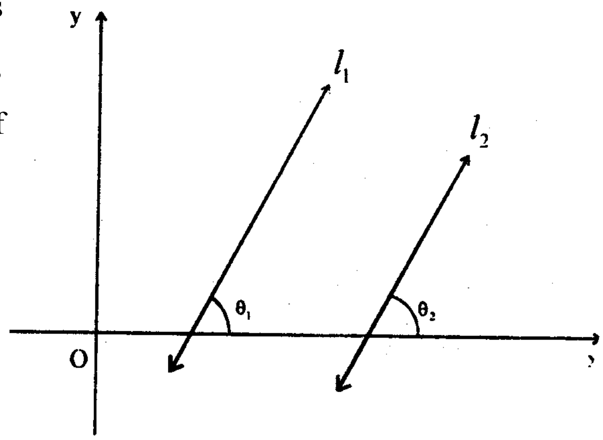

Fig. 9.15

Thus, if two lines are parallel then their slopes are equal.

Case 2: Let the lines l_1 and l_2 have equal slopes.

i.e., $m_1 = m_2 \Rightarrow \tan\theta_1 = \tan\theta_2$

$\Rightarrow \theta_1 = \theta_2 \qquad (0^0 \le \theta \le 180^0)$

$\Rightarrow l_1 \parallel l_2$

Hence, two (non-vertical) lines are parallel if and only if $m_1 = m_2$.

9.8.2 SLOPES OF PERPENDICULAR LINES

Let l_1 and l_2 be two (non-vertical)lines with their slopes m_1 and m_2 respectively. Also let θ_1 and θ_2 be their inclinations respectively.

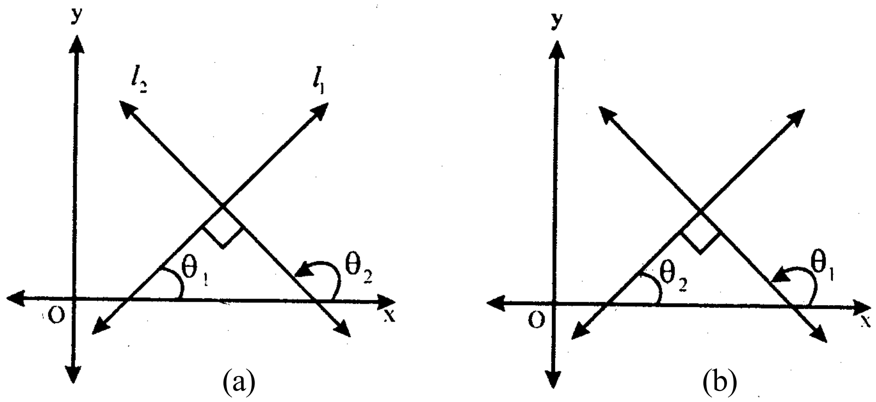

Fig. 9.16

Case-I: Let $l_1 \perp l_2$

$\Rightarrow \theta_2 = 90^0 + \theta_1 \qquad$ or $\qquad \theta_1 = 90^0 + \theta_2$

$\Rightarrow \tan\theta_2 = \tan(90 + \theta_1)$ or $\tan\theta_1 = \tan(90^0 + \theta_2)$

$\Rightarrow \tan\theta_2 = -\cot\theta_1) \qquad$ or $\qquad \tan\theta_1 = -\cot\theta_2$

$\Rightarrow \tan\theta_2 = -\dfrac{1}{\tan\theta_1} \qquad$ or $\qquad \tan\theta_1 = -\dfrac{1}{\tan\theta_2}$

$\Rightarrow$ In both the cases, we have

$$\tan\theta_1 \tan\theta_2 = -1$$

or $$m_1 m_2 = -1$$

Thus, if two lines are perpendicular then the product of their slopes is equal to -1.

Case - II :

Let the two lines l_1 and l_2 be such that the product of their slopes is -1.

i.e., $m_1 m_2 = -1$

$\Rightarrow \tan\theta_1 \tan\theta_2 = -1$

$$\tan\theta_1 = -\frac{1}{\tan\theta_2} = -\cot\theta_2 = \tan(90^0 + \theta_2)$$

$\Rightarrow$ Either $\theta_1 = 90^0 + \theta_2$ and $\theta_2 = 90^0 + \theta_1$

In both cases $l_1 \perp l_2$

Hence, two (non-vertical) lines are perpendicular if and only if $m_1 m_2 = -1$.

Example 9.18 : Show that the line passing through the points A(5, 6) and B(2, 3) is parallel to the line passing, through the points C(9, –2) and D(6, –5).

Solution: Slope of the line AB $= \dfrac{3-6}{2-5} = \dfrac{-3}{-3} = 1$

and slope of the line CD $= \dfrac{-5+2}{6-9} = \dfrac{-3}{-3} = 1$

As the slopes are equal

$\therefore$ AB || CD.

Example 9.19 : Show that the line passing through the points A(2, –5) and B(–2, 5) is perpendicular to the line passing through the points L(6,3) and M(1,1).

Solution: Here

m_1 = Slope of the line AB $= \dfrac{5+5}{-2-2} = \dfrac{10}{-4} = \dfrac{-5}{2}$

and m_2 = slope of the line LM $= \frac{1-3}{1-6} = \frac{2}{5}$

Now $m_1 \,.\, m_2 = \frac{-5}{2} \times \frac{2}{5} = -1$

Hence, the lines are perpendicular to each other.

Example 9.20 : Using the concept of slope, show that A(4,4), B(3,5) and C are the vertices of a right triangle.

Solution: Slope of line AB = $m_1 = \frac{5-4}{3-4} = -1$

Slope of line BC = $m_2 = \frac{-1-5}{-1-3} = \frac{3}{2}$

and Slope of line AC = $m_3 = \frac{-1-4}{-1-4} = 1$

Now $m_1 \times m_3 = -1$

$\Rightarrow$ AB $\perp$ AC

$\Rightarrow$ ΔABCis a right-angled triangle.

Hence, A(4,4), B(3,5) and C(−1, −1) are the vertices of right triangle.

Example 9.21 : What is the value of y so that the line passing through the points A(3, y) and B(2,7) is perpendicular to the line passing through the point C(−1, 4) and D(0, 6) ?

Solution: Slope of the line AB = $m_1 = \frac{7-4}{2-3} = y-7$

Slope of the line CD $= m_2 = \frac{6-4}{0+1} = 2$

Since the lines are perpendicular,

$\therefore\ m_1 \times m_2 = -1$

or $(y-7) \times 2 = -1$

or $2y - 14 = -1$

or $2y = 13$

or $y = \frac{13}{2}$

EXERCISE 9.7

1. Show that the line joining the points $(2, -3)$ and $(-4, 1)$ is

   (i) parallel to the line joining the points $(7, -1)$ and $(0, 3)$

   (ii) perpendicular to the line joining the points $(4, 5)$ and $(0, -2)$.

2. Find the slope of a line parallel to the line joining the points $(4, 1)$ and $(2, 3)$

3. The line joining the points $(-5, 7)$ and $(0, -2)$ is perpendicular to the line joining the points $(1, 3)$ and $(4, x)$. Find x.

4. A$(-2, 7)$, B$(1, 0)$, C$(4, 3)$ and D$(1, 2)$ are the vertices of quadrilateral ABCD. Show that the sides of ABCD are parallel.

5. Using the concept of the slope ofa line, show that the points A$(6, -1)$, B$(5, 0)$ and C$(2, 3)$ are collinear. [Hint: slopes of AB, BC and CA must be equal.]

6. Find k so that line passing through the points $(k, 9)$ and $(2,7)$ is parallel to the line passing through the points $(2, -2)$ and $(6, 4)$.

7. Using the concept of slope of a line, show that the points $(-4, -1)$, $(-2, -4)$, $(4, 0)$ and $(2, 3)$taken in the given order are the vertices of a rectangle.

8. The vertices ofa triangle ABC are A$(-3, 3)$, B$(-1, -4)$ and C$(5, -2)$, M and N are the midpoints of AB and AC. Show that MN is parallel to BC and $MN = \frac{1}{2} BC$.

9.9 INTERCEPTS MADE BY A LINE ON AXES

If a line l (not passing through the Origin) meets x-axis at A and y-axis at B as shown in Fig. 9.17, then

(i) OA is called the x-intercept or the intercept made by the line on x-axis.

(ii) OB is called y-intercept or the intercept made by the line on y-axis.

(iii) OA and OB taken together in this order are called the intercepts made by the line l on the axes.

(iv) AB is called the portion of the line intercepted between the axes.

(v) The coordinates of the point A on x-axis are $(a, 0)$ and those of point B are $(0, b)$

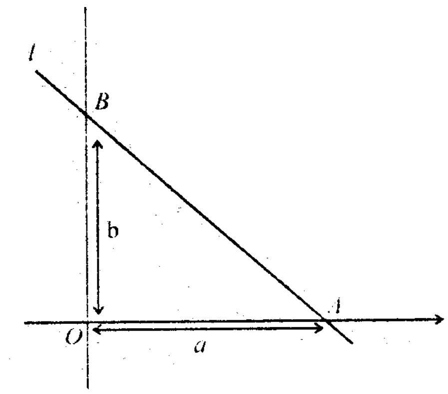

Fig. 9.17

To find the intercept of a line in a given plane on x-axis, we put $y = 0$ in the given equation of a line and the value of x so obtained is called the x intercept.

To find the intercept of a line on y-axis we put $x = 0$ and the value of y so obtained is called the y intercept.

Note:

1. *A line which passes through origin makes no intercepts on axes.*
2. *A horizontal line has no x-intercept and vertical line has no y-intercept.*
3. *The intercepts on x- axis and y-axis are usually denoted by a and b respectively. But if only y-intercept is considered, then it is usually denoted by c.*

Example 9.22 : If a line is represented by $2x + 3y = 6$ find its x and y intercepts.

Solution: The given equation of the line is

$$2x + 3y = 6 \qquad .... (i)$$

Putting $x = 0$ in (i), we get

$$y = 2$$

Thus, y-intercept is 2.

Again putting $y = 0$ in (i), we get

$$2x = 6 \quad \Rightarrow \quad x = 3$$

Thus, x-intercept is 3.

EXERCISE 9.8

1. Find x andy intercepts, if the equations of lines are:

(i) $x + 3y = 6$ (ii) $7x + 3y = 2$ (iii) $\frac{x}{2a} + \frac{y}{2b} = 1$

(iv) $ax + by = c$ (v) $\frac{Y}{2} - 2x = 8$ (vi) $\frac{y}{3} - \frac{2x}{3} = 7$

9.10 LOCUS OF A POINT

9.10.1 DEFINITION OF THE LOCUS OF A POINT

Locus of a point is the path traced by the point when moving under a given condition or conditions. Thus, locus of a point is a path of definite shape. It may be a straight line, circle or any other curve.

For Example: (i) The locus of a point in a plane which moves such that it is at constant distance from a fixed point in the plane is a circle as shown in Fig. 9.18 *(a)*.

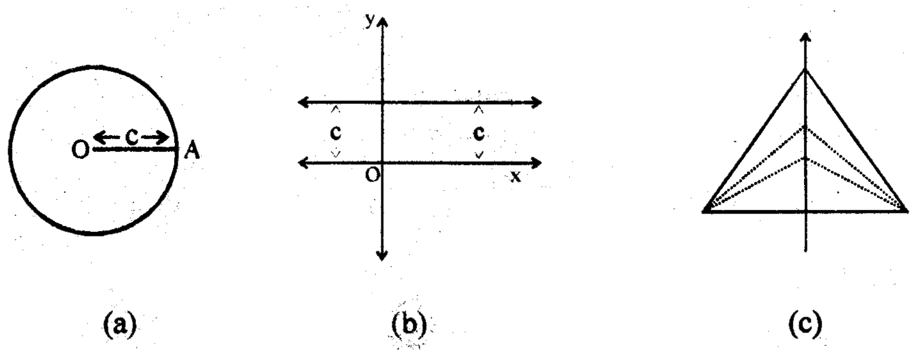

Fig. 9.18

(ii) The locus of a point which moves such that it is always at a constant distance from x-axis is a pair of straight lines parallel to x-axis. [See Fig. 9.18 (b)]

(iii) The locus of a point in a plane which moves such that it is always at a constant distance from the two fixed points in the same plane is perpendicular bisector of the line segment joining the two points. [See Fig. 9. 18(c)].

From the defmition and the examples of a locus, we observe that

(a) Every point which satisfies the given condition or conditions is a point on the locus.

(b) Every point of the locus must satisfy the given condition or conditions.

9.10.2 EQUATION OF LOCUS

The equation of locus of a moving point (x,y) is an algebraic relation between x and y satisfying the given conditions of motion of a point.

The coordinates (x,y) of the moving point which generates the locus are called current coordinates. The point covers all the positions on the locus and is called the general point.

Let us take an example:

Let P(4,3) and Q(7, 11) be two points. Let us try to locate a point R which is equidistant from both the points P and Q.

Let the Coordinates of R be (x,y).

Then $PR = \sqrt{(x-4)^2+(y-3)^2} = \sqrt{x^2+y^2-8x-6y+25}$

$$QR = \sqrt{(x-7)^2+(y-1)^2} = \sqrt{x^2+y^2-14x-22y+170}$$

$\therefore$ PR = QR

$$\sqrt{x^2+y^2-8x-6y+25} = \sqrt{x^2+y^2-14x-22y+170}$$

Squaring, we get

$$x^2+y^2-8x-6y+25 = x^2+y^2-14x-22y+170$$

$$\Rightarrow \quad 6x+16y-145=0$$

This is called the equation of the locus of a point R which is equidistant from the points P and Q.

From the above, we observe the following working rule.

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9.10.3 WORKING RULE TO FIND THE EQUATION OF THE LOCUS OF A POINT

(i) Take any point (x, y) on the locus.

(ii) Write the given geometrical form in the terms of x and y and known constant or constants and simplify it, if necessary.

(iii) Express the given condition in mathematical form in the terms of x and y and known constant or constants and simplify it, if necessary.

(iv) The equation so obtained is the equation of the required locus.

Example 9.23: Find the equation of locus of points which are thrice as far from $(-a, 0)$ as from $(a, 0)$.

Solution: Let $P(x, y)$ be any point on locus. Also let $A(-a, 0)$ and $B(a, 0)$ be the two given points.

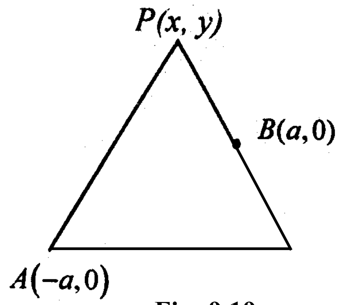

Fig. 9.19

Then by the given condition.

$$PA = 3PB$$

$$PA^2 = 9PB^2$$

$$\Rightarrow \quad (x + a)^2 + y^2 = 9[(x - a)^2 + y^2)]$$

$$\Rightarrow \quad x^2 + 2ax + a^2 + y^2 = 9[x^2 - 2ax + a^2 + y^2]$$

$$\Rightarrow \quad 8x^2 - 20ax + 8a^2 + 8y^2 = 0$$

Thus, $2x^2 + 2y^2 - 5ax + 2a^2 = 0$. is the required equation of the locus.

Example 9.24 : Find the equation of the locus of a point such that the sum of its distances from $(0, 2)$ and $(0, -2)$ is 6.

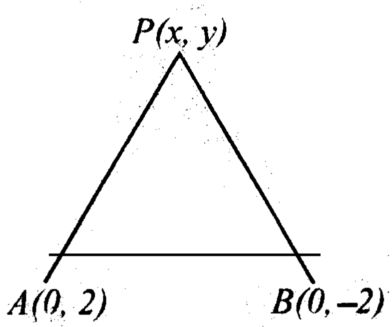

Fig. 9.20

Solution: Let P(x,y) be any point on the locus. Also let A(0,2) and B(0,–2) be the given points.

From the given condition, we have

$$PA + PB = 6$$

$$\therefore \quad \sqrt{x^2+(y-2)^2} + \sqrt{x^2+(y+2)^2} = 6$$

or $$\sqrt{x^2+y^2-4y+4} = 6-\sqrt{x^2+y^2+4y+4}$$

Squaring both sides, we get

$$x^2+y^2-4y+4 = 36+x^2+y^2+4y+4-12\sqrt{x^2+y^2+4y+4}$$

or $$-8y-36 = -12\sqrt{x^2+y^2+4y+4}$$

or $$2y+9 = 3\sqrt{x^2+y^2+4y+4}$$

Squaring both sides again, we get

$$(2y+9)^2 = 9(x^2+y^2+4y+4)$$

or $$4y^2+36y+81 = 9x^2+9y^2+36y+36$$

or $$9x^2+4y^2 = 45.$$

which is the required equation of the locus.

Example 9.25: A(3, 1) and B(–2, 4) are the two vertices of a triangle ABC. Find the equation of the locus of the centroid of the triangle, if the third vertex C is a point of the locus whose equation is $3x - 4y = 8$.

Solution: Let C(a,b) be the third vertex of the triangle ABC. Since C(a,b) lies on the locus whose equation is $3x - 4y = 8$.

$$\therefore \quad 3a - 4b = 8 \qquad \text{.... (i)}$$

Let (h, k) be the centroid of the triangle ABC.

$$\therefore \quad h = \frac{3-2+a}{3} \text{ and } k = \frac{1+4+b}{3}$$

$$\Rightarrow \quad a = 3h - 1 \text{ and } b = 3k - 5$$

Substituting these values of a and $b,$ we get

$$3(3h - 1) - 4(3k\ 5) = 8$$

$$\Rightarrow \quad 3h - 4k + 3 = 0$$

Hence, the locus of the point G (h, k) is $3x - 4y + 3 = 0$

Example 9.26:(2, −2) is a point of the locus whose equation is $y^2 = ax$ If (8, b)is also a point of locus, find b.

Solution: Since (2, −2) is a point of the locus whose equation is $y^2 = ax$

$$\therefore\ (-2)^2 = 2a \quad \Rightarrow a = 2$$

$\therefore$ The equation of the locus is $y^2 = 2x$.

As (8, b) is also a point of this locus

$$\therefore \quad b^2 = 2 \times 8$$

$$\Rightarrow \quad b^2 = 16 \qquad \Rightarrow b = \pm 4$$

$\therefore$ Hence, the value of $b = \pm 2$.

EXERCISE 9.9

1. Find the locus of a point which is equidistant from the points (3, 4) and (−4, 6).
2. Find the locus of a point equidistant from the points (4,2) and the x-axis.
3. Find the equation of locus ofa point which moves so that the distance from the point (4,1) is twice its distance from the point (1,5).
4. A (2,3) and B (0,2) are the coordinates of the two vertices of a triangle. Find the locus of a point P such that the area of the triangle PAB = 3 sq. units.
5. Find the equation of the locus of a point which moves so that the sum of the squares of its distances from the point (2,3) and (−3, 4) is 16.
6. Find the locus of a point which moves such that the sum of its distances from the points (3,0) and (−3,0) is less than 9.

7. If $(h, 0)$ is a point of the locus whose equation is $x^2 + y^2 - 6x + 8y - 36 = 0$.

8. If $\left(\frac{3}{2}, -2\right)$ is a point of thelocus whose equation is $y^2 = ax$ find b if $(b, 6)$ is also a point of the locus.

9.11 TRANSFORMATION OF AXES

LEARNING OUTCOMES

A plane extends infinitely in all directions. By drawing X-axis and Y-axis, and dividing the infinite plane into four quadrants, we represent any point in the plane as an odered pair of real numbers.

It is to be noted that these axes can be chosen arbitrarily and therefore the position of these axes in the plane is not fixed. They can be changed When the position of axes is changed, the coordinates of a point also get changed correspondigly. Consequently equations of curves will also be changed. This process of transformation of axes will be of great advantage to solve some problems very easily.

The axes can be transformed or changed usually in the following ways:

(i) Translation of axes

(ii) Rotation of axes

(iii) Translation and rotation of axes.

9.11.1 Translation of Axes

9.11.1.1 Definition [Translation of Axes]

The transformation obtained, by shifting the origin to a given different point in the plane, without changing the directions of coordinate axes therein is called a translation of axes.

9.11.2 Changes in the coordinates by a translation of axes

Let $\overrightarrow{OX}, \overrightarrow{OY}$ be the given coordinate axes. Suppose the origin O is shifted to $O' = (h, k)$ by the translation of the axes $\overrightarrow{O'X'}, \overrightarrow{O'Y'}$. Let $\overrightarrow{O'X'}, \overrightarrow{O'Y'}$ be the new axes as shown in the figure below. Then with refer--ence to $\overrightarrow{O'X'}, \overrightarrow{O'Y'}$ the point O' has coordinates (0, 0).

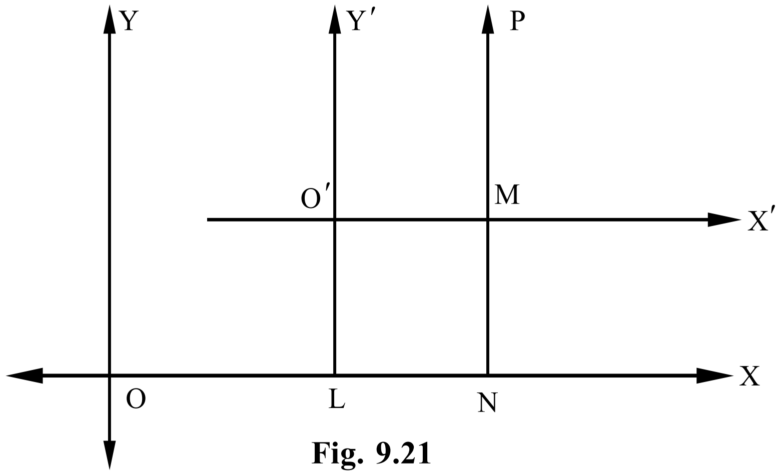

Fig. 9.21

Let P be a point with coordinates (x, y) in the system $\overrightarrow{OX}, \overrightarrow{OY}$ and with coordinates $(x'\, y')$ in the system $\overrightarrow{O'X'}, \overrightarrow{O'Y'}$.

Then $\quad O'L = k$ and $OL = h$

Now, $x = ON = OL + LN$

$= OL + O'M$

$= h + x'$

and $\quad y = PN = PM + MN$

$= PM + O'L$

$= y' + k$

Thus, $\quad x = x' + h,\ y = y' + k$

or $\quad x' = x - h,\ y' = y - h$

9.11.3 Note

If the origin is shifted to (h, k) by translation of axes, then,

(i) the coordinates of a point $P(x, y)$ are transformed as $P(x - h, y - k)$ and,

(ii) the equation $f(x, y) = 0$ of the curve is transformed as $f(x' + h, y' + k) = 0$.

9.11.4 Examples

Example 1

When the origin is shifted to $(-3, 5)$ by translation of axes, let us find the coordinates of $(2, 4)$ with respect to the new axes.

Sol : Here $(h, k) = (-3, 5)$

Let $(x, y) = (2, 4)$ be shifted to (x', y') by the translation of axes.

Then $(x', y') = (x - h, y - k) = (2 - (-3), 4 - 5)$

$$= (5, -1)$$

Example 2

When the origin is shifted to $(3, 1)$ by the translation of axes, let us find the transformed equation $x^2 + 4xy + 6y^2 = 0$.

Sol : Here $(h, k) = (3, 1)$

We get $x = x' + 3$ and $y = y' + 1$ in the given equation.

i.e., $(x' + 3)^2 + 4\,(x' + 3)\,(y' + 1) + 6(y' + 1)^2 = 0$

On simplifying the above equation, we get

$$x'^2 + 4\,x'\,y' + 6\,y'^2 + 10\,x' + 24y' + 27 = 0$$

This equation can be written (dropping dashes) as :

$$x^2 + 4xy + 6y^2 + 10x + 24y + 27 = 0.$$

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9.12 ROTATION OF AXES

9.12.1 Definition (Rotation of Axes)

The transformation obtained, by rotating both the coordinate axes in the plane by an equal angle, without changing the position of the origin is called a Rotation of axes.

9.12.2 Changes in the coordinates when the axes are Rotated Through an Angle 'θ'

Let P = (x, y) with reference to the axes $\overrightarrow{OX}, \overrightarrow{OY}$. Let the axes be rotated through an angle 'θ' in the positive direction about the origin O, to get the new system $\overrightarrow{OX'}, \overrightarrow{OY'}$ as shown in figure. With reference to the new axes $\overrightarrow{OX'}, \overrightarrow{OY'}$, and P$(x', y')$.

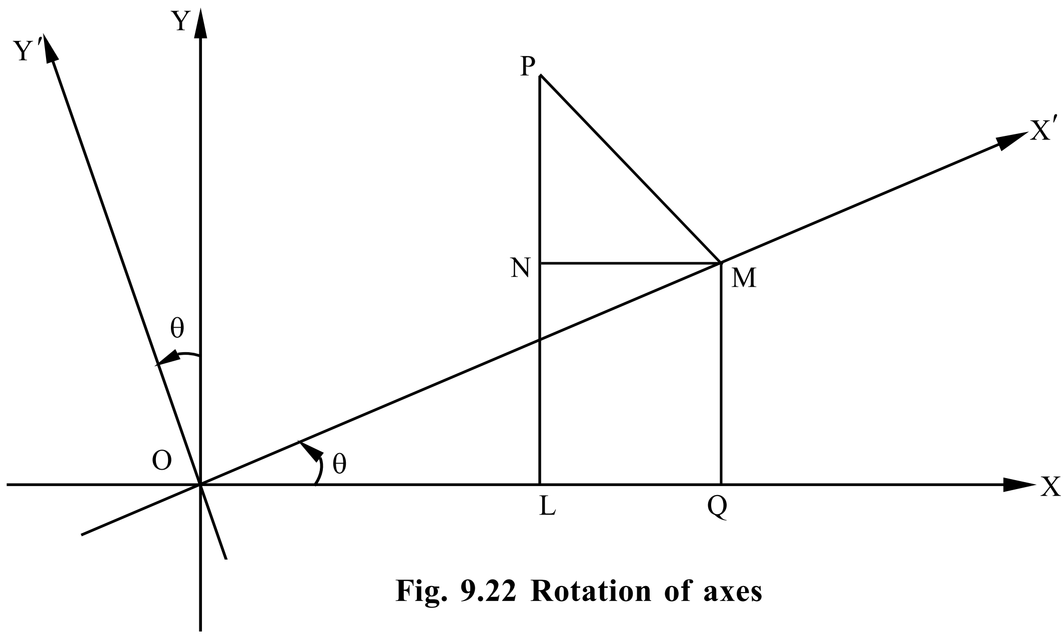

Fig. 9.22 Rotation of axes

Since the angle of rotation is 'θ', we have

$$\underline{|X}OX' = \underline{|Y}OY' = \theta$$

Let L, M be the feet of the perpendiculars drawn from P upon $\overrightarrow{OX}, \overrightarrow{OX'}$. The angle between the two straight lines is equal to the angle between their perpendiculars.

Hence, $\underline{|L}PM = \underline{|X}OX' = \theta$

Let N be the foot of the perpendicular from M to $\overline{PL}$

$$\begin{aligned} \text{Now,} \quad x &= OL = OQ - LQ \\ &= OQ - NM \\ &= OM \cos\theta - PM \sin\theta \\ &= x' \cos\theta - y' \sin\theta \qquad \text{...(1)} \end{aligned}$$

$$\begin{aligned} \text{Also} \quad y &= PL = PN + NL \\ &= PN + MQ \\ &= PM \cos\theta + OM \sin\theta \\ &= y' \cos\theta - x' \sin\theta \qquad \text{...(2)} \end{aligned}$$

Therefore

$$x = x' \cos\theta - y' \sin\theta$$

$$y = x' \sin\theta + y' \cos\theta \qquad \text{...(3)}$$

From the above equations, the values of x', y' can be found as :

$$x' = x \cos\theta + y \sin\theta$$

$$y' = -x \sin\theta + y \cos\theta \qquad \text{...(4)}$$

The results in (3) & (4) can be easily remebered by the following table

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

9.12.3 Note

When the axes are rotated through an angle 'θ' then

i) the coordinates of a point P(x, y) are transformed as

$P(x', y') = P(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$ and

ii) the equation $f(x, y) = 0$ of the curve is transformed as :

$$f(x'\cos\theta - y'\sin\theta,\ x'\sin\theta + y'\cos\theta) = 0$$

Example 3

Let us find the coordinates of P(–1, 2) with reference to the new axes, when the axes are rotated through an angle of 60^0.

Sol : Let $P(x, y) = (-1, 2)$ and (x', y') be the coordinates of P in the new system.

$$x' = (-1)\cos 60^0 + 2(\sin 60^0) = -\frac{1}{2} + 2\frac{\sqrt{3}}{2} = \frac{2\sqrt{3}-1}{2}$$

$$y' = -(-1)\sin 60^0 + 2\cos 60^0 = \frac{\sqrt{3}}{2} + 2\left(\frac{1}{2}\right) = \frac{\sqrt{3}+2}{2}$$

Therefore, the new coordinates of P are : $\left(\frac{2\sqrt{3}-1}{2}, \frac{\sqrt{3}+2}{2}\right)$

Example 4:

The origin is shifted to (1, 6) by the translation of axes. If the coordinates of the point P changes to (3, 5), find the coordinates of 'P' in the original system.

Sol : Here $(h, k) = (1, 6)$; $(x'\ y') = (3, 5)$

Let the required point to $P(x, y)$

then $x = x' + h = 3 + 1 = 4$

$y = y' + k = 5 + 6 = 11$

$\therefore$ The coordinates of P in the original system are :

$P(x, y) = P(x' + h, y' + k) = P(4, 11).$

Example 5

When the axes are rotated through an angle 60^0, the new coordinates of a point P is $(3, 4)$, then find the original coordinates of P.

Sol : Here $(x', y') = (3, 4)$; $\theta = 60^0$

Let the coordinates of P in original system be $P(x, y)$.

Then, $x = x' \cos\theta - y' \sin\theta$

$x = 3\cos 60^0 - 4\sin 60^0$

$$= \frac{3}{2} - \frac{4\sqrt{3}}{2} = \frac{3-4\sqrt{3}}{2}$$

Also $y = x' \sin\theta + y' \cos\theta$

$= 3\sin 60^0 + 4\cos 60^0$

$$= 3\left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{1}{2}\right) = \frac{3\sqrt{3}+4}{2}$$

$\therefore$ The coordinates of P in the original system

$$= \left(\frac{3-4\sqrt{3}}{2}, \frac{3\sqrt{3}+4}{2}\right)$$

Note :

1. The point to which the origin is to be shifted by the translation of axes so as to remove the first degree terms form the equation :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ where $h^2 \neq ab$ is $\left(\dfrac{hf-bg}{ab-h^2}, \dfrac{gh-af}{ab-h^2}\right)$.

2. The point to which the origin is to be shifted by the translation of axes so as to remove the first degree terms from the equation

   $ax^2 + by^2 + 2gx + 2fy + c = 0$ where $a \neq 0$, $b \neq 0$ is $\left(\dfrac{-g}{a}, \dfrac{-f}{b}\right)$.

3. The angle through which the axes are to be rotated to remove 'xy' term from the equation $ax^2 + 2hxy + by^2 = 0$, is :

$$\theta = \frac{1}{2}\,\mathrm{Tan}^{-1}\left(\frac{2h}{a-b}\right), \quad \text{if } a \neq b$$

$$= \frac{\pi}{4}, \quad \text{if } a = b.$$

EXERCISE 9.10

1. The origin is shifted to (2, 3) by the translation of axes if the coordinates of the point P changes as:

   (i) (0, 0) (ii) (4, −3)

   Also, find the coordinates of 'P' in the original system.

2. When the origin is shifted to (4,−5) by the translation of axes, find the coordinates of the following points with reference to the new axes.

   (i) (−2, 4) (ii) (3, 2) (iii) (0, 3)

3. Find the point to which the origin is to be shifted so that the point (5, −1) may change to (4, 3).

4. Find the point to which the origin is to be shifted so as to remove the first degree terms from the equation : $4x^2 + 9y^2 - 8x + 36y + 4 = 0$.

5. When the axes are rotated through an angle 30^0, find the new coordinates of the following points :

   (i) $(0, 5)$ (ii) $(-2, 4)$ (iii) $(0, 0)$

6. When the origin is shifted to the point $(2, 3)$, the transformed equation of a curve is $x^2 + 3xy - 2y^2 + 17x - 7y - 11 = 0$. Find the original equation of the curve.

7. If the point P changes to $(4, -3)$ when the axes are rotated through an angle of 135^0, find the coordinates of P with respect to the original system.

8. When the axes are rotated through an angle $\frac{\pi}{4}$, find the transformed equation of $3x^2 + 10xy + 3y^2 = 9$.

KEY WORDS

- Distance between any two points (x_1, y_1) and (x_2, y_2) is

  $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

- Coordinates of the point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio $m_1 : m_2$ are

  $\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_2y_2 + m_2y_1}{m_1 + m_2}\right)$

- Coordinates of the point dividing the line segment joining the the points (x_1, y_1) and (x_2, y_2) externally are in the ratio $m_1 : m_2$ are

  $\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_2y_2 - m_2y_1}{m - n}\right)$

- Coordinates of the mid point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

MODULE - II
Coordinate Geometry

Notes

- The area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\left|\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]\right|$$

- Three points A, B, and C are collinear if the area of the triangle formed by them is zero.
- If θ is the angle which a line makes with the positive direction of x-axis, then the slope of the line is $m = \text{Tan } \theta$.
- Slope (m) of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
- A line with the slope m_1 is parallel to the line with slope m_2 if $m_1 = m_2$.
- A line with the slope m_1 is perpendicular to the line with slope m_2 if $m_1 \times m_2 = -1$.
- If a line l (not passing through the origin) meets x- axis at A and y- axis at B then OA is called the x- intercept and OB is called the y- intercept.
- Locus of a point is the path traced by it when moving under given condition or conditions
- If the origin is shifted to (h, k) by translation of axes, then
  (i) The co-ordinates (x, y) of a point P are transformed as $(x - h, y - k)$ and
  (ii) The equation $f(x, y) = 0$ of the curve is transformed as $f(x' + h, y' + k = 0$
- If the axes are rotated through an angle 'θ' then
  (i) The coordinates (x, y) of a point P are transformed as $(x', y') = (x \cos \theta + y \sin \theta, - x \sin \theta + y \cos \theta)$ and
  (ii) The equation $f(x, y) = 0$ of the curve is transformed as
  $f(x' \cos \theta - y' \sin \theta, x' \sin \theta + y' \cos \theta) = 0.$

SUPPORTIVE WEB SITES

http : //www.wikipedia.org

http:// math world . wolfram.com

PRACTICE EXERCISE

1. Find the distance between the pairs of points:

   (a) $(2, 0)$ and $(1, \cot\theta)$ (b) $(-\sin A, \cos A)$ and $(\sin B, \cos B)$

2. Which of the following sets of points form a triangle?

   (a) $(3, 2), (-3, 2)$ and $(0, 3)$ (b) $(3, 2), (3, -2)$ and $(3, 0)$

3. Find the midpoint of the line segment joining the points $(3, -5)$ and $(-6, 8)$.

4. Find the area of the triangle whose vertices are:

   (a) $(1, 2), (-2, 3), (-3, -4)$ (b) $(c, a), (c + a, a), (c - a, a)$

5. Show that the following sets of points are collinear (by showing that area formed is 0).

   (a) $(-2, 5), (2, -3)$ and $(1, 0)$ (b) $(a, b + c), (b, c + a)$ and $(c, a + b)$

6. If $(-3, 12), (7, 6)$ and (x, a) are collinear, find x.

7. Find the area of the quadrilateral whose vertices are $(4, 3), (-5, 6), (0, 7)$ and $(3, -6)$.

8. Find the slope of the line through the points

   (a) $(1, 2), (4, 2)$ (b) $(4, -6), (-2, -5)$

9. What is the value of y so that the line passing through the points $(3, y)$ and $(2, 7)$ is parallel to the line passing through the points $(-1, 4)$ and $(0, 6)$?

10. Without using Pythagoras theorem, show that the points (4, 4), (3, 5) and $(-1, -1)$ are the vertices of a right-angled triangle.

11. Using the concept of slope, determine which of the following sets of points are collnear:

    (i) $(-2, 3)$, $(8, -5)$ and $(5, 4)$ (ii) $(5, 1)$, $(1, -1)$ and $(11, 4)$

12. If $A(2, -3)$ and $B(3, 5)$ are two vertices ofarectangle ABCD, find the slope of

    (i) BC (ii) CD (iii) DA

13. A quadrilateral has vertices at the points (7, 3), (3, 0), $(0, -4)$ and $(4, -1)$. Using slopes, show that the mid-points of the sides of the quadrilatral form a parallelogram.

14. Find the x-intercepts of the following lines:

    (i) $2x - 3y = 8$ (ii) $3x - 7y + 9 = 0$ (iii) $x - \frac{y}{2} = 3$

15. Find the equation of the locus of a point equidistant from the points (2, 4) andy-axis.

16. Find the equation of the locus of a point which is equidistant from the points $(a + b, a - b)$ and $(a - b, a + b)$.

17. Is $A(a, 0)$, $B(-a, 0)$ are two fixed points, find the locus of a point P which moves so that $3|PA| = 2|PB|$.

18. Find the equation of the locus ofa pointP if the sum of squares of its distances from (1,2) and (3,4) is 25 units.

ANSWERS

EXERCISE 9.1

1. (a) $\sqrt{58}$ (b) $\sqrt{2(a^2+b^2)}$

EXERCISE 9.2

1. (a) $\left(\frac{1}{2}, 4\right)$ (b) (2, 5) 2. (1, 4)

3. (a) (2, 6) 4. $\left(3, \frac{5}{3}\right), \left(4, \frac{1}{3}\right)$

EXERCISE 9.3

1. (a) $\frac{25}{2}$ sq. units (b) 12 sq. units (c) $\frac{a^2}{2}$ sq. units

2. $k = \frac{5}{3}$ 3. 80 sq. units 4. $\frac{41}{2}$ sq. units

EXERCISE 9.4

5. $k = 3$ 6. $k = \frac{1}{2}, -1$

EXERCISE 9.5

1. (i) $\sqrt{3}$ (ii) $-\frac{1}{\sqrt{3}}$ 2. $-\sqrt{3}$

3. $-\sqrt{3}$

EXERCISE 9.6

1. -3 2. 5 3. $-\frac{7}{3}$ 4. $\frac{5}{3}$

EXERCISE 9.7

2. $\frac{1}{3}$ 3. $\frac{14}{3}$ 6. $k = \frac{10}{3}$

EXERCISE 9.8

1. (i) x-intercept = 6 y-intercept = 2

   (ii) x-intercept = $\frac{2}{7}$ y-intercept = $\frac{2}{3}$

   (iii) x-intercept = $2a$ y-intercept = $2b$

   (iv) x-intercept = $\frac{c}{a}$ y-intercept = $\frac{c}{b}$

   (v) x-intercept = -4 y-intercept = 16

   (vi) x-intercept = $-\frac{21}{3}$ y-intercept = 21

EXERCISE 9.9

1. $14x - 4y + 27 = 0$
2. $x^2 - 8x - 4y + 20 = 0$
3. $3x^2 + 3y^2 - 38y + 87 = 0$
4. $x - 2y = 2$
5. $x^2 + y^2 + x - 7y + 11 = 0$

6. $20x^2 + 36y^2 < 405$

7. $3 \pm 3\sqrt{5}$

8. $\frac{27}{2}$

EXERCISE 9.10

1. (i) (2, 3) (ii) (6, 0)

2. (i) (−6, 9) (ii) (−1, 7) (iii) (−4, 8)

3. (1, −4)

4. (1, −2)

5. (i) $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$ (ii) $(2-\sqrt{3}, 1+2\sqrt{3})$ (iii) (0, 0)

6. $x^2 + 3xy - 2y^2 + 4x - y - 20 = 0$

7. $\left(\frac{-1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$

8. $8x^2 - 2y^2 = 9$

PRACTICE EXERCISE

1. (a) cosec θ (b) $2\sin\frac{A+B}{2}$

2. None of the given sets forms a triangle.

3. $\left(-\frac{3}{2}, \frac{3}{2}\right)$

4. (a) 11 sq. unit (b) a^2 sq. unit

6. $\frac{51-5a}{3}$ 7. 29 sq. unit

8. (a) 0 (b) $-\frac{1}{6}$

9. $y = 3$ 11. Only (ii)

12. (i) $-\frac{1}{8}$ (ii) 8 (iii) $-\frac{1}{8}$

14. (i) 4 (ii) -3 (iii) 3

15. $y^2 - 8y - 4x + 20 = 0$ 16. $x - y = 0$

17. $5x^2 + 5y^2 - 26ax + 5a^2 = 0$

18. $2x^2 + 2y^2 - 8x - 12y + 5 = 0$

Chapter

10

STRAIGHT LINES

LEARNING OUTCOMES

After studying this chapter, student will be able to :

- derive equations of a line parallel to either of the coordinate axes;
- derive equations in different forms (slope-intercept, point -slope, two point, intercept, parametric and perpendicular) of a line;
- find the equation of a line in the above forms under given conditions;
- state that the general equation of first degree represents a line;
- express the general equation of a line into
  (i) slope-intercept form (ii) intercept form and
  (iii) perpendicular form;
- derive the formula for the angle between two lines with given slopes;
- find the angle between two lines with given slopes;
- derive the conditions for parallelism and perpendicularity of two lines;
- determine whether two given lines are parallel or perpendicular;
- derive an expression for fmding the distance of a given point from a given line;
- calculate the distance of a given point from a given line;

- derive the equation of a line passing through a given point and parallel/ perpendicular to a given line;
- write the equation of a line passing through a given point and:

  (i) parallel or perpendicular to a given line

  (ii) with given x-intercept or y-intercept

  (iii) passing through the point of intersection of two lines; and
- prove various geometrical results using coordinate geometry.

PREREQUISITES

- Congruence and similarity of traingles

INTRODUCTION

In the high school mathematics, the student is familiar with basic concepts of coordinate geometry such as point in a plane distance between two points, the section formula, the area of the triangle in terms of the coordinates of its vertices, slope of line standard form equation of a line and condition for two lines parallel and perpendicular.

Now in this chapter we shall find equations of striaght line in different forms and various results relating to line in detail and try to solve problems based on the results.

10.1 STRAINGHT LINE PARALLEL TO AN AXIS

If you stand in a room with your arms stretched, we can have a line drawn on the floor parallel to one side. Another line perpendicular to this line can be drawn intersecting the first line between your legs.

In this situation the part of the line in front of you and going behind you is they-axis and the one being parallel to your arms is the x-axis.

The direction part of the y-axis in front of you is positive and behind you is negative.

The direction of the part x-axis to your right is positive and to that to your left is negative.

Now, let the side facing you be at b units away from you, then the equation of this edge will be $y = b$ (parallel to x-axis)

where b is equal in absolute value to the distance from the$x$-axis to the opposite side.

If $b > 0$, then the line lies in front of you, i.e., above the x-axis.

If $b < 0$, then the line lies behind you, i.e., below the x-axis.

If $b = 0$, then the line passes through you and is the x-axis itself.

Again, let the side of the right of you is at c units apart from you, then the equation of this line will be $x = c$ (parallel to y-axis)

where c is equal in absolute value, to the distance from the y-axis on your right.

If $c > 0$, then the line lies on the right of you, i.e., to the right of$y$-axis.

If $c < 0$, then the line lies on the left of you, i.e., to the left of y-axis

If $c = 0$, then the line passes through you and is the y-axis.

Example 10.1: Find the equation of the lines passing through (2,3) and is

(i) parallel to x-axis. (ii) parallel to y-axis .

Solution:

(i) The equation of any line parallel to x - axis is $y = b$

Since it passes through (2,3), hence $b = 3$

$\therefore$ The required equation of the line is $y = 3$

(ii) The equation of any line parallel to y-axis is $x = c$

Since it passes through (2, 3), hence $c = 2$

The required equation of the line is $x = 2$.

Example 10.2 : Find the equation of the line passing through (−2, −3) and

(i) parallel to x-axis (ii) parallel to y-axis

Solution:

(i) The equation of any line parallel to x-axis is $y = b$

Since it passes through (–2, –3), hence $-3 = b$

$\therefore$ The required equation of the line is $y = -3$

(ii) The equation of any line parallel to y-axis is $x = c$

Since it passes through (–2, –3), hence $-2 = c$

$\therefore$ The required cquation of the line *is* $x = -2$

EXERCISE 10.1

1. If we fold and press the paper then what will the crease look like
2. Find the slope of a line which makes an angle of

   a) 45° with the positive direction of x - axis.

   b) 45° with the positive direction of y - axis.

   c) 45° with the negative direction of x - axis.
3. Find the slope ofa line joining the points (2, −3) and (3,4).
4. Determine x so that the slope of the line through the points (2,5) and (7, x) is 3.
5. Find the equation of the line passing through (−3, −4) and

   a) parallel to x-axis. b) parallel to y-axis.
6. Find the equation of a line passing through (5,–3) and perpendicular to x-axis.
7. Find the equation of the line passing through (-3,–7) and perpendicular to y-axis.

10.2 DERIVATION OF THE EQUATION OF STRAIGHT LINE IN VARIOUS STANDARD FORMS

So far we have studied about the inclination, slope of a line and the lines parallel to the axes. Now the questions is, can we find a relationship between x and y, where (x, y) is any arbitrary point on the line?

The relationship between x and y which is satisfied by the co-ordinates of arbitrary point on the line is called the equation of a straight line. The equation of the line can be found in various forms under the given conditions, such as

(a) When we are given the slope of the line and its intercept on y - axis.

(b) When we are given the slope of the line and it passes through a given point.

(c) When the line passes through two given points.

(d) When we are given the intercepts on the axes by the line.

(e) When we are given the length of perpendicular from origin on the line and the angle which the perpendicualr makes with the positive direction of x-axis.

(f) When the line passes through a given point making an angle a with the positive direction of x-axis. (Parametric form).

We will discuss all the above cases one by one and try to find the equation of line in its standard forms.

(A) SLOPE-INTECEPT FORM

Let AB be a straight line making an angleθ with x-axis and cutting off an intercept OD = c from OY.

As the line makes intercept OD = c on y-axis, it is called y-intercept.

Let AB intersect OX' at T.

Take any point P(x, y) on AB. Draw PM $\perp$ OX.

The OM = x, MP = y

Draw DN $\perp$ MP

MODULE - II
Coordinate
Geometry

Notes

From the right-angled triangle DNP, we have

$$\tan\theta = \frac{NP}{DN} = \frac{MP - MN}{OM}$$

$$\tan\theta = \frac{y - OD}{OM}$$

$$\tan\theta = \frac{y - c}{x}$$

$$\therefore\ y = x\tan\theta + c$$

$$\tan\theta = m(\text{slope})$$

$$\therefore\ y = mx + c$$

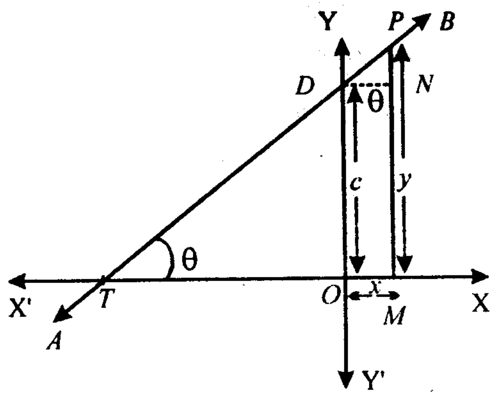

Fig. 10.1

Since, this equation is true for every point on AB, and clearly for no other point in the plane, hence it represents the equation of the line AB.

Note:

1. When $c = 0$ and $m \neq 0 \Rightarrow$ the line passes through the origin and its equation is $y = mx + c$
2. When $c = 0$ and $m = 0 \Rightarrow$ the line coincides withx- axis and its equation is of the form $y = 0$
3. When $c \neq 0$ and $m = 0 \Rightarrow$ the line is parallel to x-axis and its equation is of the form $y = c$

Example 10.3 : Find the equation of a line with slope 4 and y - intercept 0.

Solution: Putting $m = 4$ and $c = 0$ in the slope intercept form of the equation, we get $y = 4x$

This is the desired equation of the line.

Example 10.4 : Detennine the slope and the y-intercept of the line whose equation is $8x + 3y = 5$.

Solution : The given equation of the line is $8x + 3y = 5$

or, $$y = -\frac{8}{3}x + \frac{5}{3}$$

Comparing this equation with the equation $y = mx + c$ (Slope intercept fonn) we get

$$m = -\frac{8}{3} \text{ and } c = \frac{5}{3}$$

Therefore, slope of the line is $-\frac{8}{3}$ and its

y-intercept is $\frac{5}{3}$.

Example 10.5 : Find the equation of the line cutting off an intercept of length 2 from the negative direction of the axis of y and making an angle of 120^0 with the positive direction x-axis.

Solution: From the slope intercept fonn of the line

$$\therefore \quad y = x\tan 120^0 + (-2)$$

$$= -\sqrt{3}\,x - 2$$

or $\quad y + \sqrt{3}\,x + 2 = 0$

Here $m = \tan 120^0$ and $c = -2$, because the intercept is cut on the negative side of y-axis.

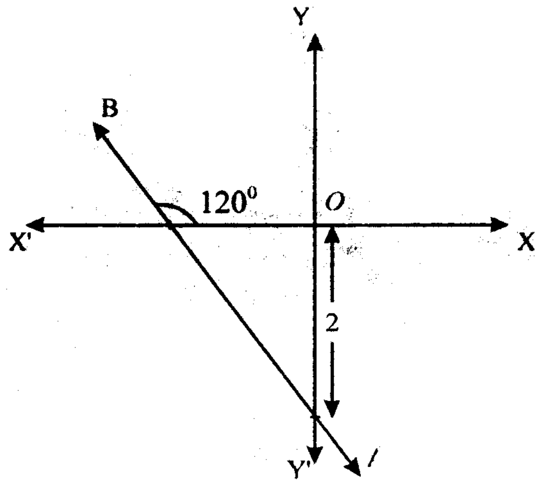

Fig. 10.2

(b) **POINT -SLOPE FORM**

Here we will find the equation of a line passing through a given point $A(x_1, y_1)$ and having the slope m.

Let $P(x, y)$ be any point other than A on given the line. Slope ($\tan\theta$) of the line joining $A(x_1, y_1)$ and $P(x,y)$ is given by

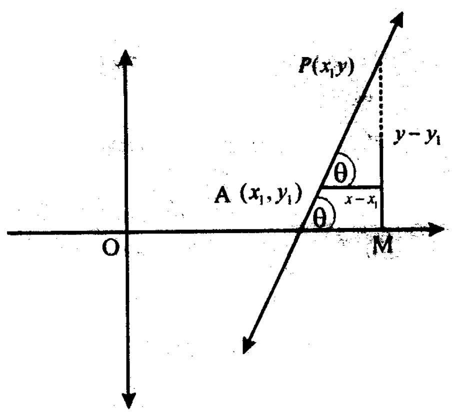

Fig. 10.3

$$m = \tan\theta = \frac{y - y_1}{x - x_1}$$

The slope of the *lineAP* is given to be *m*.

$$\therefore\ m = \frac{y - y_1}{x - x_1}$$

$\therefore$ The equation of the required line is

$$y - y_1 = m(x - x_1)$$

Example 10.6 : Determine the equation of the line passing through the point (2, –1) and having slope $\frac{2}{3}$.

Solution : Putting $x_1 = 2,\ y_1 = -1$ and $m = \frac{2}{3}$ in the equation of the point-slope form of the line we get

$$y - (-1) = \frac{2}{3}(x - 2)$$

$$\Rightarrow\ y + 1 = \frac{2}{3}(x - 2)$$

$$y = \frac{2}{3}x - \frac{7}{3}$$

$\therefore$ which is the required equation of the line.

(C) TWO POINT FORM

Let $A(x_1, y_1)$ and $B(x_2, y_2)$be two given distinct points.

Slope of the line passing through these points is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}\ (x_2 \neq x_1)$$

From the equation of line in point slope form, we get

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}\ (x - x_1)$$

which is the required equation of the line in two-point form.

Example 10.7 : Find the equation of the line passing through (3, −7) and (−2, −5).

Solution : The equation of a line passing through two points (x_1, y_1) and (x_2, y_2)

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{...(i)}$$

Since $x_1 = 3; y_1 = -7$, and $x_2 = -2, y_2 = -5$equation (i) becomes,

$$y + 7 = \frac{-5+7}{-2-3}(x-3)$$

or $$y + 7 = \frac{2}{-5}(x-3)$$

or $$2x + 5y + 29 = 0$$

(d) INTERCEPT FORM

We want to find the equation of a line which cuts off given intercepts on both the co-ordinate axes

Let PQ be a line meeting x-axis in A and y-axis in B. Let OA = a, OB = b.

Then the co-ordinates of A and B are $(a,0)$ and $(0, b)$ respectively.

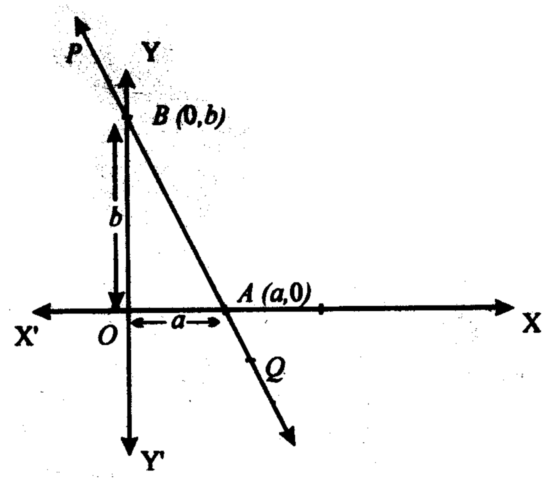

Fig. 10.4

The equation of the line joining A and B is

$$y - 0 = \frac{b-0}{0-a}(x-a)$$

or $$y = \frac{-b}{a}(x-a) \quad \text{or} \quad \frac{y}{b} = \frac{-x}{a} + 1$$

or $\frac{x}{a}+\frac{y}{b}=1$

This is the required equation of the line having intercepts a and b on the axes.

Example 10.8: Find the equation of a line which cuts off intercepts 5 and −3 on x and y axes respectively.

Solution: The intercepts are 5 and −3 on x and y axes respectively.

i.e., $a = 5$, $b = -3$

The required equation of the line is

$$\frac{x}{5}+\frac{y}{-3}=1$$

$$3x - 5y - 15 = 0$$

Example 10.9 : Find the equation of a line which passes through the point (3, 4) and makes intercepts on the axes equl in magnitude but opposite in sign.

Solution : Let the x-intercept and y-intercept be a and $-a$ respectively

$\therefore$ The equation of the line is

$\frac{x}{a}+\frac{y}{-a}=1$

$x - y = a$ (i)

Since (i) passes through (3, 4)

$\therefore\quad 3 - 4 = a$

or $a = -1$

Thus, the required equation of the line is

$$x - y = -1$$

or $\quad x - y + 1 = 0.$

Example 10.10 : Determine the equation of the line through the point (−1, 1) and parallel to x - axis.

Solution: Since the line is parallel to x-axis its slope ia zero. Therefore from the point slope form of the equation, we get

$$y - 1 = 0[x - (-1)]$$

$$y - 1 = 0$$

which is the required equation of the given line.

Example 10.11: Find the intercepts made by the line $3x - 2y + 12 = 0$ on the coordinate axes.

Solution: Equation of the given line is $3x - 2y = -12$

Dividing by -12, we get

$$\frac{x}{-4} + \frac{y}{6} = 1$$

Comparing it with the standard equation of the line in intercept form, we fmd $a = -4$; and $b = 6$. Hence the intercepts on the x-axis and y-axis repectively are – 4. and 6.

Example 10.12 : The segment of a line, intercepted between the coordinate axes is bisected at the point P(x_1, y_1). Find the equation of the line

Solution : Let P(x_1, y_1) be the middle point or the segment *CD* of the *line AB* intercepted between the axes. Draw PM $\perp$OX

$\therefore$ OM = x_1 and MP = y_1

$\therefore$ OC = $2x_1$ and OD = $2y_1$

Now, from the intercept form of the line

$$\frac{x}{2x_1} + \frac{y}{2y_1} = 1$$

or, $$\frac{x}{x_1} + \frac{y}{y_1} = 2.$$

which is the required equation of the line.

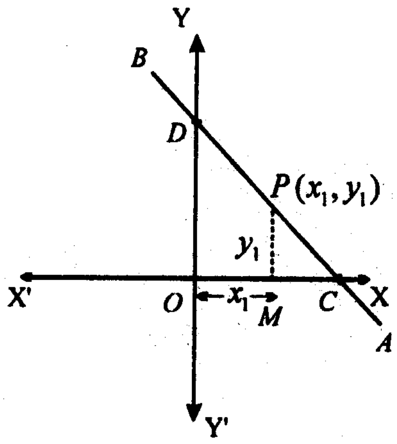

Fig. 10.5

(e) PERPENDICULAR FORM (NORMAL FORM)

We now derive the equation of a line when p be the length of perpendicular from the origin on the line and α, the angle which this perpendicular makes with the positive direction of x - axis are given.

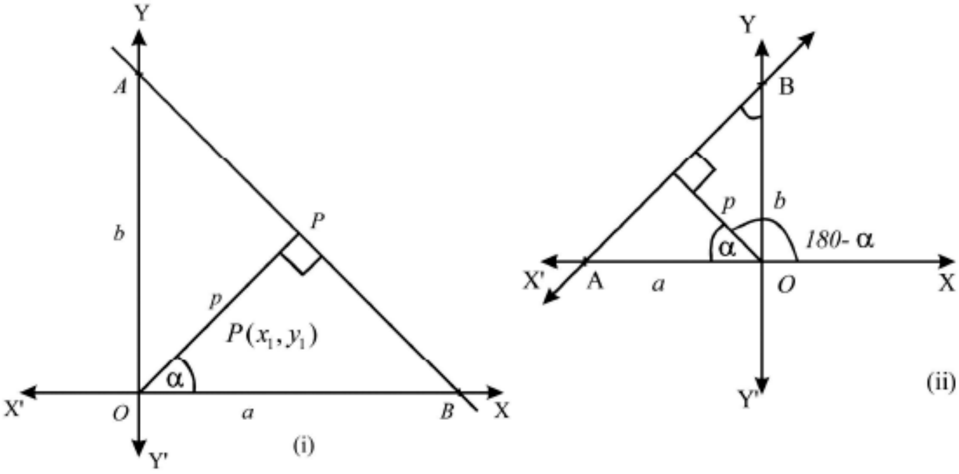

Fig. 10.6

(i) LetA B be the given line cutting off intercepts a and b on x-axis and y-axis respectively. Let OP be perpendicular from origin O on AB and $\angle POB = \alpha$ (See Fig. 10.6 (i))

$$\therefore \frac{p}{a} = \cos\alpha \Rightarrow a = p\sec\alpha$$

and $\frac{p}{b} = \sin\alpha \Rightarrow b = p\,\text{cosec}\,\alpha$

$\therefore$ The equation of line AB is

$$\frac{x}{p\sec\alpha} + \frac{y}{p\,\text{cosec}\,\alpha} = 1$$

or $x\cos\alpha + y\,\text{cosec}\,\alpha = p.$

(ii) $\frac{p}{a} = \cos(180-\alpha) = -\cos\alpha$ (From Fig. 10.6(ii))

$\Rightarrow a = -p\sec\alpha$

similary, $b = p\,\text{cosec}\,\alpha.$

$\therefore$ The equation of line AB is $\frac{x}{-a}+\frac{y}{b}=1$

$$x \cos \alpha + y \sin \alpha = p.$$

Note: (1) p is the length of perpendicular from the origin on the line and in always taken to be positive.

(2) α is the angle between positive direction of x-axis and the line perpendicular from the origin to the given line.

Example 10.13: Determine the equation of the line with $\alpha = 135^0$ and perpendicular distance $p = \sqrt{2}$ from the origin.

Solution: From the standard equation of the line in normal form

$$x \cos 135^0 + y \sin 135^0 = \sqrt{2}$$

or $$\frac{-x}{\sqrt{2}}+\frac{y}{\sqrt{2}}=\sqrt{2}$$

or $$-x + y - 2 = 0$$

or $$x - y - 2 = 0$$

which is the required equation of the straight line.

Example 10.14 : Find the equation of the line whose perpendicular distance from the origin is 6 units and the perpendicular from the origin to line makes an angle of 30° with the positive direction of x-axis.

Solution: Here $\alpha = 30^0, \quad p = 6$

$\therefore$ The equation of the line is

$$x \cos 30^0 + y \sin 30^0 = 6$$

or, $$x\left(\frac{\sqrt{3}}{2}\right)+y\left(\frac{1}{2}\right)=6$$

or $$\sqrt{3}x+y=12$$

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(F) PARAMETRIC FORM

We now want to find the equation of a line through a given point Q(x_1, y_1) which makes an angle α with the positive direction of x - axis in the form

$$\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}=r$$

where r is the distance of any point P (x, y) on the line from Q(x_1, y_1).

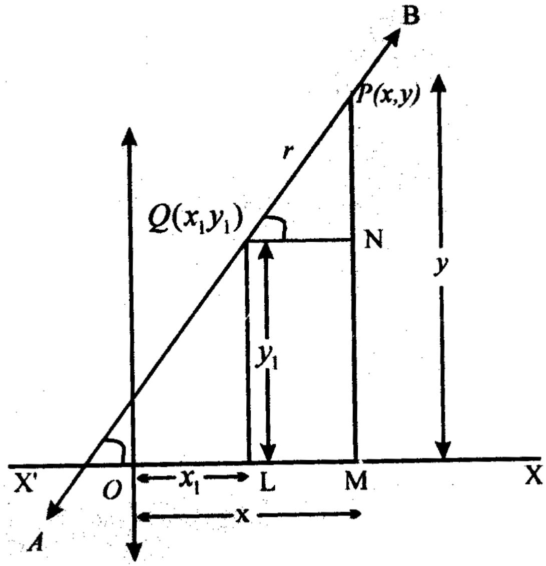

Fig. 10.7

Let AB be a line passing through a given point Q(x_1, y_1) making an angle a with the positive direction of x-axis.

Let P (x,y) be any point on the line such that QP = r (say)

Draw QL, PM perpendiculars to x-axis and draw QN $\perp$ PM

From right angled ΔPNQ

$$\cos\alpha=\frac{QN}{QP},\ \sin\alpha=\frac{PN}{QP}$$

But QN = LM = OM − OL = $x - x_1$,

and PN = PM − MN = PM − QL = $y - y_1$

$$\therefore \cos\alpha=\frac{x-x_1}{r},\ \sin\alpha=\frac{y-y_1}{r}$$

or $$\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}=r$$

$\therefore$ which is the required equation of the line.

Note:

1. The co-ordinates of any point on this line can be written as ($x_1 + r\cos\alpha$, $y_1 + r\sin\alpha$). Clearly coordinates of the point depend on the value of r. This variable r is called parameter.

2. The equation $x = x_1 + r \cos \alpha$, $y = y_1 + r \sin \alpha$ are called the parametric equations of the line.

3. The value of 'r' is positive for all points lying on one side of the given point and negative for all points lying on the other side of the given point.

Example 10.15 : Find the equation of a line passing through A (–1, –2) and making an angle of 30° with the positive direction of x-axis, in the parametric form. Also find the coordinates of a point P on it at a distance of 2 units from the point A.

Solution: Here $x_1 = -1$; $y_1 = -2$ and $\alpha = 30^0$

$\therefore$ The equation of the line is

$$\frac{x+1}{\cos 30^0} = \frac{y+2}{\sin 30^0}$$

or, $$\frac{x+1}{\frac{\sqrt{3}}{2}} = \frac{y+2}{\frac{1}{2}}$$

Since AP = 2, $r = 2$

$$\therefore \quad \frac{x+1}{\frac{\sqrt{3}}{2}} = \frac{y+2}{\frac{1}{2}} = 2$$

$$\Rightarrow x = \sqrt{3} - 1, \; y = -1$$

Thus the coordinates of the point P are $\left(\sqrt{3} - 1, -1\right)$.

Example 10.16: Find the distance of the point (1,2) from the line $2x - 3y + 9 = 0$ measured along a line making an angle of 45° with x - axis.

Solution : The equation of any line through A (1,2) making an angle of 45^0 with x-axis is

$$\frac{x-1}{\cos 45^0} = \frac{y-2}{\sin 45^0} = r$$

$$\frac{x-1}{\frac{1}{\sqrt{2}}}=\frac{y-2}{\frac{1}{\sqrt{2}}}=r$$

Any point on it is $\left(1+\frac{r}{\sqrt{2}},\ 2+\frac{r}{\sqrt{2}}\right)$

It lies on the given line $2x - 3y + 9 = 0$

if $\quad 2\left(1+\frac{r}{\sqrt{2}}\right)-3\left(2+\frac{r}{\sqrt{2}}\right)+9=0$

or, $\quad 2-6+9-\frac{r}{\sqrt{2}}=0$

or, $\quad r = 5\sqrt{2}$

Hence the required distance is $5\sqrt{2}$.

EXERCISE 10.2

1. (a) Find the equation of a line with slope 2 and y - intercept is equal to -2.

   (b) Determine the slope and the intercepts made by the line on the axes whose equation is $4x + 3y = 6$.

2. Find the equation of the line cutting off an interecept $\frac{1}{\sqrt{3}}$ on negative direction of axis of y and inclined at 120^0 to the positive direction of x-axis.

3 Find the slope and y - intercept of the line whose equation is $3x - 6y = 12$

4. Determine the equation of the line passing through the point $(-7, 4)$ and having the slope $-\frac{3}{7}$.

5. Determine the equation of the line passing through the point (1, 2) which makes equal angles with the two axes.

6. Find the equation of the line passing through (2, 3) and parallel to the line joining the points (2, −2) and (6,4).

7. (a) Determine the equation of the line through (3, −4) and (−4,3).

   (b) Find the equation of the diagonals of the rectangle ABCD whose vertices are A (3,2), B (11, 8), C (8, 12) and D (0, 6).

8. Find the equation of the medians of a triangle whose vertices are (2,0), (0, 2) and (4,6).

9. Find the equation of the line which cuts off intercepts of length 3 units and 2 units on *x-axis* and y-axis respectively.

10. Find the equation of a line such that the segment between the coordinate axes has its mid point at the point (1,3).

11. Find the equation of a line which passes through the point (3, −2) and cuts off positive intercepts on x and y axes in the ratio of 4 : 3.

12. Determine the equation of the line whose perpendicular from the origin is of length 2 units and makes an angle of 45° with the positwe direction of x-axis.

13. If p is the length of the perpendicular segment from the origin, on the line whose intercepts on the axes are a and b, then show that $\frac{1}{P^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

14. Find the equation of a line passing through A (2, 1) and making an angle 45° with the positive direction of x-axis in parametric form. Also find the coordinates of a point P on it at a distance of 1 unit from the point A.

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10.3 GENERAL EQUATION OF FIRST DEGREE

You know that *a* linear equation in two variables x and y is given by

$$Ax + By + C = 0 \qquad \text{... (1)}$$

In order to understand its graphical representation, we need to take the following three cases.

Case 1: (When both A and B are equal to zero)

In this case C is automaticaly zero and the equation does not exist.

Case 2: (when $A = 0$ and $B \neq 0$)

In this case the equation (1) becomes $By + C = 0$

or $y = -\frac{C}{B}$ and is satisfied by all points lying on a line which is parallel to x-axis and the y-coordinate of every point on the line is $-\frac{C}{B}$. Hence this is the equation of a straight line. The case where $B = 0$ and $A \neq 0$ can be treated similarly.

Case 3: (When $A \neq 0$ and $B \neq 0$)

We can solve the equation (1) for y and obtain.

$$y = -\frac{A}{B}x - \frac{C}{B}$$

Clearly, this represents a straight line with slope $-\frac{A}{B}$ and y-intercept equal to $-\frac{C}{B}$.

10.3.1 CONVERSION OF GENERAL EQUATION OF A LINE INTO VARIOUS FORMS

If we are given the general equation of a line, in the form $Ax + By + C = 0$, we will see how this can be converted into various forms studied before.

10.3.2 CONVERSION INTO SLOPE-INTERCEPT FORM

We are given a first degree equation in x and y as $Ax + By + C = 0$

Are you able to find slope and y - intercept?

Yes, indeed, if we are able to put the general equation in slope-intercept form. For this purpose, let us re-arrange the given equation as.

$Ax + By + C = 0$ as

$By = -Ax - C$

or $y = -\frac{A}{B}x - \frac{C}{B}$ (Provided $B \neq 0$)

which is the required form. Hence, the slope $= -\frac{A}{B}$, y-intercept $= -\frac{C}{B}$.

Example 10.17: Reduce the equation $x + 7y - 4 = 0$ to the slope - intercept form.

Solution: The given equation is

$$x + 7y - 4 = 0$$

or $7y = -x + 4$

or $y = -\frac{1}{7}x + \frac{4}{7}$

which is the converted form of the given equation in slope-intercept form

Example 10.18 : Find the slope and y intercept of the line $x + 4y - 3 = 0$.

Solution : The given equation is

$$x + 4y - 3 = 0$$

or $4y = -x + 3$

or $y = -\frac{1}{4}x + \frac{3}{4}$

Comparing it with slope-intercept form, we have

slope $= -\frac{1}{4}$, y - intercept $= \frac{3}{4}$.

10.3.3 CONVERSION INTO INTERCEPT FORM

Suppose the given first degree equation in x and y is

$$Ax + By + C = 0 \quad ...(i)$$

In order to convert (i) in intercept form, we re arrange it as

$$Ax + By = -C$$

or $$\frac{Ax}{-C}+\frac{By}{-C}=1$$

or $$\frac{x}{\left(-\frac{C}{A}\right)}+\frac{y}{\left(-\frac{C}{B}\right)}=1 \quad \text{(Provided } A \neq 0 \text{ and } B \neq 0)$$

which is the requied converted form. It may be noted that intercept on x - axis $x = -\frac{C}{A}$ and intercept on y-axis $= -\frac{C}{B}$.

Example 10.19: Reduce $3x + 5y = 7$ into the intercept form and find its intercepts on the axes.

Solution: The given equation is

$$3x + 5y = 7$$

or, $$\frac{3}{7}x+\frac{5}{7}y=1$$

or $$\frac{x}{\left(\frac{7}{3}\right)}+\frac{y}{\left(\frac{7}{5}\right)}=1$$

$\therefore$ The x- intercept $= \frac{7}{3}$ and, y - intercept $= \frac{7}{5}$.

Example 10.20 : Find x andy-intercepts for the line $3x - 2y = 5$.

Solution: The given equation is

$$3x - 2y = 5$$

or $$\frac{3}{5}x-\frac{2}{5}y=1$$

or $$\frac{x}{\left(\frac{5}{3}\right)}+\frac{y}{\left(-\frac{5}{2}\right)}=1$$

Thus, the required x - intercept $= \frac{5}{3}$

and y-intercept $= -\frac{5}{2}$.

10.3.4 CONVERSION INTO PERPENDICULAR FORM

Let the general first degree equation in x and y be

$$Ax + By + C = 0 \qquad ...(i)$$

We will convert this general equation in perpendicular form. For this purpose let us re-write the given equation (i) as $Ax + By = -C$

Multiplying both sides of the above equation by λ, we have

$$\lambda Ax + \lambda By = -\lambda C \qquad ...(ii)$$

Let us choose λ such that $(\lambda A)^2 + (\lambda B)^2 = 1$

or $\lambda = \dfrac{1}{\sqrt{A^2+B^2}}$ (Taking positive sign)

Substituting this value of λ in (ii), we have

$$\frac{Ax}{\sqrt{A^2+B^2}} + \frac{By}{\sqrt{A^2+B^2}} = -\frac{C}{\sqrt{A^2+B^2}} \qquad ...(iii)$$

This is required conversion of (i) in perpendicular form. Two cases arise according as Cis negative or positive.

(i) If $C < 0$ the equation (ii) is the required form.

(ii) If $C > 0$ the R. H. S. of the equation of (iii) is negative.

$\therefore$ We shall multiply both sides of the equation of (iii) by -1.

$\therefore$ The required form will be

$$-\frac{Ax}{\sqrt{A^2+B^2}} - \frac{By}{\sqrt{A^2+B^2}} = -\frac{C}{\sqrt{A^2+B^2}}$$

Thus, length of perpendicular from the origin $= \dfrac{|C|}{\sqrt{A^2+B^2}}$

Inclination of the perpendicular with the positwe direction of x -axis is given by $\cos\theta = \mp\dfrac{A}{\sqrt{A^2+B^2}}$

or $$\sin\theta = \left(\mp\frac{B}{\sqrt{A^2+B^2}}\right)$$

where the upper sign is taken for $C > 0$ and the lower sign for $C < 0$. If $C = 0$, the line passes through the origin and there is no perpendicular from the origin on the line.

With the help of the above three cases, we are able to say that

"The general equation of first degree in x and y always represents a straight line provided A and B are not both zero simultaneously. "

Is the converse of the above statement true? *The converse of the above statement is that* every straight line can be expressed as a general equation of first degree in x and y.

In this lesson we have studied about the various forms of equation of straight line. For example, let us take some of them as $y = mx + c$, $\frac{x}{a}+\frac{y}{b}=1$ and $x\cos\alpha + y\sin\alpha = p$. Obviously, all are linear equations in x and y. We can re-arrange them as $y - mx - c = 0$, $bx + ay - ab = 0$ and $x\cos\alpha + y\sin\alpha - p=0$ respectively. Clearly, these equations are nothing but a different arrangement of general equation of first degree in x and y. Thus, we have established that

"Every straight line can be expressed as a general equation of first degree in x and y".

Example 10.21: Reduce the equation $x+\sqrt{3}y+7=0$ into perpendicular form.

Solution: The equation of given line is $x+\sqrt{3}y+7=0$ ...(i)

Comparing (i) with general equation of straight line, we have

$A = 1$ and $B = \sqrt{3}$

$$\therefore \quad \sqrt{A^2+B^2} = 2$$

Dividing equation (i) by 2, we have

$$\frac{x}{2}+\frac{\sqrt{3}}{2}y+\frac{7}{2}=0$$

or $$\left(-\frac{1}{2}\right)x+\left(-\frac{\sqrt{3}}{2}\right)y-\frac{7}{2}=0$$

or $$x\cos\frac{4\pi}{3}+y\sin\frac{4\pi}{3}=\frac{7}{2}$$

(cos θ and sin θ being both negative in the third quadrant, value of θ will lie in the third quadrant). This is the representation of the given line in perpendicular form.

Example 10.22 : Find the perpendicular distance from the origin on the line $\sqrt{3}x-y+2=0$ Also, find the inclination of the perpendicular from the origin.

Solution: The given equation is $\sqrt{3}x-y+2=0$

Dividing both sides by $\sqrt{(\sqrt{3})^2+(-1)^2}$ or 2, we have

$$\frac{\sqrt{3}}{2}x-\frac{1}{2}y+1=0$$

or $$\frac{\sqrt{3}}{2}x-\frac{1}{2}y=-1$$

Multiplying both sides by -1, we have

$$-\frac{\sqrt{3}}{2}x+\frac{1}{2}y=1$$

or $$x\cos\frac{5\pi}{6}+y\sin\frac{5\pi}{6}=1$$

(cos θ is -ve in second quadrant and sin θ is +ve in second quadrant, so value ofe lies in the second quadrant).

Thus, inclination of the perpendicular from the origin is 1500 and its length is equal to 1.

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Example 10.23 : Find the equation of a line which passess through the point (3,1) and bisects the portion of the line $3x + 4y = 12$ intercepted between coordinate axes.

Solution : First we find the intercepts on coordinate axes cut off by the line whose equation is from $3x + 4y = 12$

or $$\frac{3x}{12}+\frac{4y}{12}=1$$

or $$\frac{x}{4}+\frac{y}{3}=1$$

Hence, intercepts on x-axis and y-axis are 4 and 3 respectively.

Thus, the coordinates of the points where the line meets the coordinate axes are A (4,0) and B (0, 3).

$\therefore$ Mid -point of AB is $\left(2, \frac{3}{2}\right)$

Hence the equation of the line through (3, 1) and $\left(2, \frac{3}{2}\right)$ is

$$y-y_1 = \frac{\frac{3}{2}-1}{2-3}\ (x-3)$$

or $$y-1=-\frac{1}{2}\ (x-3)$$

or $$2(y-1)+(x-3)=0$$

or $$2y-2+x-3=0$$

or $$x+2y-5=0$$

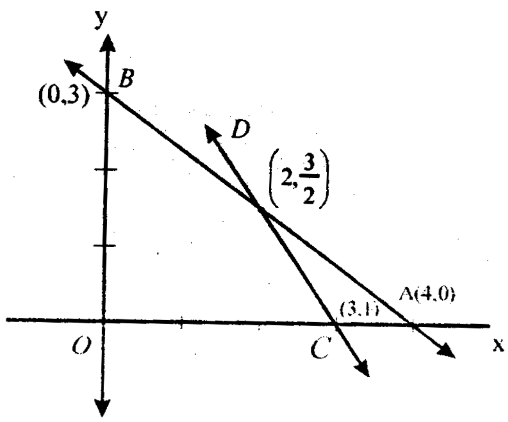

Fig. 10.8

Example 10.24 : Prove that the line through (8, 7) and (6, 9) cuts off equal intercepts on coordinate axes.

Solution: The equation of the line passing through (8, 7) and (6, 9) is

$$y-7=\frac{9-7}{6-8}\ (x-8)$$

or $\quad y - 7 = -(x - 8)$

or $\quad x + y = 15$

or $\quad \dfrac{x}{15} + \dfrac{y}{15} = 1$

Hence, intercepts on both axes are 15 each.

Example 10.25 : Find the ratio in which the line joining (−5, 1) and (1, −3) divides the join of (3,4) and (7, 8).

Solution : The equation of the line joining C(−5, 1) and D(1, −3) is

$$y - 1 = \frac{-3-1}{1+5}\ (x+5)$$

or $$y - 1 = \frac{-4}{6}\ (x+5)$$

or $\quad 3y - 3 = -2x - 10$

or $\quad 2y + 3y + 7 = 0 \qquad ...(i)$

Let line (i) divide the join of A (3,4) andB (7,8) at the point P.

If the required ratio is λ : 1 in which line (i) divides the join of A(3,4) and B(7,8), then the coordinates of P are

$$\left(\frac{7\lambda+3}{\lambda+1}, \frac{8\lambda+4}{\lambda+1}\right)$$

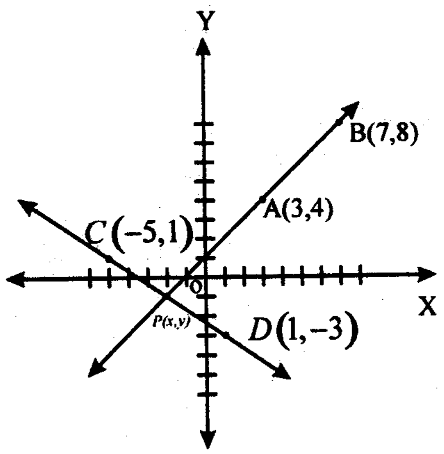

Fig. 10.9

Since P lies on the line (i), we have

$$2\left(\frac{7\lambda+3}{\lambda+1}\right) + 3\left(\frac{8\lambda+4}{\lambda+1}\right) + 7 = 0$$

$\Rightarrow\ 14\lambda + 6 + 24\lambda + 12 + 7\lambda + 7 = 0$

$\Rightarrow\ 45\lambda + 25 = 0 \Rightarrow\ \lambda = -\dfrac{5}{9}$

∴ Hence, the line joining (−5, 1) and (1, −3) divides the join of (3, 4) and (7,8) externallyin the ratio 5 : 9.

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10.3.5 Intersection of two straight lines.

Theorem: If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ represents two intersecting lines, then their point of intersection is

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

Proof: Consider the straight lines $a_1x + b_1y + c_1 = 0$ ...(1)

$$a_2x + b_2y + c_2 = 0 \qquad ...(2)$$

Since lines intersect, slopes are not equal.

i.e., $\frac{-a_1}{b_1} \neq \frac{-a_2}{b_2} \Rightarrow a_1b_2 - a_2b_1 \neq 0$

Let (α, β) be the intersecting point of (1), (2)

then $a_1\alpha + b_1\beta + c_1 = 0$ ...(3)

$a_2\alpha + b_2\beta + c_2 = 0$ ...(4)

Applying the rule of cross - multiplication to (3) and (4) we obtain.

$$\begin{array}{cccc} \alpha & \beta & 1 & \\ b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$

$$\Rightarrow \frac{\alpha}{b_1c_2 - b_2c_1} = \frac{\beta}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \alpha = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad \beta = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

$\therefore$ Intersecting point $(\alpha, \beta) = \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$

10.3.6 Theorem : The ratio in which the straight line ax + by + c = 0 divides the line segment joining the points

$A(x_1, y_1), B(x_2, y_2)$ is $-(ax_1 + by_1 + c) : (ax_2 + by_2 + c)$

Proof: Let the straight line

$ax + by + c = 0$ divides $\overline{AB}$

in the radio $l : m$ at P

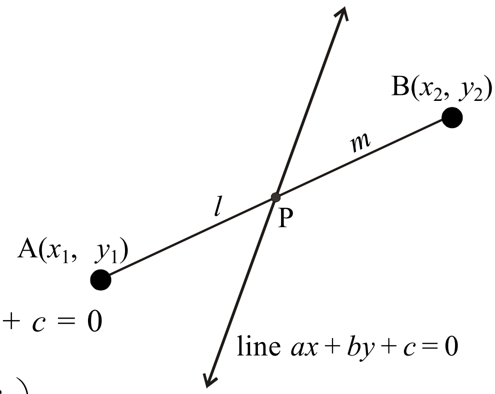

Fig. 10.10

$$P = \left(\frac{lx_2 + mx_1}{l+m}, \frac{ly_2 + my_1}{l+m}\right)$$

P is a point on a line $ax + by + c = 0$

$$\therefore\ a\left(\frac{lx_2 + mx_1}{l+m}\right) + b\left(\frac{ly_2 + my_1}{l+m}\right) + c = 0$$

i.e., $\quad a(lx_2 + mx_1) + b(ly_2 + my_1) + c(l + m) = 0$

$$l(ax_2 + by_2 + c) + m(ax_1 + by_1 + c) = 0$$

Hence $l : m = -(ax_1 + by_1 + c) : (ax_2 + by_2 + c)$

Note :

(i) X-axis devides $\overline{AB}$ in the ratio - $y_1 : y_2$

(ii) Y-axis devides $\overline{AB}$ in the ratio - $x_1 : x_2$

10.3.7 Intersecting lines (or) concurrent lines

If three or more lines have only one common point, then the lines are called intersecting lines (or) concurrent lines.

The common point is called intersecting point (or) concurrent point.

10.3.7(1) Condition for Concurrent lines

Theorem : Let $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3$ be three straight lines, no two of which are parallel.

Then these lines are concurrent iff.

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

Proof : Given lines are $\quad a_1x + b_1y + c_1 = 0 \quad$ ...(1)

$$a_2x + b_2y + c_2 = 0 \quad \text{...(2)}$$

$$a_3x + b_3y + c_3 = 0 \quad \text{...(3)}$$

Point of intersection of (2) and (3) is

$$P\left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}, \frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}\right)$$

Given lines are concurrent.

$\Leftrightarrow$ The point P lines on (1)

$$\Leftrightarrow a_1\left(\frac{b_2c_3 - b_3c_2}{a_2b_3 - a_3b_2}\right) + b_1\left(\frac{c_2a_3 - c_3a_2}{a_2b_3 - a_3b_2}\right) + c_1 = 0$$

$$\Leftrightarrow a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$$

$$\Leftrightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

10.3.8 Family of straight lines

Theorem : Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ represent two intersecting lines. Then

(i) The equation $\lambda_1 L_1 + \lambda_2 L_2 = 0$ for parametric values of λ_1 and λ_2 with $\lambda_1^2 + \lambda_2^2 \neq 0$ represents a family of lines passing through the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$.

(ii) Conversely, the equation of any straight line passing through the point of intersection of the given straight lines is of the form $\lambda_1 L_1 + \lambda_2 L_2 = 0$ for some real λ_1 and λ_2 such that $\lambda_1^2 + \lambda_2^2 \neq 0$.

10.3.9 Concurrent lines - properties related to a triangle.

10.3.9(1) Theorem : The medians of a triangle are concurrent.

Proof : Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC.

Let D, E, F mid points of the sides $\overline{BC}, \overline{CA}, \overline{AB}$ respectively.

$$\therefore\ D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$

$$E = \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$$

$$F = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

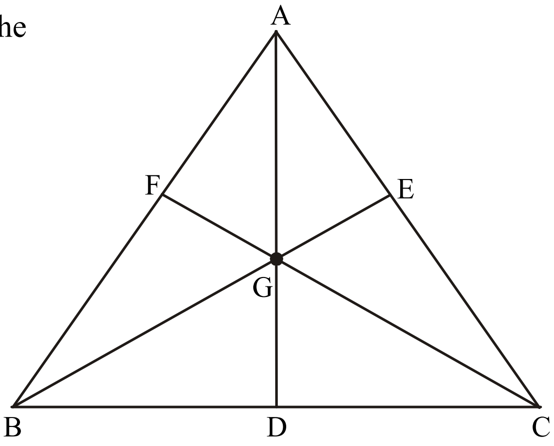

Fig. 10.11

Slope of $\overleftrightarrow{AD}$ is $= \dfrac{y_2+y_3-2y_1}{x_2+x_3-2x_1}$

Equation of $\overleftrightarrow{AD}$ is $y - y_1 = \dfrac{y_2+y_3-2y_1}{x_2+x_3-2x_1}(x - x_1)$

$\Rightarrow (y - y_1)(x_2 + x_3 - 2x_1) = (y_2 + y_3 - 2y_1)(x - x_1)$

$\Rightarrow L_1 \equiv (x - x_1)(y_2 + y_3 - 2y_1) - (y - y_1)(x_2 + x_3 - 2x_1) = 0$

Similarly equations of $\overleftrightarrow{BE}, \overleftrightarrow{CF}$ are L_2, L_3

$L_2 \equiv (x - x_2)(y_3 + y_1 - 2y_2) - (y - y_2)(x_3 + x_1 - 2x_2) = 0$

$L_3 \equiv (x - x_3)(y_1 + y_2 - 2y_3) - (y - y_3)(x_1 + x_2 - 2x_3) = 0$

Hence $1.L_1 + 1.L_2 + 1.L_3 = 0$

$\Rightarrow \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$ Here $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$\Rightarrow$ Medians $L_1 \equiv 0,\ L_2 \equiv 0,\ L_3 \equiv 0$ are concurrent.

The concurrent point G is called centroid of triangle ABC.

10.3.9(2)

Theorem : The altitudes of a triangle are concurrent.

Proof: Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are vertias of the triangle.

Let $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ be the altitudes of the triangle ABC drawn from the vertices A, B and C respectively.

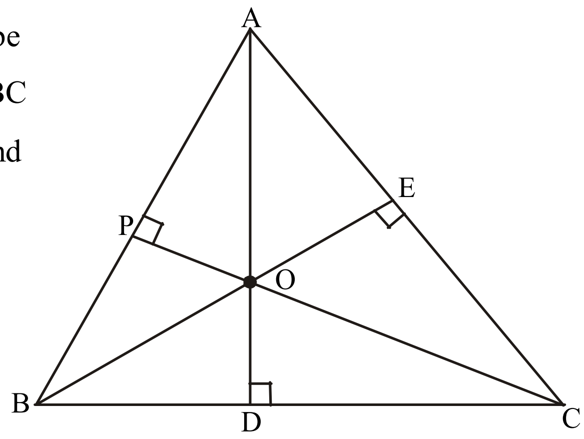

Fig. 10.12

Slope of $\overline{BC}$ is $\dfrac{y_3 - y_2}{x_3 - x_2}$

Then $\overline{AD} \perp \overline{BC}$, slope of $\overline{AD}$ is $-\dfrac{x_3 - x_2}{y_3 - y_2}$

Equation of Attitude through a is

$$y - y_1 = -\frac{(x_3 - x_2)}{y_3 - y_2}(x - x_1)$$

$\Rightarrow (y - y_1)(y_3 - y_2) = -(x_3 - x_2)(x - x_1)$

$\Rightarrow L_1 \equiv (x - x_1)(x_3 - x_2) + (y - y_1)(y_3 - y_2) = 0$

Similarly Equations of Attitudes through B, C are

$L_2 \equiv (x - x_2)(x_1 - x_3) + (y - y_2)(y_1 - y_3) = 0$

$L_3 \equiv (x - x_3)(x_1 - x_2) + (y - y_3)(y_1 - y_2) = 0$

Hence $1.L_1 + 1.L_2 + 1.L_3 = 0$

$\Rightarrow \lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$ Here $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$\therefore$ Attitudes $L_1 \equiv 0$, $L_2 \equiv 0$, $L_3 \equiv 0$ are concurrent.

Note that the concurrent point "O" is called the orthocentre of the triangle ABC.

10.3.9(3)

Theorem : The perpendicular bisectors of the sides of a triangle are concurrent.

Proof: Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are vertias of the triangle.

Let D, E, F midpoints of the sides $\overline{BC}, \overline{CA}, \overline{AB}$ respectively.

Let the perpendicular bisectors of the sides $\overline{BC}, \overline{CA}, \overline{AB}$ meets at S.

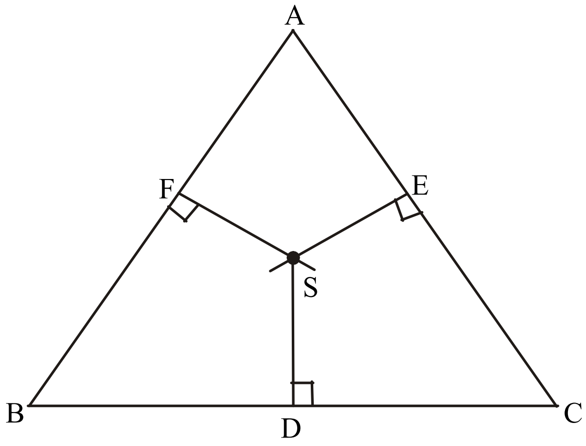

Fig. 10.13

$$D = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right), E = \left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right)$$

$$F = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Slope of $\overline{BC} = \dfrac{y_3 - y_2}{x_3 - x_2}$

Slope of perpendicular bisector of $\overline{BC}$ is $= \dfrac{-(x_3 - x_2)}{y_3 - y_2}$

Equation of perpendicular bisector of $\overline{SD}$ of $\overline{BC}$

is $\quad y - \dfrac{y_2 + y_3}{2} = \dfrac{-(x_3 - x_2)}{y_3 - y_2}\left(x - \dfrac{x_2 + x_3}{2}\right)$

$\Rightarrow\ 2y(y_3 - y_2) - (y_2 + y_3)(y_3 - y_2) = -2x(x_3 - x_2) + (x_3 - x_2)(x_2 + x_3)$

$\Rightarrow\ L_1 \equiv 2x(x_2 - x_3) + 2y(y_2 + y_3) - (x_2^2 - x_3^2) - (y_2^2 - y_3^2) = 0$

Similarly the equations of perpendicular bisectors $\overline{SE}$, $\overline{SF}$ of $\overline{AC}$, $\overline{AB}$ are

$L_2 \equiv 2x(x_3 - x_1) + 2y(y_3 - y_1) - (x_3^2 - x_2^2) - (y_3^2 - y_1^2) = 0$

$L_3 \equiv 2x(x_1 - x_2) + 2y(y_1 - y_2) - (x_1^2 - x_2^2) - (y_1^2 - y_2^2) = 0$

Now $1.L_1 + 1.L_2 + 1.L_3 = 0$ here $\lambda_1 = \lambda_2 = \lambda_3 = 1$

$\therefore$ The perpendicular bisectors of the sides of the triangle are concurrent.

Note that, the concurrent point of the perpendicular bisectors of the sides of the triangle is cicumcentre.

Example 10.25(i) : Find the point of intersection of the lines $x - 3y + 6 = 0$, $2x + 3y - 10 = 0$

Solution : Let $p(\alpha, \beta)$ be the intersecting point.

$$\therefore \ \alpha - 3\beta + 6 = 0, \ \ 2\alpha + 3\beta - 10 = 0$$

$$\begin{array}{ccccc} & \alpha & \beta & 1 & \\ -3 & 6 & & 1 & -3 \\ 3 & -10 & & 2 & 3 \end{array}$$

$$\frac{\alpha}{30-18} = \frac{\beta}{12+10} = \frac{1}{3+6} \Rightarrow \alpha = \frac{12}{9}, \ \beta = \frac{22}{9}$$

$$\therefore \text{ Intersecting point } \left(\frac{4}{3}, \frac{22}{9}\right).$$

Example 10.25(ii) : Find the value of p, if the straight lines $x + p = 0$, $y + 2 = 0$ and $3x + 2y + 5 = 0$ are concurrent.

Solution: Given straight lines $x + p = 0$, $y + 2 = 0$, $3x + 2y + 5 = 0$ are concurrent.

$$\Rightarrow \begin{vmatrix} 1 & 0 & p \\ 0 & 1 & 2 \\ 3 & 2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow 1(5 - 4) + 0(0 - 6) + p(0 - 3) = 0$$

$$\Rightarrow p = \frac{1}{3}$$

Example 10.25(iii) (a) Find the ratios in which (a) the X-axis

(b) the Y-axis devides the line segment $\overline{AB}$ joining A(2, −3) and B(3, −6).

(a) X-axis divides $\overline{AB}$ in the ratio $-y_1 : y_2$

$$= -(-3) : (-6)$$

$$= -1 : 2$$

(b) Y-axis divides $\overline{AB}$ in the ratio $-x_1 : x_2$

$= -2 : 3.$

Example 10.25(iv) : Find the ratio in which the straight line $3x - 4y - 7 = 0$ divides the line join the points (2, −7) and (−1, 3).

Solution : Straight line $3x - 4y - 7 = 0$ divides the line joining the points (2, −7) and (−1, 3) in the ratio

$= -[3(2) - 4(-7) - 7] \;:\; [3(-1) - 4(3) - 7]$

$= -(6 + 28 - 7) : (-3 - 12 - 7)$

$= -27 : -22$

$= 27 : 22$

Example 10.25(v) : Find the circumcentre of the triangle whose vertices are (−2, 3), (2, −1), (4, 0)

Solution: Let the vertices of the triangle be A(−2, 3), B(2, −1), C(4, 0)

The mid points of the sides $\overline{BC}$, $\overline{CA}$ respectively

$D\left(3, \frac{-1}{2}\right), E\left(1, \frac{3}{2}\right)$

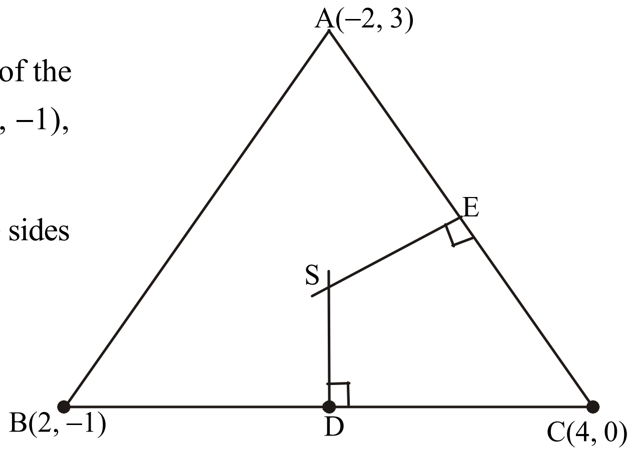

Fig. 10.14

Slope of $\overleftrightarrow{BC} = \frac{0+1}{4-2} = \frac{1}{2}$

$\therefore$ Slope of $\overleftrightarrow{SD}$ is -2 $\quad \left(\because \overleftrightarrow{SD} \perp \overleftrightarrow{BC}\right)$

So equation of $\overleftrightarrow{SD}$ is $y + \frac{1}{2} = -2(x-3)$

$\Rightarrow 4x + 2y - 11 = 0 \qquad ...(1)$

Slope of $\overleftrightarrow{AC} = \frac{0-3}{4+2} = \frac{-1}{2}$

$\therefore$ Slope of $\overleftrightarrow{AC}$ is +2 $\quad \left(\because \overleftrightarrow{SE} \perp \overleftrightarrow{AC}\right)$

So equation of $\overleftrightarrow{SE}$ is $y - \frac{3}{2} = 2(x-1)$

$$4x - 2y - 1 = 0 \qquad ...(2)$$

Solving the equations (1) and (2) we obtain circumcentre S = $\left(\frac{3}{2}, \frac{5}{2}\right)$

Example 10.25(vi) : Find the orthocentre of the triangle whose vertices are (–2, –1), (6, –1) and (2, 5).

Solution: Let A(–2, –1), B(6, –1), C(2, 5) be the vertices of the given triangle.

Let be $\overleftrightarrow{AD}$ perpendicular drawn from A to $\overleftrightarrow{BC}$ and $\overleftrightarrow{BE}$ be the perpendicular drawn from B to $\overleftrightarrow{AC}$

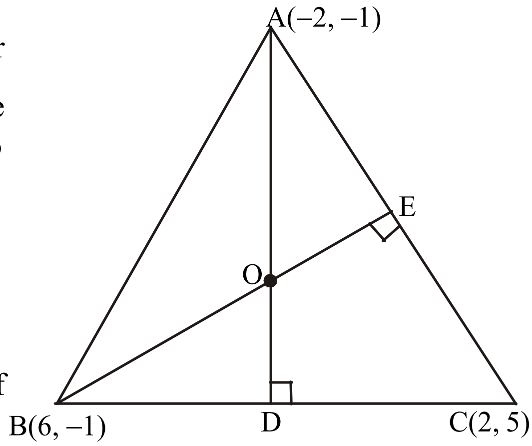

Fig. 10.15

Slope of $\overleftrightarrow{BC}$ = $\frac{5+1}{2-6} = \frac{-3}{2}$

Since $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$, slope of $\overleftrightarrow{AD}$ is = $\frac{2}{3}$

So the equation of $\overleftrightarrow{AD}$ is $y + 1 = \frac{2}{3}(x + 2)$

$$\Rightarrow 2x - 3y + 1 = 0 \qquad ...(1)$$

Slope of $\overleftrightarrow{AC}$ = $\frac{5+1}{2+2} = \frac{3}{2}$

Since $\overleftrightarrow{BE} \perp \overleftrightarrow{AC}$, slope of $\overleftrightarrow{BE}$ is = $-\frac{2}{3}$

So the equation $\overleftrightarrow{BE}$ is $y + 1 = -\frac{2}{3}(x - 6)$

$$2x + 3y - 9 = 0 \qquad ...(2)$$

Solving equations (1) and (2) we obtain.

Orthocentre $O = \left(2, \frac{5}{3}\right)$.

EXERCISE 10.3

1. Under what condition, the general equation $Ax + By + C = 0$ of first degree in x and y represents a line?
2. Reduce the equation $2x + 5y + 3 = 0$ to the slope intercept form.
3. Find the x and y intercepts for the following lines:

   (a) $y = mx + C$ (b) $3y = 3x + 8$ (c) $3x - 2y + 12 = 0$
4. Find the length of the line segment AB intercepted by the straight line $3x - 2y + 12 = 0$between the two axes.
5. Reduce the equation $x \cos \alpha + y \sin \alpha = p$ to the intercept form of the equation and also find the intercepts on the axes.
6. Reduce the following equations into normal form.

   (a) $3x - 4y + 10 = 0$ (b) $3x - 4y = 0$
7. Which of the lines $2x - y + 3 = 0$ and $x - 4y - 7 = 0$ is nearer from the origin?
8. Find the value of k, if the lines $2x - 3y + k = 0$, $3x - 4y - 13 = 0$ and $8x - 11y - 33 = 0$ are concurrent.
9. Find the point of intersection of the lines.

   $3x - y - 5 = 0$ and $x + 2y - 4 = 0$.
10. Find the circum centre of the triangle whose vertices are (1, 3), (–3, 5) and (5, –1).
11. Find the orthocentre of the triangle whose vertices are (–5, –7), (13, 2) and (–5, 6).

MODULE - II
Coordinate Geometry

Notes

10.4 ANGLE BETWEEN TWO LINES

We will now try to find out the angle between the arms of the divider or clock when equation of the arms are known. Two methods are discussed below.

10.4.1 When two lines with slopes m_1 and m_2 are given

Let AB and CD be two straight lines whose equations are

$y = m_1 x + c_1$ and $y = m_1 x + c_2$ respectively.

Let P be the point of intersection of AB and CD

$\angle XEB = \theta_1$ and $\angle XFD = \theta_2$. Then

$\tan\theta_1 = m_1$ and $\operatorname{Tan}\theta_2 = m_2$

Let θ be the angle $\angle EPF$ between AB and CD.

Now $\theta_1 = \theta_2 + \theta$

or $\theta = \theta_1 - \theta_2$

$\therefore \tan\theta = \tan(\theta_1 - \theta_2)$

or $$\tan\theta = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

Hence $$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

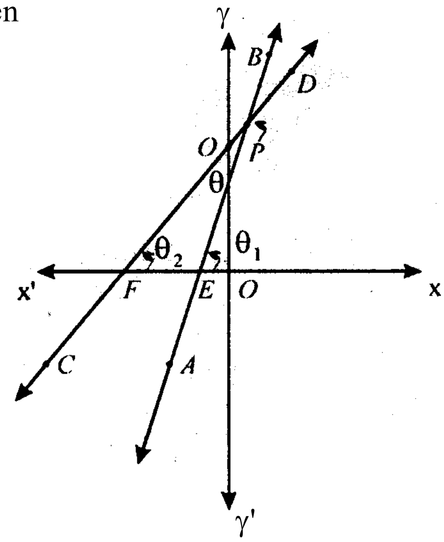

Fig. 10.16

which gives tangent of the angle θ between two given lines in terms of their slopes.

From this result we can find the angle between any two given lines when their slopes are given.

Note:

1. If $\dfrac{m_1 - m_2}{1 + m_1 m_2} > 0$ then angle between lines is **acute**

2. If $\frac{m_1 - m_2}{1 + m_1 m_2} < 0$ then angle between lines is **obtuse.**

3. $\frac{m_1 - m_2}{1 + m_1 m_2} = 0$

   i.e., $m_1 = m_2$, then lines are either ***coincident or parallel.***

   i.e, Two lines are parallel if they differ only by a constant term.

4. If the denominator of $\frac{m_1 - m_2}{1 + m_1 m_2}$ is zero.

   i.e., $m_1 m_2 = -1$, then lines are ***perpendicular.***

Example 10.26 : Find the angle between the lines $7x - y = 1$ and $6x - y = 11$.

Solution: Let θ be the angle betwen the lines $7x - y = 1$ and $6x - y = 11$. The equations of these lines can be written as $y = 7x - 1$ and $y = 6x - 11$

Here, $m_1 = 7$, $m_2 = 6$, where m_1 and m_2 are respective slopes of the given lines.

$$\tan\theta = \frac{7-6}{1+42} = \frac{1}{43}$$

The angle θ between the given lines is given by $\tan\theta = \frac{1}{43}$

Example 10.27 : Find the angle between the lines $x + y + 1 = 0$ and $2x - y - 1 = 0$

Solution: Putting the equations of the given lines in slope intercept form, we get

$$y = -x - 1 \text{ and } y = 2x - 1$$

$\therefore$ $m_1 = -1$ and $m_2 = 2$. where m_1 and m_2 are respectively slopes of the given lines.

$$\therefore \tan\theta = \frac{-1-2}{1+(-1)(2)} = 3$$

The angle θ between the given lines is given by $\tan\theta = 3$.

Example 10.28 : Are the straight lines $y = 3x - 5$ and $3x + 4 = y$ parallel or perpendicular?

Solution: For the first line $y = 3x - 5$, the slope $m_1 = 3$ and for the second line $y = 3x + 4$ the slope $m_2 = 3$. Since the slopes of both the lines are same for both the lines, hence the lines are parallel.

Example 10.29: Are the straight lines $y = 3x$ and $y = -\frac{1}{3}x$ parallel or perpendicular?

Solution: The slope m_1 of the line $y = 3x$ is 3 and slope m_2 of the second line

$y = -\frac{x}{3}$ is $-\frac{1}{3}$.

Since, $m_1 m_2 = -1$, the lines are perpendicular.

10.4.2 When the Equation of two lines are in general form

Let the general equations of two given straight lines be given by

$$A_1x + B_1y + C_1 = 0 \quad ...(i)$$

and $$A_2x + B_2y + C_2 = 0 \quad ...(ii)$$

Equations (i) and (ii) can be written into their slope intercept from as

$$y = -\frac{A_1}{B_1}x - \frac{C_1}{B_1} \quad \text{and} \quad y = -\frac{A_2}{B_2}x - \frac{C_2}{B_2}$$

Let m_1 and m_2 be their respective slopes and using the formula, we get.

i.e., $$\tan\theta = \frac{A_2B_1 - A_1B_2}{A_1A_2 + B_1B_2}$$

Thus, by using this formula we can calculate the angle between two lines when they are in general form.

Note: (i) If $A_2B_1 - A_1B_2 = 0$

or $A_1B_2 - A_2B_1 = 0$

i.e., $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0$ then the given lines are parallel.

(ii) If $A_1A_2 + B_1B_2 = 0$ then the given lines are perpendicular.

Example 10.30: Are the straight lines $x - 7y + 12 = 0$ and $x - 7y + 6 = 0$ parallel?

Solution : Here, $A_1 = 1$, $B_1 = -7$, $A_2 = 1$ and $B_2 = -7$

Since $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} 1 & -7 \\ 1 & -7 \end{vmatrix} = 0$

Example 10.31: Determine whether the straight lines $2x + 5y - 8 = 0$ and $5x - 2y - 3 = 0$ are parallel or perpendicular?

Here $A_1 = 2$, $B_1 = 5$, $A_2 = 5$ and $B_2 = -2$

Solution: Here $A_1A_2 + B_1B_2 = (2) \times (5) + 5 \times (-2) = 0$

Hence the given lines are perpendicular.

EXERCISE 10.4

1. What is the condition for given lines $y = m_1x + C_1$ and $Ax + By + C = 0$

   (a) to be parallel? (b) to be perpendicular?

2. Find the angle between the lines $4x + y = 3$ and $\frac{x}{2} + y = \frac{4}{7}$.

3. (a) Write the condition that the two lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are

   (i) parallel (ii) perpendicular.

   (b) Are the straight lines $x - 3y = 7$ and $2x - 6y - 16 = 0$ parallel?

   (c) Are the straight lines $x = y + 1$ and $x = -y + 1$ perpendicular?

10.5 DISTANCE OF A GIVEN POINT FROM A GIVEN LINE

In this section, we shall discuss the concept of finding the distance of a given point from a given line or lines.

Let $P(x_1, y_1)$ be the given point and l be the line $Ax + By + C = 0$

Let the line l intersect x axis and y axis R and Q respectively.

Draw $PM \perp L$ and let $PM = d$

Let the coordinates of M be (x_2, y_2)

$$d = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}} \qquad ...(i)$$

$\therefore$ M lies on l

$\therefore\ Ax_2 + By_2 + C = 0$

or $\quad C = -(Ax_3 + By_2) \qquad ...(ii)$

The coordinates of R and Q are $\left(-\frac{C}{A}, 0\right)$ and $\left(0, -\frac{C}{B}\right)$ respectively.

The slope of QR $= \dfrac{0 + \frac{C}{B}}{-\frac{C}{B} - 0} = -\dfrac{A}{B}$ and

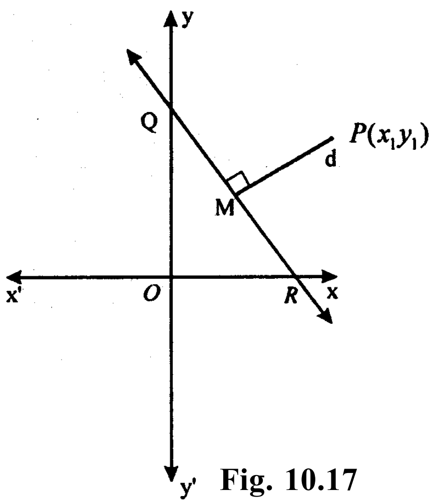

Fig. 10.17

the slope of $PM = \dfrac{y_2 - y_1}{x_2 - x_1}$

As $PM \perp QR \Rightarrow \dfrac{y_2 - y_1}{x_2 - x_1} \times \left(-\dfrac{A}{B}\right) = -1$

or $\quad \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{B}{A} \quad ...(iii)$

From (iii) $\dfrac{x_1 - x_2}{A} = \dfrac{y_1 - y_2}{B} = \dfrac{\sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}}}{\sqrt{A^2 + B^2}} \quad ...(iv)$

(U sing properties of Ratio and Proportion)

Also $\dfrac{x_1 - x_2}{\text{A}} = \dfrac{y_1 - y_2}{\text{B}} = \dfrac{\text{A}(x_1 - x_2) + \text{B}(y_1 - y_2)}{\sqrt{\text{A}^2 + \text{B}^2}}$ ...(v)

From (iv) and (v), we get

$$\frac{\sqrt{\left\{(x_1 - x_2)^2 + (y_1 - y_2)^2\right\}}}{\sqrt{(\text{A}^2 + \text{B}^2)}} = \frac{\text{A}(x_1 - x_2) + \text{B}(y_1 - y_2)}{\text{A}^2 + \text{B}^2}$$

or $$\frac{d}{\sqrt{(\text{A}^2 + \text{B}^2)}} + \frac{(\text{A}x_1 + \text{B}y_1) - (\text{A}x_2 + \text{B}y_2)}{\text{A}^2 + \text{B}^2} \quad \text{[Using (i)]}$$

or $$\frac{\text{A}x_1 + \text{B}y_1 + \text{C}}{\sqrt{(\text{A}^2 + \text{B}^2)}} \quad \text{[Using (ii)]}$$

Since the distance is always positive, we can write

$$d = \left|\frac{\text{A}x_1 + \text{B}y_1 + \text{C}}{\sqrt{\text{A}^2 + \text{B}^2}}\right|$$

Note : *The perpendicular distance of the origin* (0, *0) from*

$$\text{A}x + \text{B}y + \text{C} = 0 \ \textit{is} \ \frac{\text{A}(0) + \text{B}(0) + \text{C}}{\sqrt{\text{A}^2 + \text{B}^2}} = \frac{\text{C}}{\sqrt{\text{A}^2 + \text{B}^2}}$$

Example 10.32 : Find the points on the x-axis whose perpendicular distance from the straight line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ is a.

Solution : Let $(x_1, 0)$ be any point on x-axis.

Equation of the given line is $bx + ay - ab = 0$. The perpendicular distance of the point $(x, 0)$ from the given line is

$$a = \pm \frac{bx_1 + a.0 - ab}{\sqrt{a^2 + b^2}}$$

$$\therefore \ x_1 = \frac{a}{b}\left\{b \pm \sqrt{a^2 + b^2}\right\}$$

Thus, the point on x-axis is $\left(\dfrac{a}{b}\left(b \pm \sqrt{a^2 + b^2}\right), 0\right)$

10.5.1 Distance between parallel lines

Theorem: The distance between the parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\dfrac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$

Proof :

Let $P(x_1, y_1)$ be the point on the line $L_1 : ax + by + c_1 = 0$

Let $L_2 : ax + by + c_2 = 0$ be the other line.

Now distance between parallel lines $L_1 = 0$ and $L_2 = 0$ is equal to PM.

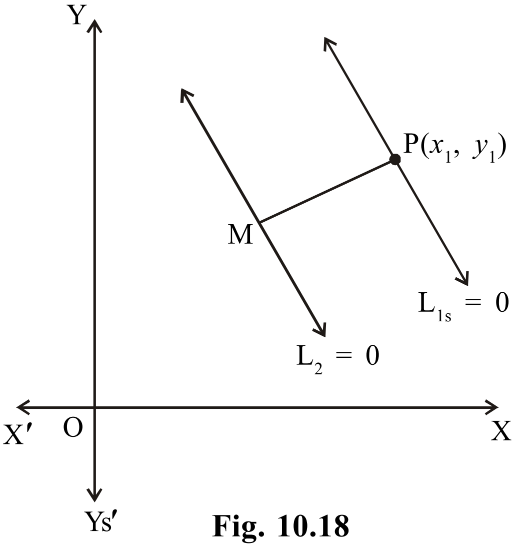

Fig. 10.18

PM = Perpendicular distance from $P(x_1, y_1)$ to $L_2 = 0$

$$= \frac{|ax_1 + by_1 + c_2|}{\sqrt{a^2+b^2}}$$

$$= \frac{|ax_1 + by_1 + c_1 + c_2 - c_1|}{\sqrt{a^2+b^2}}$$

$$= \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}} \quad (\because P(x_1, y_1) \text{ is point on } L_1 = 0 \text{ i.e. } ax_1 + by_1 + c_1 = 0)$$

$\therefore$ distance between parallel lines is $\dfrac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$.

10.5.2 Foot of the perpendicular from a point to the line.

Theorem : If $Q(h, k)$ is the foot of the perpendicular from $P(x_1, y_1)$ on the straight line $ax + by + c = 0$ then

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2+b^2}$$

Proof : Let $Q(h, k)$ be the foot of the perpendicular from $P(x_1, y_1)$ on the line $ax + by + c = 0$.

Since Q(h, k) lies on $ax + by + c = 0$

$\Rightarrow ah + bk + c = 0$ ...(1)

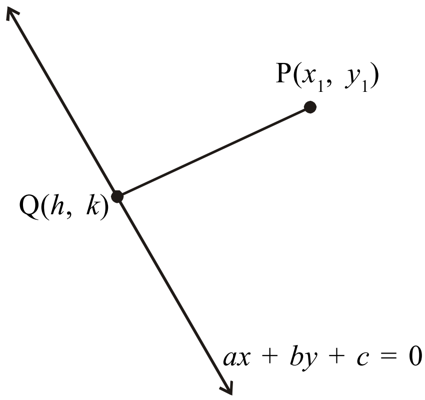

Fig. 10.19

(Slope of the line $ax + by + c = 0$)

X slope of $\overleftrightarrow{PQ} = -1$.

$$\frac{-a}{b} \times \frac{k - y_1}{h - x_1} = -1$$

$$\Rightarrow \frac{h - x_1}{a} = \frac{k - y_1}{b}$$

Let $\frac{h - x_1}{a} = \frac{k - y_1}{b} = \lambda$ ...(2)

$\therefore\ h = x_1 + a\lambda,\ \ k = y_1 + b\lambda$

substituting in (1)

$a(x_1 + a\lambda) + b(y_1 + b\lambda) + c = 0$

$$\Leftrightarrow \lambda = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2} \quad ...(3)$$

from (2) and (3)

$$\Rightarrow \frac{h - x_1}{a} = \frac{k - y_1}{b} = -\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

10.5.3 Image of a point on a line

Theorem : If Q(h, k) is the image of the point P(x_1, y_1) w.r.to the straight line $ax + by + c = 0$ then $\frac{h - x_1}{a} = \frac{k - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$

Proof : Let Q(h, k) be the image of the point P(x_1, y_1) w.r.to the line

$ax + by + c = 0$.

Let R be the mid point of $\overline{PQ}$

$$R = \left(\frac{h + x_1}{2}, \frac{k + y_1}{2}\right)$$

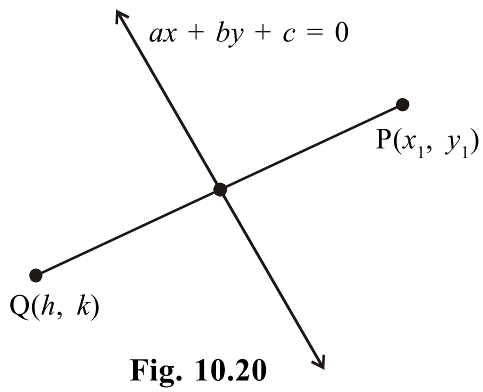

Fig. 10.20

Since R lies on L = $ax + by + c = 0$;

$$a\left(\frac{h+x_1}{2}\right)+b\left(\frac{k+y_1}{2}\right)+c=0;$$

$$ah + by + 2c = -(ax_1 + by_1) \qquad ...(1)$$

$\because$ (Slope of line $ax + by + c = 0$) $\times$ (Slope of $\overleftrightarrow{PQ}$ $= -1$)

$$\frac{-a}{b}\times\frac{y_1-k}{x_1-h}=-1$$

$$\Rightarrow \frac{h-x_1}{a} = \frac{k-y_1}{b}$$

Let $\frac{h-x_1}{a} = \frac{k-y_1}{b}=\lambda \Rightarrow x_1+a\lambda,\ k = y_1+b\lambda$

substitutes in (1)

$$a(x_1 + a\lambda) + b(y_1 + b_1) + 2c = -(ax_1 + by_1)$$

$$\lambda = \frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

hence $\frac{h-x_1}{a}=\frac{k-y_1}{b}=-\frac{2(ax_1+by_1+c)}{a^2+b^2}$.

Example 10.32(i) : Find the distance between the parallel lines

$$x - 2y + 3 = 0,\ 4x - 8y - 4 = 0$$

Solution: Given lines are $x - 2y + 3 = 0$

and $4x - 8y - 4 = 0 \Rightarrow x - 2y - 1 = 0$

Here $c_1 = 3$, $c_2 = -1$, $a = 1$, $b = -2$

distance between parallel lines $= \frac{|c_2-c_1|}{\sqrt{a^2+b^2}}$

$$= \frac{|-1-3|}{\sqrt{1^2+(-2)^2}} = \frac{4}{5}$$

Example 10.32(ii) Find the foot of the perpendicular from (−1, 3) on the straight line $5x - y - 18 = 0$

Solution: Let (h, k) be the foot of the perpendicular from (−1, 3) on the line $5x - y - 18 = 0$

$$\therefore \frac{h+1}{5} = \frac{k-3}{-1} = -\frac{(5(-1)-(3)-18)}{5^2+(-1)^2}$$

$$\frac{h+1}{5} = \frac{k-3}{-1} = -\frac{(-26)}{26}$$

$$\Rightarrow h + 1 = 5, \quad k - 3 = -1$$

$$\Rightarrow h = 4, \quad k = 2$$

$$\Rightarrow (h, k) = (4, 2).$$

Example 10.32(iii) : Find the image of (3, 4) w.r.to the line

$$3x + 4y + 5 = 0.$$

Solution: Let (h, k) be the image of (3, 4) w.r.to the line $3x + 4y + 5 = 0$.

$$\therefore \frac{h-3}{3} = \frac{k-4}{4} = \frac{-2(3\times3+4\times4+5)}{3^2+4^2}$$

$$\Rightarrow \frac{h-3}{3} = \frac{k-4}{4} = \frac{-60}{25} = \frac{-12}{5}$$

$$\Rightarrow h-3 = -\frac{36}{5}, \; k-4 = \frac{-48}{5}$$

$$\Rightarrow h = -\frac{21}{5}, \; k = \frac{-28}{5}$$

$$\therefore \text{ Image } = \left(-\frac{21}{5}, \frac{-28}{5}\right)$$

EXERCISE 10.5

1. Find the perpendicular distance of the point (2,3) from $3x + 2y + 4 = 0$.
2. Find the points on the axis of y whose perpendicular distance from the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is b.

3. Find the points on the axis of y whose perpendicular distance from the straight line $4x + 3y = 12$ is 4.

4. Find the perpendicular distance of the origin from $3x + 7y + 14 = 0$.

5. What are the points on the axis of x whose perpendicular distance from the straight line $\frac{x}{4}+\frac{y}{4}=1$ is 3?

6. Find the foot of the perpendicular from (4, 1) on the line $3x - 4y + 12 = 0$.

7. If the foot of the perpendicular from (−4, 5) on the line is (2, −3) then find the equation of line.

8. Find the image of the point (2, 3) w.r.to the line $4x - 5y + 8 = 0$.

9. Find the distance between the parallel line.

   (i) $3x - 4y = 12, \quad 3x - 4y = 7$

   (ii) $3y - 5x + 4 = 0, \quad 10x - 6y - 9 = 0$

10.6 EQUATION OF PARALLEL (OR PERPENDICULAR) LINES

Till now, we have developed methods to find out whether the given lines are prallel or perpendicular. In this section, we shall try to find, the equation of a line which is parallel or perpendicular to a given line.

10.6.1 EQUATION OF A STRAIGHT LINE PARALLEL TO THE GIVEN LINE

$$Ax + By + C = 0 \qquad ...(i)$$

Let $\quad A_1x + B_1y + C_1 = 0$

be any line parallel to the given line

$$Ax + By + C = 0$$

The condition for parallelism of (i) and (ii) is ...(ii)

$$\frac{A_1}{A} = \frac{B_1}{B} = K_1 \quad \text{(say)}$$

$\Rightarrow \quad A_1 = AK_1, \quad B_1 = BK_1$

with these values of A_1 and B_1, (i) gives

$$AK_1x + BK_1y + C_1 = 0$$

or $$Ax + By + \frac{C_1}{K_1} = 0$$

or $$Ax + By + K = 0$$

where $$K = \frac{C_1}{K_1} \qquad \text{...(iii)}$$

This is a line parallel to the given line. From equations (ii) and (iii) we observe that

(i) coefficients of x and y are same

(ii) constants are different, and are to evaluated from given conditions

Example 10.33 : Find equation of the straight line, which passes through the point (1,2) and which is parallel to the straight line $2x + 3y + 6 = 0$.

Solution: Equation of any straight line parallel to the given equation can be written if we put

(i) the coefficients of x and y as same as in the given equation.

(ii) constant to be different from the given equation, which is to be evaluated under given condition.

Thus, the required equation of the line will be

$$2x + 3y + K = 0 \quad \text{for some constant } K$$

Since it passes through the point (1, 2) hence

$$2 \times 1 + 3 \times 2 + K = 0$$

or $K = -8$

$\therefore$ Required equation of the line is $2x + 3y = 8$.

10.7 STRAIGHT LINE PERPENDICULAR TO THE GIVEN LINE

$$Ax + By + C = 0$$

Let $A_1x + B_1y + C_1 = 0$ ..(i)

be any line perpendicular to the given line

$Ax + By + C = 0$

Condition for perpendicularity of lines (i) and (ii) is ...(ii)

$$AA_1 + BB_1 = 0$$

$$\Rightarrow \frac{A_1}{B} = \frac{-B_1}{A} = K_1 \quad \text{(say)}$$

$$\Rightarrow A_1 = BK_1 \text{ and } B_1 = -AK_1$$

With these values of A_1 and B_1, (i) gives

$$Bx - Ay + \frac{C_1}{K_1} = 0$$

or $Bx - Ay + K = 0$ where $K = \dfrac{C_1}{K_1}$ ...(iii)

Hence, the line (iii) is perpendicular to the given line (ii)

We observe that in order to get a line perpendicular to the given line we have to follow the following procedure:

(i) Interchange the coefficients of x and y

(ii) Change the sign of one of them.

(iii) Change the Constant term to a new constant K (say), and evaluate it from given condition.

Example 10.34 : Find the equation of the line which passes through the point (1,2) and is perpendicular to the line $2x + 3y + 6 = 0$.

Solution: Following the procedure given above, we get the equation of line perpendicular to the given equation as

$$3x - 2y + K = 0 \qquad \text{...(i)}$$

(i) passes through the point (1,2), hence

$3 \times 1 - 2 \times 2 + K = 0$ or $K = 1$

$\therefore$ Required equation of the straight line is $3x - 2y + 1 = 0$.

Example 10.35 : Find the equation of the line which passes through the point (x_2, y_2) and is perpendicular to the straight line $yy_1 = 2a_1(x + x_1)$.

Solution: The given straight line is $yy_1 = 2ax - 2ax_1 = 0$ ...(i)

Any straight line perpendicular to (i) is $2ay + xy_1 + C = 0$

This passes through the point (x_2, y_2)

$$\therefore \quad 2ay_2 + x_2y_1 + C = 0$$

$$\Rightarrow \quad C = -2ay_2 - x_2y_1$$

$\therefore$ Required equation of the spraight line is

$$2a(y - y_2) + y_1(x - x_2) = 0$$

EXERCISE 10.6

1. Find the equation of the straight line which passes through the point (0, −2) and is parallel to the straight line $3x + y = 2$.
2. Find the equation of the straight line which passes through the point (−1, 0) and is parallel to the straight line $y = 2x + 3$.
3. Find the equation of the straight line which passes through the point (0, −3) and is perpendicular to the straight line $x + y + 1 = 0$.
4. Find the equation of the line which passes through the point (0, 0) and is perpendicular to the straight line $x + y = 3$.

MODULE - II
Coordinate Geometry

Notes

5. Find the equation of the straight line which passes through the point (2, –3) and is perpendicular to the given straight line $2a(x+2)+3y=0$.

6. Find the equation of the line which has x - intercept -8 and is perpendicular to the line $3x + 4y - 17 = 0$

7. Find the equation of the line whose y-intercept is 2 and is parallel to the line $2x - 3y + 7 = 0$.

8. Prove that the equation of a straight line passing throngh ($a\cos^3\theta$, $a\sin^3\theta$) and perpendicular to the $x\sec\theta + y\operatorname{cosec}\theta = a$ is $x\cos\theta - y\sin\theta = a\cos 2\theta$.

10.8 APPLICATION OF COORDINATE GEOMETRY

Let us take some examples to show how coordinate geometry can be gainfully used to prove some geometrical results.

Example 10.36: Prove that the diagonals of a rectangle are equal.

Solution: Let us take origin as one vertex of the rectangle without any loss of generality and take one side along x-axis of length a and another side of length b along y-axis, as shown in Fig. 10.12. Then the coordinates of the points A, B and C are $(a, 0)$, (a, b) and $(0, b)$ respectively

$$\therefore \text{ Length of diagonal AC } = \sqrt{(a-0)^2+(0-b)^2} = \sqrt{a^2+b^2}$$

$$\text{Length of diagonal OB } = \sqrt{(a-0)^2+(0-b)^2} = \sqrt{a^2+b^2}$$

$$\therefore \quad AC = OB$$

i.e., the diagonals of the rectangle OA BC are equal. Thus, in general, we can say that the length of the diagonals ofarectangle are equal.

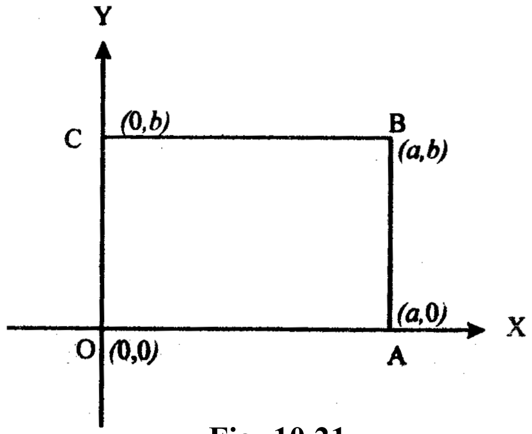

Fig. 10.21

Example 10.37 : Prove that angle in a semi circle is a right angle.

Solution : Let 0 (0,0) be the centre of the circle and $(-a, 0)$and O(0, 0) be the end- points of one of its diameter. Let the circle intersects the y-axis at $(0, a)$ [see Fig. 10.13]

$\therefore$ Slope of BQ $= \dfrac{a-0}{0-a} = 1 = m_1$

Slope of AQ $= \dfrac{0-a}{a-0} = -1 = m_2$

As $m_1\, m_2 = -1$ $\Rightarrow$ AQ is perpendicular to BQ

$\therefore$ BQ A is a right angle.

Again, let $P(a\cos\theta, a\sin\theta)$ be a general point on the circle.

$\therefore$ Slope of PB $= \dfrac{a\sin\theta}{a\cos\theta + a} = \dfrac{\sin\theta}{1+\cos\theta} = m_1$ (say)

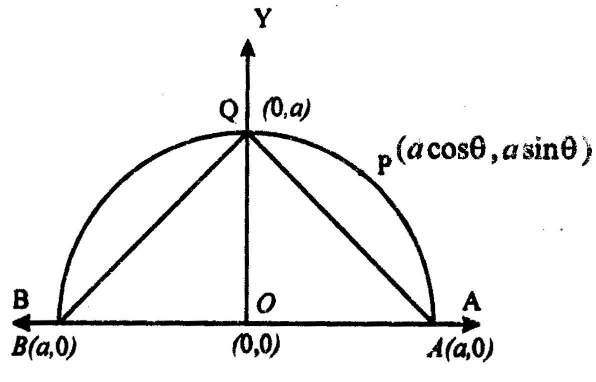

Fig. 10.22

Slope of PA $= \dfrac{0 - a\sin\theta}{a - a\cos\theta} = \dfrac{-\sin\theta}{1-\cos\theta} = m_2$ (say)

and $m_1.m_2 = \left(\dfrac{\sin\theta}{1+\cos\theta}\right)\left(\dfrac{-\sin\theta}{1-\cos\theta}\right)$

$$= -\frac{\sin^2\theta}{1-\cos^2\theta}$$

$$= -\frac{\sin^2\theta}{1-\cos^2\theta} = -1.$$

$\therefore$ P A is perpendicular to PB.

$\therefore$ $\angle$BPA is a right angle.

Example 10.38 : Medians drawn from two vertices to equal sides of an isosceles triangle are equal.

Solution: Let OAB be an equilateral triangle with vertices (0,0), $(a,0)$ and $\left(\dfrac{a}{2}, a\right)$

Let C and D be the mid-points of AB and OB respectively.

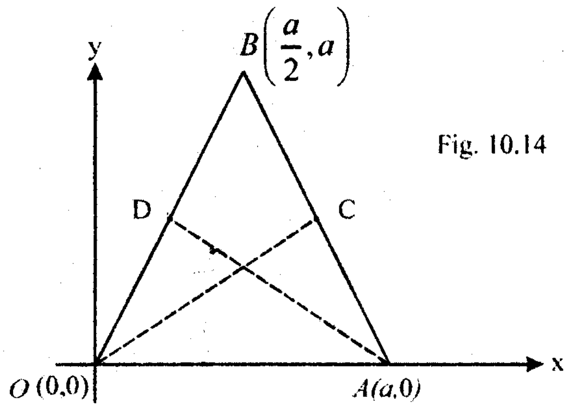

Fig. 10.23

$\therefore$ C has coordinates $\left(\dfrac{3a}{4}, \dfrac{a}{2}\right)$ and

D has coordinates $\left(\dfrac{a}{4}, \dfrac{a}{2}\right)$

$$\therefore \text{ Length of OC } = \sqrt{\left(\frac{3a}{4}\right)^2+\left(\frac{a}{2}\right)^2} = \sqrt{\frac{9a}{16}+\frac{a^2}{4}} = \frac{a}{4}\sqrt{13}$$

$$\text{and Length AD } = \sqrt{\left(a-\frac{a}{4}\right)^2+\left(a-\frac{a}{2}\right)^2} = \frac{a}{4}\sqrt{13}$$

$\therefore$ Length of OC = length of AD

Hence the proof.

Example 10.39 : The line segment joining the mid-points of two sides ofa triangle is parallel to the third side and is half of it.

Solution: Let the coordinates of the vertices be O(0,0), A(a, 0) and B(b, c). Let C and D be the mid-points of OB and AB respectively

$$\therefore \text{ The coordinates of C are } \left(\frac{b}{2},\frac{c}{2}\right)$$

$$\text{The coordinates of D are } \left(\frac{a+b}{2},\frac{c}{2}\right)$$

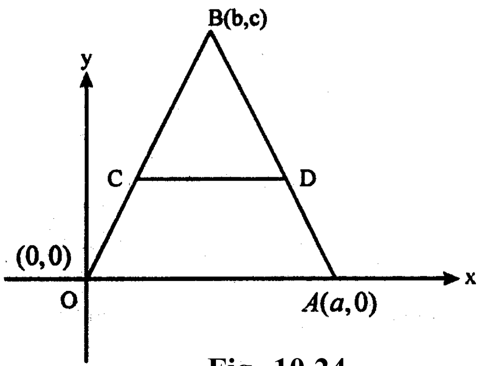

Fig. 10.24

$$\therefore \text{ Slope of CD } = \frac{\frac{c}{2}-\frac{c}{2}}{\frac{a+b}{2}-\frac{b}{2}} = 0$$

Slope of OA, i.e, x - axis = 0

$\therefore$ CD is parallel to OA

$$\text{Length of CD } = \sqrt{\left(\frac{a+b}{2}-\frac{b}{2}\right)^2}+\sqrt{\left(\frac{c}{2}-\frac{c}{2}\right)^2} = \frac{a}{2}$$

Length of OA = a

$$\therefore \text{ CD } = \frac{1}{2}\text{OA.}$$

which proves the above result.

Example 10.40 : If the diagonals of *a* quadrilateral are perpendicular and bisect each other, then the quadrilateral is *a* rhombus.

Solution: Let the diagonals be represented along OX and OY as shown in Fig. 10.16 and suppose that the lengths of their diagonals be 2*a* and 2*b* respectively.

$$OA = OC = a \text{ and } OD = OB = b$$

$$AD = \sqrt{OA^2 + OD^2} = \sqrt{a^2 + b^2}$$

$$AB = \sqrt{OB^2 + OA^2}$$

$$= \sqrt{a^2 + b^2}$$

Similarly $BC = \sqrt{a^2 + b^2} = CD$

As $AB = BC = CD = DA$

$\therefore$ A BCD is a rhumbus.

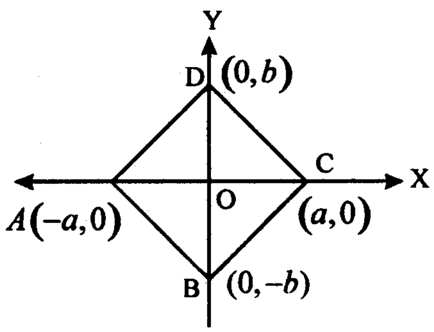

Fig. 10.25

EXERCISE 10.7

Using coordinate geometry, prove the following geometrical results:

1. If two medians of a triangle are equal, then the triangle is isosceles.
2. The diogonals of a square are equal and perpendicular to each other.
3. If the diagonals of a parallelogram are equal, then the parallelogram is a rectangle.
4. If D is the mid-point of base BC of ΔABC then $AB^2 + AC^2 = 2(AD^2 + BD^2)$
5. In a triangle, if a line is drawn parallel to one side, it divides the other two sides proportionally.
6. If two sides of a triangle are unequal, the greater side has a greater angle opposite to it.

7. The sum of any two sides of a triangle is greater than the third side.
8. In a triangle, the medians pass throngh the same point.
9. The diagonal of a paralleogram divides it into two triangles of equal area.
10. In a parallelogram, the pairs of opposite sides are of equal length.

10.8 COMBINED EQUATION OF A PAIR OF STRAIGHT LINES

LEARNING OUTCOMES

After studying this lesson you will be able to know :

- The point of intersection and angle between the two lines when their combined equation is given.
- A second degree homogeneous equation x and y represents a pair of lines.
- Combined equation of pair of straight lines which passes through origin.
- Combined equation of pair of straight lines which does not passes through origin.

INTRODUCTION

Given the equations of two straight lines, the methods of finding their point of intersection and the angle between them were discussed in previous chapters. Here, we shall find the conditions under which a second degree equation in 'x' and 'y' represents a pair of straight lines.

10.8.1 Combined equation of a pair of straight lines

Let 'L_1' and 'L_2' denote two straight lines and let their equations be $L_1 \cong a_1x + b_1y + c_1 = 0$ and $L_2 = a_2x + b_2y + c_2 = 0$ i.e., which are the linear in 'x' and 'y' (i.e., a_1, b_1 are not both zero and a_2, b_2 are not both zero).

Consider the equation $(a_1x + b_1y + c_2)(a_2x + b_2y + c_2) = 0$

Now, $p(\alpha, \beta)$ is a point on the locus represented by eq(1).

$\Rightarrow$ $(a_1\alpha + b_1\beta + c_1)(a_2\alpha + b_2\beta + c_2) = 0.$

$\Rightarrow$ $a_1\alpha + b_1\beta + c_1 = 0$ (or) $a_2\alpha + b_2\beta + c_2 = 0.$

$\Rightarrow$ 'p' lies on L_1 (or) 'p' lies on L_2.

We therefore conclude that the locus or graph of the eq.(1) is the pair of straight lines L_1 and L_2. We say that equation (1) is the combined equation or simply the equation of 'L_1' and 'L_2'.

Example 10.41 : The equation $6x^2 + 11xy - 10y^2 = 0$ represents the pair of straight lines $3x - 2y = 0$, and $2x + 5y = 0$. since $(3x - 2y)(2x + 5y) \equiv 6x^2 + 11xy - 10y^2$.

Similarly, we consider the euqation $(3x + 2y - 1) = 0$

& $2x - 3y + 1 = 0$

Since $(3x + 2y - 1)(2x - 3y + 1) \equiv 6x^2 - 5xy - 6y^2 + x + 5y - 1 = 0$, the equation $6x^2 - 5xy - 6y^2 + x + 5y - 1 = 0$ represents the pair of straight lines $3x + 2y - 1 = 0$ and $2x - 3y + 1 = 0$

10.8.2 Definition

If a, b, h are real numbers, not all zero, then $H \cong ax^2 + 2hxy + by^2 = 0$ is called the homogeneous equation of second degree in 'x' and 'y' and

$S \cong ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is called the general equation of second degree in 'x' and 'y' .

Note:

(i) If a, b and h are not all zero, then the equation $H \cong ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines if and only if $h^2 \geq ab$

(ii) If $h^2 \geq ab$, then we can write $H \cong (l_1x + m_1y)(l_2x + m_2y)$ so that $l_1 \, l_2 = a$, $m_1 \, m_2 = b$ and $l_1m_2 + l_2m_1 = 2h$. Also, $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ are the straight lines represented by H = 0.

10.8.3 Angle between pair of lines

When the equations of two straight lines are given separately, finding the angle between them was discussed in previous chapters. This section aims at finding the angle between a pair of straight lines when their combined equation is given.

10.8.3.1 Theorem

Let the euqation $ax^2 + 2hxy + by^2 = 0$ represent a pair of straight lines. Then the angle θ between the lines is given by :

$$\cos\theta = \frac{a+b}{\sqrt{(a-b)^2 + 4h^2}}$$

Note

(i) The equation $H \cong ax^2 + 2hxy + by^2 = 0$ (i.e., H = 0) represents a pair of coincident lines if $h^2 = ab$.

Now, the lines given by H = 0 are perpendicular iff (if and only if)

$\cos\theta = 0$

$\Leftrightarrow a + b = 0$ (sum of coefficients of x^2 & y^2 in H = 0 is zero)

(ii) If $a + b \neq 0$, then the lines representd by H = 0 are not perpenduculat and in such a situation, and angle θ between the lines is also given by formula :

$$\text{Tan}\theta = \frac{2\sqrt{h^2 - ab}}{a+b}, \text{ because } \text{Cos}\,\theta = \frac{a+b}{\sqrt{(a-b)^2 + 4h^2}} \text{ gives}$$

$$\text{Sin}\theta = \frac{2\sqrt{h^2 - ab}}{\sqrt{(a-b)^2 + 4h^2}}$$

Example 10.42 : Find the angle between the straight lines represented by the equation : $3x^2 - 4xy + 1y^2 = 0$.

Solution : Given equation : $3x^2 - 4xy + 1y^2 = 0$.

Comparing the given equation with $a\,x^2 + 2h\,xy + b\,y^2 = 0$, we find $a = 3$, $b = 1$, $h = -\frac{4}{2} = -2$ Therefore, angle θ between the given pair of lines is given by :

$$\text{Tan}\,\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{4-(3)(1)}}{3+1} = \frac{2\sqrt{1}}{4} = \frac{1}{2}$$

Hence, the acute angle between the lines is $= \text{Tan}^{-1}\left(\frac{1}{2}\right)$.

10.8.4 Bisectors of Angles

Note :

Let the euqation of two intersecting lines be : $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$. Then the equations of the bisectors of the angles between $L_1 = 0$, $L_2 = 0$ are :

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Example 10.43 : Find the equations of the straight lines bisecting the angles between the lines : $5x + y + 3 = 0$, $x + 5y + 2 = 0$.

Solution : The equations of the straight lines bisecting the angles between the given by

$$\left[\frac{5x + y + 3}{\sqrt{26}}\right] \pm \left[\frac{x + 5y + 2}{\sqrt{26}}\right] « 0$$

$\Rightarrow \quad 5x + 5 + 3 + (x + 5y + 2) = 0$

(or) $\quad 6x + 6y + 5 = 0, \quad 4x - 4y + 1 = 0$

EXERCISE 10.8

1. Find the acute angle between the pair of lines represented by the following equations.

   (i) $12x^2 + 25xy + 12y^2 = 0$

   (ii) $4x^2 - 7xy - 11y^2 = 0$

   (iii) $x^2 - 2xy \sec\alpha + y^2 = 0$

   (iv) $x^2 + 2xy \cot\alpha - y^2 = 0$

2. Show that the pair of bisectors of the angles between the straight lines $(ax + by)^2 = c(bx - ay)^2, c > 0$ are parallel and perpendiculars to the lines $ax + by + k = 0$.

3. Find the centroid and the area of the triangle formed by the following lines.

   (i) $2y^2 - xy - 6x^2 = 0, \; x + y + 4 = 0$

   (ii) $3x^2 - 4xy + y^2 = 0, \; 2x - y = 6$

10.8.5 Pair of lines - Second degree General equation

If the second degree euqation $S \cong ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ in the two variables 'x' and 'y' represents a pair of straight lines, then:

(i) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

(ii) $h^2 \geq ab, g^2 \geq ac, f^2 \geq bc$

Note : If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, then the equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the drigin and parallel to the former pair of lines.

10.8.6 Conditions for parallel lines distance between them

$S \cong ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines. Then the angle between this pair of lines is the same as the angle between in the pair of lines represented by $H \cong ax^2 + 2hxy + by^2 = 0$

Hence an angle between the pair of lines $S = 0$ is :

$$\mathrm{Cos}^{-1}\left[\frac{a+b}{\sqrt{(a-b)^2+4h^2}}\right] = \mathrm{Tan}^{-1}\left[\frac{2\sqrt{h^2-ab}}{a+b}\right], \quad \text{if } (a+b) > 0$$

$$= \mathrm{Tan}^{-1}\left[\frac{2\sqrt{h^2-ab}}{a+b}\right] + \pi, \quad \text{if } (a+b) < 0$$

$$= \frac{\pi}{2}, \quad \text{if } a+b = 0$$

Therefore, the lines represented by $S = 0$ are parallel if $h^2 = ab$, perpendicular if $a + b = 0$ & intersecting if $h^2 > ab$.

10.8.6.1 Theorem

If the equation $S \cong ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel straight lines, then (i) $h^2 = ab$ (ii) $af^2 = bg^2$ and (iii) the distance between the parallel lines

$$= 2\sqrt{\frac{g^2-ac}{a(a+b)}} = 2\sqrt{\frac{f^2-bc}{b(a+b)}}.$$

10.8.7 Point of intersection of the pair of lines.

10.8.7.1 Theorem

If the equation $S \cong ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines intersecting at the origin, then $g = f = c = 0$

10.8.7.2 Theorem

If the equation $S \cong ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of intersecting straight lines, then their point of intersection is

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right)$$

Example 10.44 : Find the point of intersection of pair of straight lines represented by : $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$

Solution : Comparing this equation with the general equation of the second degree in 'x' and 'y',

We get $a = 3,\ b = 2\ c = 2$

Also $2f = 5 \Rightarrow f = 5/2$

$2g = 5 \Rightarrow g = 5/2$

$2h = 7 \Rightarrow h = 7/2$

$\therefore$ point of intersection $\left(\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right)$

$$\frac{hf - bg}{ab - h^2} = \frac{\frac{7}{2}\cdot\frac{5}{2} - 2.\frac{5}{2}}{6 - \frac{49}{4}}$$

$$= \frac{35 - 20}{24 - 49} = \frac{15}{-25} = \frac{-3}{5}$$

$$\frac{gh - af}{ab - h^2} = \frac{\frac{5}{2}\cdot\frac{7}{2} - 3.\frac{5}{2}}{6 - \frac{49}{4}}$$

$$= \frac{35 - 30}{24 - 49} = \frac{5}{-25} = \frac{-1}{5}$$

$\therefore$ Point of Intersection $= \left(\dfrac{-3}{5}, \dfrac{-1}{5}\right)$.

Example 10.45 :

Find the value of k for which the equation $x^2 - y^2 + 2x + 2y + k = 0$ represents a pair of straight lines.

Solution : A necessary condition for the given equation to represent a pair of lines is

$$a = 1 \qquad g = 1$$

$$b = -1 \qquad h = 0$$

$$c = k \qquad f = 1$$

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

therefore $1(-1)(k) + 2(1)\,(1)\,(0) - 1(1)^2 - (-1)\,(1)^2$

$$- k(0)^2 = 0$$

$$- k - 1 + 1 = 0$$

$$k = 0$$

Example 10.46 :

Find the value of k, if the equation $2x^2 + k\,xy - 6y^2 + 3x + y + 1 = 0$ represents a pair of straight lines. Find the point of intersection of lines and the angle between the straight lines for this value of k.

Sol : By comparing the given equation with $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ here

$$a = 2, \qquad 2f = 1 \Rightarrow f = \frac{1}{2}$$

$$b = -6, \qquad 2g = 3 \Rightarrow g = \frac{3}{2}$$

$$c = 1, \qquad 2h = k \Rightarrow h = \frac{k}{2}$$

$$-12 + 2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{k}{2}\right) \; 2\left(\frac{1}{4}\right) - 6\left(\frac{9}{4}\right) - \frac{k^2}{4} = 0$$

$$-48 + 3k - 2 + 54 - k^2 = 0$$

$$-k^2 + 3k + 4 = 0 = \; k^2 - 3k - 4 = 0$$

$$(k - 4)\,(k + 1) = 0$$

$$k = 4 \text{ (or) } -1$$

Case (1) : If $k = -1$

Point of intersection = $\left(\dfrac{hf - bg}{ab-h^2}, \dfrac{gh-af}{ab-h^2}\right)$

$$\frac{hf - bg}{ab-h^2} = \frac{-\frac{1}{2}.\frac{1}{2} - (-6)\frac{3}{2}}{2(-6) - \frac{1}{4}} = \frac{-1+36}{-49} = \frac{35}{-49} = \frac{-5}{7}$$

$$\frac{gh - af}{ab-h^2} = \frac{\frac{3}{2}\left(-\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)}{-12 - \frac{1}{4}} = \frac{-3-4}{-49} = \frac{1}{7}$$

$\therefore$ Point of intersection $= \left(\dfrac{-5}{7}, \dfrac{1}{7}\right)$

Angle between the lines $= \cos^{-1} \dfrac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}}$

$$= \cos^{-1} \frac{|2-6|}{\sqrt{(2+6)^2+4}} = \cos^{-1}\left(\frac{4}{\sqrt{65}}\right)$$

Case (ii): If $k = 4$. then

$$\frac{hf - bg}{ab-h^2} = \frac{2\left(\frac{1}{2}\right) + 6\frac{3}{2}}{-12-4} = \frac{1+9}{-16} = \frac{-5}{8}$$

$$\frac{gh - af}{ab-h^2} = \frac{\frac{3}{2}(2) - 2.\frac{1}{2}}{-12-4} = \frac{2}{-16} = -\frac{1}{8}$$

Point of intersection P $= \left(-\dfrac{5}{8}, -\dfrac{1}{8}\right)$

Angle $\cos\alpha = \dfrac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}}$

$$= \frac{|2-6|}{\sqrt{(2+6)^2+16}}$$

$$= \frac{4}{4\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\alpha = \cos^{-1}\frac{1}{\sqrt{5}}.$$

EXERCISE 10.9

1. Show that the equation $x^2 - y^2 - x + 3y - 2 = 0$ represents a pair of perpendicular lines, and find their equations.

2. Find the distances between the following pairs of parallel straight lines:

   (i) $9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$

   (ii) $x^2 + 2\sqrt{3}\,xy + 3y^2 - 3x - 3\sqrt{3}\,y - 4 = 0$

3. Find the point of intersection of the pair of straight lines $2(x^2 - y^2) + 3xy + 7x + 9y + 5 = 0$.

4. Find the value of 'a' if the equation $ax^2 + 3xy - 2y^2 - 5x + 5y + 8 = 0$ represents a pair of perpendicular lines.

5. Find the angle between lines $ax^2 + 2hxy - ay^2 = 0$.

10.8.8 Homogenising a second degree equation with a first degree equation in *x* and *y*.

Generally, the locus of a second degree equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

is called the second degree curve.

This second degree curve can be either pair of straight lines (or) circle (or) conic. Any such curve has atmost two points of intersection with a given straight line is not contained in the locus (1).

In this section we find the combined equation of the pair of lines joining the origin to the points of intersection of a second degree curve with a given straight line.

Let the straight line $lx + my + n = 0$, $(n \neq 0)$ intersect the second degree curve (1) in two distinct points A and B. Then the equation $\overleftrightarrow{AB}$ can be taken as $\dfrac{lx+my}{-n} = 1$ ...(2)

we homogenise equation (1) with equation (2) as follows.

Let us write (1) as

$ax^2 + 2hxy + by^2 + 2gx(1) + 2fy(1) + c(1)^2 = 0$

$$\Rightarrow ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx+my}{-1}\right)^2$$

$$+ 2fy\left(\frac{lx+my}{-1}\right) + c\left(\frac{lx+my}{-1}\right)^2 = 0 \quad ...(3)$$

$$\Rightarrow a'x^2 + 2h'xy + b'y^2 = 0$$

where $a' = a - \dfrac{2gl}{n} + \dfrac{cl^2}{n^2}$,

$$h' = h - \frac{gm}{n} - \frac{fl}{n} + \frac{clm}{n}$$

$$b' = b - \frac{2fm}{n} + \frac{cm^2}{n^2}$$

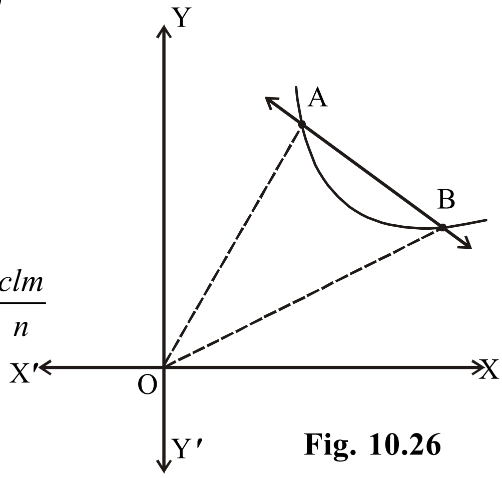

Fig. 10.26

Therefore equation (3) represents the combined equation of $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$.

Example 10.47 : Show that the lines joining the origin to the points of intersection of the curve $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ and the straight line $x - y - \sqrt{2} = 0$ are mutually perpendicular.

Solution: Given curve $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ ...(i)

Given straight line $x - y - \sqrt{2} = 0$ ...(ii)

Let the straight line meet the given curve at A and B.

Then the equation of $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ is obtained by homogenising the equation (1) with the help of equation (2)

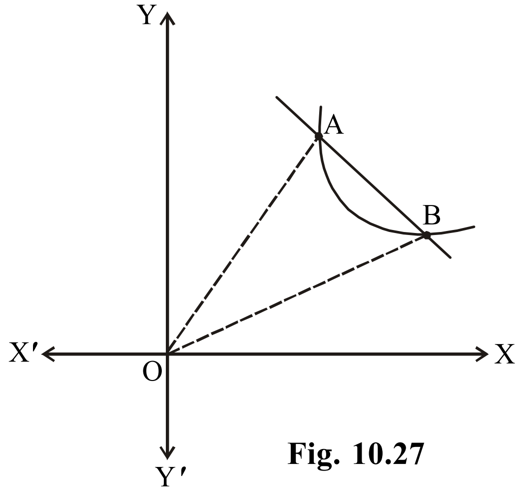

Fig. 10.27

Form $\dfrac{x-1}{\sqrt{2}}=1$

homogenising (1)

$$x^2 - xy + y^2 + 3x + 3y - 2 = 0$$

$$x^2 - xy + y^2 + 3x\ (1) + 3y\ (1) - 2(1)^2 = 0$$

$$x^2 - xy + y^2 + 3x\left(\frac{x-y}{\sqrt{2}}\right)+3y\left(\frac{x-y}{\sqrt{2}}\right)-2\left(\frac{x-y}{\sqrt{2}}\right)^2=0$$

$$x^2-xy+y^2+\frac{3x^2-3y^2}{\sqrt{2}}-(x^2-2xy+y^2)=0$$

that is, $3x^2+\sqrt{2}xy-3y^2=0$

which is the equation of pair of lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$.

Since the sum of the coefficients of x^2 and y^2 in this equation zero, the lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ are mutually perpendicular..

Example 10.48 : Find the condition if the lines joining the origin to the points of intersection of curve. $3x^2 - xy + 3y^2 + 2x - 3y + 4 = 0$ and the line $2x + 3y = k$ are mutually perpendicular.

Solution: Given curve $3x^2 - xy + 3y^2 + 2x - 3y + 4 = 0$ ...(1)

straight line $2x + 3y = k$ ...(2)

Let the straight line meet the curve at A and B then the equation of $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$ is obtained by homogenising (1) with help of (2)

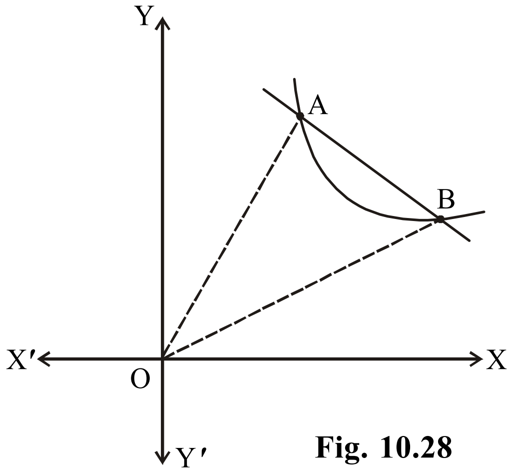

Fig. 10.28

from (2) $\dfrac{2x+3y}{k}=1$

homogenising (1)

$$3x^2 - xy + 3y^2 + 2x(1) - 3y\,(1) + 4(1)^2 = 0$$

$$3x^2 - xy + 3y^2 + 2x\left(\frac{2x+3y}{k}\right) - 3y\left(\frac{2x+3y}{k}\right) + 4\left(\frac{2x+3y}{k}\right)^2 = 0$$

$$3x^2 - xy + 3y^2 + \frac{4x^2-9y^2}{k} + 4\left(\frac{4x^2+9y^2+12xy}{k^2}\right) = 0$$

$$\Rightarrow \left(3+\frac{4}{k}+\frac{16}{k^2}\right)x^2 + \left(\frac{48}{k^2}-1\right)xy + \left(3-\frac{9}{k}+\frac{36}{k^2}\right)y^2 = 0$$

which is the equation of $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$

Since $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ are mutually perpendicular

x^2 coefficient + y^2 coefficient = 0

$$3+\frac{4}{k}+\frac{16}{k^2}+3-\frac{9}{k}+\frac{36}{k^2} = 0$$

$$3k^2 + 4k + 16 + 3k^2 - 9k + 36 = 0$$

$$6k^2 - 5k + 52 = 0$$

EXERCISE 10.10

1. Find the angle between the lines joining the origin to the points of intersection of the curve $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0$ and the line $3x - y + 1 = 0$.

2. Find the values of k, if the lines joining the origin to the points of intersection of the curve $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ and the line $x + 2y = k$ are mutually perpendicular.

3. Find the condition for the chord $lx + my = 1$ of the circle $x^2 + y^2 = a^2$ to subtend a right angle at the origin.

4. Find the angle between the lines joining the origin to the points of intersection of the curve $7x^2 - 4xy + 8y^2 + 2x - 4y - 8 = 0$ and the line $3x - y = 2$.

KEY WORDS

- The equation of a line parallel to y-axis is $x = a$ and parallel to x-axis is $y = b$.
- The equation of the line which cuts off intercept c on y-axis and having slope m is $y = mx + c$
- The equation of the line passing through $A(x_1, y_1)$ and having the slope m is $y - y_1 = m(x - x_1)$.
- The equation of the line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$
- The equation of the line which cuts off intercepts a and b on x-axis and y-axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$
- The equation of the line in normal or perpendicular form is

  $x \cos \alpha + y \sin \alpha = p$. where p is the length of perpendicular from the origin to the line and α is the angle which this perpendicular makes with the positive direction of the x-axis.

- The equation of a line in the parametnc form is $\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}=r$ where r is the distance of any point $P(x, y)$ on the line from a given point $A(x_1, y_1)$ on the given line and α is the angle which the line makes with the positive direction of the x-axis.

- The general equation of first degree *in* x and y always represents a straight line provided A and B are not both zero simultaneously.

- From general equation $Ax + By + C = 0$we can evaluate the following:

  i) Slope of the line = $-\frac{A}{B}$

  ii) x-intercept == $-\frac{C}{A}$

  iii) y-intercept = $-\frac{C}{B}$

  iv) Length of perpendicular from the origin to the line = $\frac{|C|}{\sqrt{(A^2+B^2)}}$

  v) Inclination of the perpendicular from the origin is given by

  $$\cos\alpha = \frac{\mp A|}{\sqrt{A^2+B^2}} \ ; \quad \sin\alpha = \frac{\mp B}{\sqrt{A^2+B^2}}$$

  where the upper sign is taken for $C > 0$ and the lower sign for $C < 0$; but if $C = 0$ then either only the upper sign or only the lower sign are taken.

- If θ be the angle between two lines with slopes m_1, m_2 then

  $$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

  (i) Lines are parallel if $m_1 = m_2$

  (ii) Lines are perpendicular if $m_1m_2 = -1$

(B) Angle θ between two lines of general form is:

$$\tan\theta = \frac{A_2B_1 - A_1B_2}{A_1A_2 + B_1B_2}$$

(i) Lines are parallel if $A_1B_2 - A_2B_1 = 0$ or $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0$

(ii) Lines are perpendicular if $A_1A_2 + B_1B_2 = 0$.

- Distance of a given point (x_1, y_1) from a given line $Ax + By + C = 0$

  is $d = \left| \dfrac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$

- Equation of a line parallel to the line $Ax + By + C = 0$ is

  $Ax + By + K = 0$

- Equation of a line perpendicular to the line $Ax + By + C = 0$ is $Bx - Ay + K = 0$.

- If a, b, h are not all zero, then the equation $H \equiv ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines if and only if $h^2 \geq ab$.

- Let $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines. Then the angle θ between them is given by

$$\cos\theta = \frac{a+b}{\sqrt{(a-b)^2 + 4h^2}},$$

  If the above lines are mululy perpendicular if and only if $a + b = 0$.

- The general equation of second degree in x and y,

  $S \equiv a\,x^2 + 2hxy + b\,y^2 + 2\,gx + 2fy + c = 0$ represents a pair of straight lines if and only if

  (i) $abc + 2\,fgh - a\,f^2 - bg^2 - ch^2 = 0$

  (ii) $h^2 \geq ab,\ g^2 \geq ac,\ f^2 \geq bc$.

- Equation of pair straight lines passing through the origin and parallel to the pair of straight lines $ax^2 + 2hxy + b\,y^2 + 2\,gx + 2\,fy + c = 0$ is $ax^2 + 2hxy + by^2 = 0$.
- Let $S \equiv ax^2 + 2hxy + by^2 + 2\,gx + 2\,fy + c = 0$ represents a pair of straight lines. Then their point of intersection is $\left(\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right)$.

SUPPORTIVE WEB SITES

http : //www.wikipedia.org

http:// math world . wolfram.com

PRACTICE EXERCISE

1. Find the equation of the straight line whose y-intercept is -3 and which is:

   (a) parallel to the line joining the points $(-2,3)$ and $(4, -5)$.

   (b) perpendicular to the line joining the points $(0, -5)$ and $(-1,3)$.

2. Find the equation of the line passing through the point $(4, -5)$ and

   (a) parallel to the line joining the points $(3, 7)$ and $(-2, 4)$.

   (b) perpendicular to the line joining the points $(-1, 2)$ and $(4, 6)$.

3. Show that the points $(a, 0)$ $(0, b)$ and $(3a, -2b)$ are collinear. Also fmd the equation of the line containing them.

4. A(1, 4), B(2, −3) and C(−1, −2)are the vertices of triangle ABC. Find

   (a) the equation of the median through A.

   (b) the equation of the altitude througle A.

   (c) the right bisector of the side BC.

5. A straight line is drawn through point A (2, 1) making an angle of $\dfrac{\pi}{6}$ with the positive direction of x-axis. Find the equation of the line.

6. A straight line passes through the point (2, 3) and is parallel to the line $2x + 3y + 7 = 0$ Find its equation.

7. Find the equation of the line having a and b as x-intercept and y-intercepts respectively.

8. Find the angle between the lines $y = (2-\sqrt{3})x+5$ and $y = (2+\sqrt{3})x+5$.

9. Find the angle between the lines $2x + 3y = 4$ and $3x - 2y = 7$.

10. Find the length of the per pendicular drawn from the point (3, 4) on the straight line $12(x + 6) = 5(y - 2)$.

11. Find the length of the perpendicular from (0, 1) on $3x + 4y + 5 = 0$.

12. Find the distance between the lines $2x + 3y = 4$ and $4x + 6y = 20$.

13. Find the length of the perpendicular drawn from the point $(-3, -4)$ on the line $4x - 3y = 7$.

14. Show that the product of the perpendic ulars drawn from the points on the straight line $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ is b^2.

15. Prove that the equation of the straight line which passes through the point $(a \cos^3\theta, b \sin^3\theta)$ and is perpendicular to $x \sec\theta + y \operatorname{cosec}\theta = a$ is $x \cos\theta - y \operatorname{cosec}\theta = a \cos 2\theta$.

16. Prove the following geometrical results:

    a) The altitudes of a triangle are concurrent

    b) The medians of an equaliateral triangle are equal.

    c) The area of an equalateral triangle of side a is $\frac{\sqrt{3}a^2}{4}$

    d) If G is the centroid of ΔABC and 0 is any point, then $OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 30G^2$

    e) The perpendicular drawn from the centre of a circle to a chord bi-sects the chord.

ANSWERS

EXERCISE 10.1

1. (a) Straight line 2. (a) 1 (b) -1 (c) 1
3. 7 4. 20 5.(a) $y = -4$ (b) $x = -3$
6. $x = 5$ 7. $y + 7 = 0$

EXERCISE 10.2

1. (a) $y = 2x - 2$ (b) Slope $\frac{-4}{3}$, x - intercept = $\frac{3}{2}$, y - intercept = 2
2. $\sqrt{3}y = -3x - 1$ 3. Slope = $\frac{1}{2}$, y-intercept= -2
4. $3x + 7y = 7$ 5. $y = x + 1$; $x + y - 3 = 0$
6. $3x - 2y = 0$
7. (a) $x + y = -1$

   (b) Equation of the diagonal AC = $2x - y - 4 = 0$

   Equation of the diagonal BD = $2x - 11y + 66 = 0$
8. $x - 2 = 0$, $x - 3y + 6 = 9$ $5x - 3y - 2 = 0$
9. $2x + 3y = 6$ 10. $3x + y = 6$ 11. $3x + 4y = 11$
12. $x + y = 2\sqrt{2}$

14. $\dfrac{x-2}{\frac{1}{\sqrt{2}}} = \dfrac{y-1}{\frac{1}{\sqrt{2}}}$ and the co-ordinates of the point are $\left(2+\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$

EXERCISE 10.3

1. A and B are not both simltaneously zero
2. $y = \frac{-2}{5}x - \frac{3}{5}$

3. (a) x -intercept $=\dfrac{-C}{m}$; y-intercept = C

(b) x -intercept $=\dfrac{-8}{3}$; y-intercept $=\dfrac{8}{7}$

(c) x -intercept $= -4$; y-intercept $= -6$

4. $2\sqrt{13}$ units 5. $\dfrac{x}{P\sec\alpha}+\dfrac{x}{P\,\mathrm{cosec}\,\alpha}=1$

6. (a) $-\dfrac{3}{5}x+\dfrac{4}{4}y-2=0$ (b) $-\dfrac{3}{x}+\dfrac{4}{4}y=0$

7. The fIrst line is nearer from the origin.

8. K = −7

9. (2, 1)

10. S = (−8, −10)

11. O = (−3, 2)

EXERCISE 10.4

1. (a) $m_1=\dfrac{-A}{B}$ (b) $Am_1 = B$

2. $\theta = \tan^{-1}\left(\dfrac{7}{6}\right)$

3. (a) (i) $\dfrac{A_1}{B_1}=\dfrac{A_2}{B_2}$ (ii) $A_1A_2 + B_1B_2 = 0$

(b) Parallel (c) Perpendicular.

EXERCISE 10.5

1. $d=\dfrac{16}{\sqrt{13}}$ 2. $\left(0, \dfrac{b}{a}\left(a\pm\sqrt{a^2+b^2}\right)\right)$ 3. $\left(0, \dfrac{32}{3}\right)$

4. $\dfrac{14}{\sqrt{58}}$ 5. $\left(\dfrac{3}{4}(4\pm5), 0\right)$

6. $\left(\frac{8}{5}. \frac{21}{5}\right)$ 7. $3x - 4y = 18$

8. $\left(\frac{74}{41}, \frac{133}{41}\right)$ 9. $\frac{1}{2\sqrt{34}}$

EXERCISE 10.6

1. $3x + y + 2 = 0$ 2. $y = 2x + 2$ 3. $x - y = 3$

4. $y = x$

5. $3x - 2ay = 6(a - 1)$ 7. $2x - 3y + 6 = 0$

EXERCISE 10.8

1. (i) $\cos^{-1}\frac{24}{25}$ (ii) $\cos^{-1}\left(\frac{7}{\sqrt{232}}\right)$ (iii) α

   (iv) $\frac{\pi}{2}$

3. (i) $\left(\frac{20}{9}, -\frac{44}{9}\right), \frac{56}{3}$ (ii) (0, -4), 36

EXERCISE 10.9

1. $x + y - 2 = 0,\ x - y + 1 = 0$

2. (i) $\sqrt{\frac{2}{5}}$ (ii) $\frac{5}{2}$

3. $\left(-\frac{11}{5}, \frac{3}{5}\right)$

4. 2

5. $\frac{\pi}{2}$

EXERCISE 10.10

(1) $\cos^{-1}\left(\frac{13}{\sqrt{193}}\right)$

(2) ± 1

(3) $a^2 (l^2 + m^2) = 2$

(4) 90^0

PRACTICE EXERCISE

1. (a) $4x + 3y + 9 = 0$

   (b) $x - 8y - 24 = 0$

2. (a) $3x - 5y - 37 = 0$

   (b) $5x - 8y - 60 = 0$

4. (a) $13x - y - 9 = 0$

   (b) $3x - y + 1 = 0$

   (c) $3x - y - 4 = 0$

5. $x - \sqrt{3}y = 2 - \sqrt{3}$

6. $2x + 3y + 13 = 0$

7. $bx + ay = ab$ 8. $\frac{\pi}{2}$ 9. $\frac{\pi}{2}$

10. $\frac{98}{13}$ 11. $\frac{9}{5}$

12. $\frac{6}{\sqrt{13}}$ 13. $\frac{7}{5}$

Chapter 11

CIRCLES

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- derive and find the equation of a circle with a given centre and radius;
- state the conditions under which the general equation of second degree in two variables represents a circle;
- derive and find the centre and radius of a circle whose equation is given in general form;
- find the equation of a circle passing through :

  (i) three non-collinear points

  (ii) two given points and touching any of the axes;
- derive the equation of a circle in the diameter form;
- find the equation of a circle when the end points of any of its diameter are given; and
- find the parametric representation of a circle with given centre and radius.

PREREQUISITES

- Terms and concepts connected with circle.
- Distance between two points with given coordinates.
- Equation of a straight line in different forms.

INTRODUCTION

Geometry has probably originated in ancient Egypt and Flourished in Greece. India and other countries. In the 6th century B.C., the systematic development of geometry has begun.

Mathematicians Menachmus, Archimedis' and Thales worked on the circle and a tangent to the circle. During the fifth century Euclid collected all the known works and arranged them in his famous book called "The Elements".

The shape of a wheel bangle, shape of a round clock and some cions are of a circular shape.

Notice the path in which the tip of the hand of a watch moves. (see Fig. 11.1)

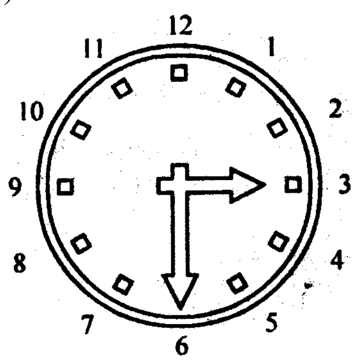

Fig. 11.1

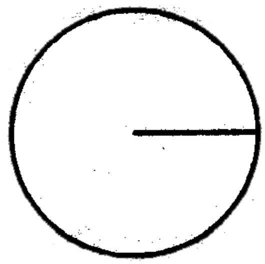

Fig. 11.2

Again, notice the curve traced out when a nail is fixed at a point and a thread of certain length is tied to it in such a way that it can rotate about it, and on the other end of the thread a pencil is tied. Then move the pencil around the fixed nail keeping the thread in a stretched position (See Fig 11.2)

Certainly, the curves traced out in the above examples are of the same shape and this type of curve is known as a *circle.*

The distance between the tip of the pencil and the point, where the nail is fixed is known as the *radius* of the circle.

In this chapter, we deal with circle and obtain its equation. We derive the equation of a chord, tangent and normal and study some important topics related to circle.

11.1 DEFINITION OF THE CIRCLE

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point in the same plane remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

11.2 EQUATION OF A CIRCLE

Can we find a mathematical expression for a given circle?

Let us try to find the equation of a circle under various given conditions.

11.2.1 WHEN COORDINATES OF THE CENTRE AND RADIUS ARE GIVEN

Let C be the centre and a be the radius of the circle. Coordinates of the centre are given to be (h, k), say.

Take any point $P(x, y)$ on the circle and draw perpendiculars CM and PN on OX. Again, draw CL perpendicular to PN.

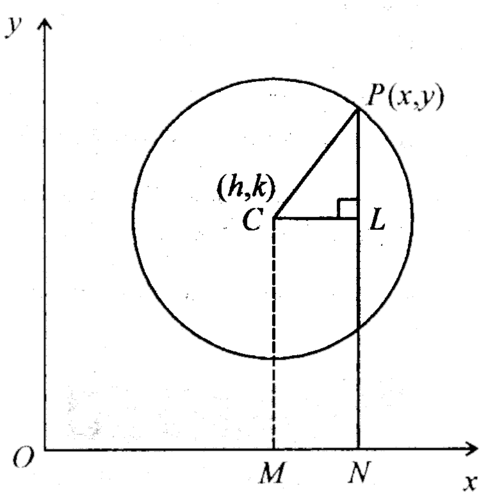

Fig. 11.3

We have

CL = MN = ON − OM = $x - h$

and $\quad PL = PN - LN = PN - CM$

$$= y - k$$

In the right angled triangle CLP, $CL^2 + PL^2 = CP^2$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = a^2 \qquad ...(1)$$

This is the required equation of the circle under given conditions. This form of the circle is known as *standard form* of the circle.

Conversely, if(x,y) is any point in the plane satisfying (l), then it is at a distance *'a'* from (h, k). So it is on the circle.

What happens when the

i) circle passes through the origin?

ii) circle does not pass through origin and the centre lies on the x-axis?

iii) circle passes through origin and the x-axis is a diameter?

iv) centre of the circle is origin?

v) circle touches the x-axis?

vi) circle touches the y-axis?

vii) circle touches both the axes?

We shall try to find the answer of the above questions one by one.

(i) In this case, since (0, 0) satisfies (l), we get

$$h^2 + k^2 = a^2$$

Hence the equation (1) reduces to

$$x^2 + y^2 - 2hx - 2ky = 0 \qquad ... (2)$$

(ii) In this case $k = 0$

Hence the equation (1) reduces to

$$(x-h)^2 + y^2 = a^2 \qquad ... (3)$$

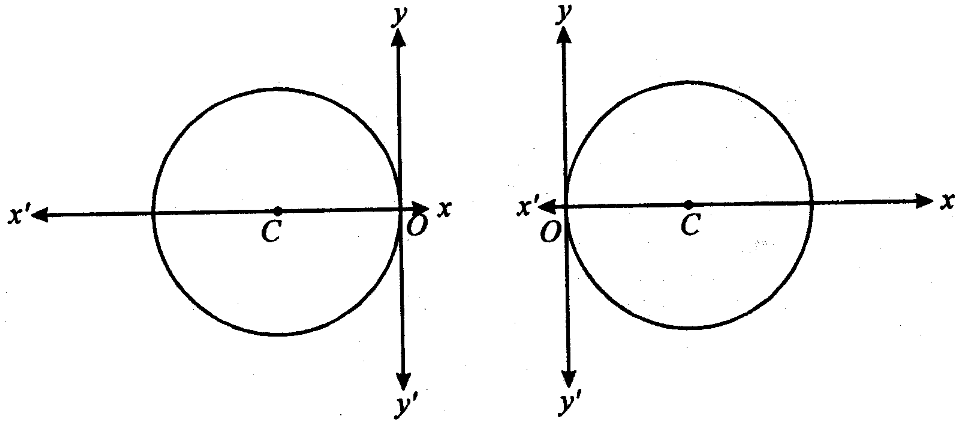

Fig. 11.4

iii) In this case $k = 0$ and $h = \pm a$ (see Fig. 11.4)

Hence the equation (1) reduces to $x^2 + y^2 \pm 2ax = 0$

... (4)

iv) In this case $h = 0 = K$

Hence the equation (1) reduces to $x^2 + y^2 = a^2$

...(5)

v) In this case $k = a$ (see Fig. 11.5)

Hence the equation (1) reduces to $x^2 + y^2 - 2hx - 2ay + h^2 = 0$

...(6)

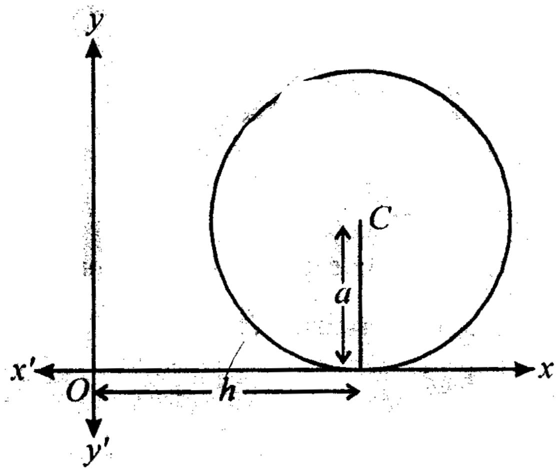

Fig. 11.5

vi) In this case $h = a$

Hence the equation (1) reduces to

$$x^2 + y^2 - 2ax - 2ky + k^2 = 0 \qquad \text{... (7)}$$

vii) In this case $h = k = a$(See Fig. 11.6)

Hence the equation (1) reduces to

$$x^2 + y^2 - 2ax - 2ay + a^2 = 0 \qquad \text{... (8)}$$

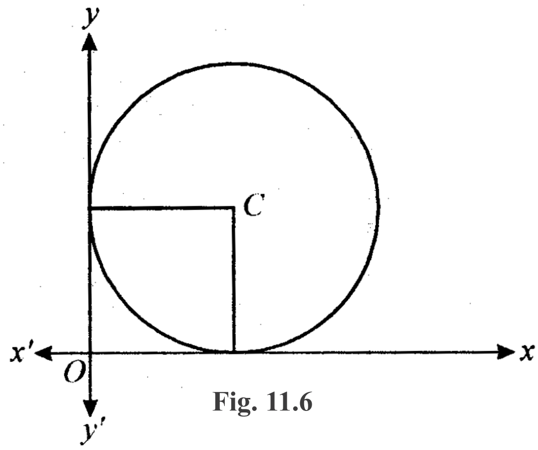

Fig. 11.6

Example 11.1 : Find the equation of the circle whose centre is (3,–4) and radius is 6.

Solution : Comparing the terms given in equation (1), we have

$$h = 3;\ k = -4,\ a = 6$$

$$\therefore \quad (x - 3)^2 + (y + 4)^2 = 6^2$$

$$\text{or} \quad x^2 + y^2 - 6x + 8y - 11 = 0$$

Example 11.2 : Find the centre and radius of the circle given by $(x + 1)^2 + (y - 1)^2 = 4$.

Solution: Comparing the given equation with $(x - h)^2 + (y - k) = a^2$ we find that

$$-h = 1,\ -k = -1;\ a^2 = 4$$

$$\therefore \quad h = -1;\ k = 1,\ a = 2$$

So the given circle has its centre (–1,1) and radius 2.

11.3 GENERAL EQUATION OF THE CIRCLE IN SECOND DEGREE IN TWO VARIABLES

The standard equation of a circle with centre (h, k) and radius r is given by

$$(x - h)^2 + (y - k)^2 = r^2 \qquad .... (1)$$

or $$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0 \qquad ... (2)$$

This is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ... (3)$$

$$\Rightarrow \quad (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$$

$$\Rightarrow (x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

$$\Rightarrow [x-(-g)]^2 + [y-(-f)]^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2 \qquad ... (4)$$

$$\Rightarrow \quad (x - h)^2 + (y - k)^2 = r^2$$

where $h = -g,\ k = -f,\ r = \sqrt{g^2 + f^2 - c}$

This shows that the given equation represents a circle with centre $(-g, -f)$, and radius $\sqrt{g^2 + f^2 - c}$.

11.3.1 CONDITONS UNDER WHICH THE GENERAL EQUATION OF SECOND DEGREE IN TWO VARIABLES REPRESENTS A CIRCLE

Let the equation be $x^2 + y^2 + 2gx + 2fy + c = 0$

(i) It is a second degree equation in x, y in which coefficients of the tenns involving x^2 and y are equal.

(ii) It contains no term involving xy.

(iii) Equation of the circle having the centre on X-axis is in the form

$x^2 + y^2 + 2gx + c = 0$ ($\because y$ - cordinate of the centre is zero)

(iv) Equation of the circle having the centre on Y-axis is in the form

$$x^2 + y^2 + 2fy + c = 0 \qquad (\because \ x \text{ coordinate of the centre is zero})$$

Note: *In solving problems, we keep the coefficients of x^2 and y^2 unity.*

Example 11.3: Find the centre and radius of the circle

$$45x^2 + 45y^2 - 60x + 36y + 9 = 0$$

Solution: Given equation can be written on dividing by 45 as

$$x^2 + y^2 \frac{-4}{5}x + \frac{4}{5}y + \frac{19}{45} = 0$$

$x^2 + y^2 + 2gx + 2fy + c = 0$ we get

$$g = -\frac{2}{3}\,;\; f = \frac{2}{5} \quad \text{and} \quad c = \frac{19}{45}$$

$\therefore$ Thus, the centre is $\left(\frac{2}{3}, -\frac{2}{5}\right)$ an radius is $\sqrt{g^2 + f^2 - c} = \frac{\sqrt{41}}{15}$

Example 11.4 : Find the equation of the circle which passes through the points (1, 0), (0, −6) and (3, 4).

Solution: Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(1)$$

Since the circle passes through three given points so they will satisfy the equation (1). Hence

$$1 + 2g + c = 0 \qquad ...(2)$$

and $\quad 36 - 12f + c = 0 \quad ...(3)$

$$25 + 6g + 8f + c = 0 \qquad ...(4)$$

Subtracting (2) from (3) and (3) from (4), we have

$$2g + 12f = 35$$

and $\quad 6g + 20f = 11$

Solving these equations for g and f, we get $g = -\frac{71}{4}$, $f = \frac{47}{8}$

Substitutingg in (2), we get $c = \frac{69}{2}$

and substituting g, f and c in (1), the required equation of the circle is

$$4x^2 + 4y^2 - 142x + 47y + 138 = 0$$

Example 11.5: Find the equation of the circles which touches the axis of x and passes through the points (1, –2) and (3, –4).

Solution: Since the circle touches the x-axis, put $k = a$ in the standard form (See result 6) of the equation of the circle, we have

$$x^2 + y^2 - 2hx - 2ay + h^2 = 0 \qquad ...(1)$$

This circle passes through the point (1, –2)

$$\therefore \quad h^2 - 2h + 4a + 5 = 0 \qquad ...(2)$$

Also, the circle passes through the point (3, – 4)

$$\therefore \quad h^2 - 6h + 8a + 25 = 0 \qquad ...(3)$$

Eliminationg *'a'* from (2) and (3), we get

$$\Rightarrow \quad h^2 + 2h - 15 = 0$$

$$h = 3 \quad \text{or} \quad h = -5$$

From (3) the corresponding values of a are –2 and –1 0 respectively. On substituting the values of h and a in (1) we get

$$x^2 + y^2 - 6x + 4y + 9 = 0 \qquad ...(4)$$

and $$x^2 + y^2 + 10x + 20y + 25 = 0 \qquad ... (5)$$

(4) and (5) represent the required equations.

11.4 EQUATION OF A CIRCLE WHEN END POINTS OF ONE OF ITS DIAMETERS ARE GIVEN

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given end points of the diamete AB (See Fig. 11.7)

Let $P(x, y)$ be any point on the circle drawn on AB as diameter. Join AP and BP Since the angle in a semi -circle is a right angle.

$$\therefore \quad AP \perp BP$$

$\therefore$ (slope of AP) × (slope of BP) = –1

Now, the slope of $AP = \dfrac{y - y_1}{x - x_1}$ and the slope of $BP = \dfrac{y - y_2}{x - x_2}$

MODULE - II
Coordinate Geometry

Notes

$$\therefore \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

or $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ ...(1)

Since (1) is true for every point on the circle, and for no other point in the plane:

$\therefore$ (1) represents the equation of the circle in ***diameter from***.

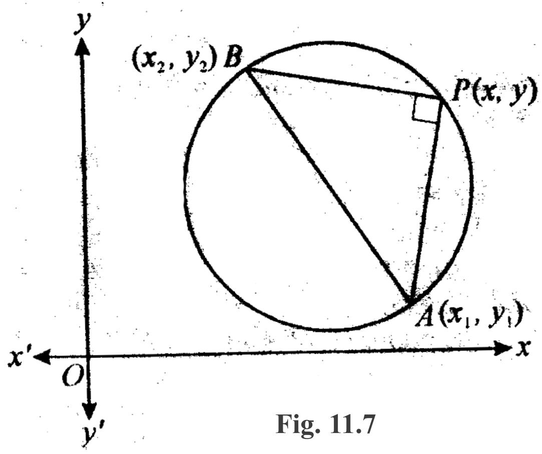

Fig. 11.7

Example 11.6: Find the equation of the circle described on the line joining the origin and the point (2, −4) as diameter.

Solution: Here $x_1 = 0,\ y_1 = 0,\ x_2 = 2,\ y_2 = -4$

Using the Equation (1) required the equation of the circle is,

$$(x - 0)(x - 2) + (y - 0)[y - (-4)] = 0$$

or $$x^2 - 2x + y^2 + 4y = 0$$

or $$x^2 + y^2 - 2x + 4y = 0$$

Example 11.7 : The equation of a chord of the circle $x^2 + y^2 - 2ax = 0$ is $y = mx$. Find the equation of the circle described on this chord as diameter.

Solution : The coordinates of the points of intersection of the circle and the given chord are,

$$(0,\ 0) \text{ and } \left(\frac{2a}{1+m^2}, \frac{2am}{1+m^2}\right).$$

Now, for the required equation of the circle these points are the end points of the diameter, so by equation (1)

$$(x-0)\left(x - \frac{2a}{1+m^2}\right) + (y-0)\left(y - \frac{2am}{1+m^2}\right) = 0$$

or $\qquad x^2 + y^2 - \dfrac{2a}{1+m^2}x - \dfrac{2am}{1+m^2}y = 0$

or $\qquad (1 + m^2)\, x^2 + (1 + m^2)\, y^2 - 2ax - 2amy = 0$

$\therefore$ This is the required equation of the circle.

11.5 EQUATION OF A CIRCLE WHEN RADIUS AND ITS INCLINATION ARE GIVEN (P ARAMATRIC FORM)

In order to fmd the equation of the circle whose centre is the origin and whose radius is r. Let P(x, y) be any point on the circle. Draw

PM $\perp$ OX

$\therefore$ OM = x; MP = y. Join OP.

Let $\quad \underline{|XOP} = \theta$ and OP = r

Now x = OM = $r \cos\theta$

and $\quad y$ = MP = $r \sin\theta$

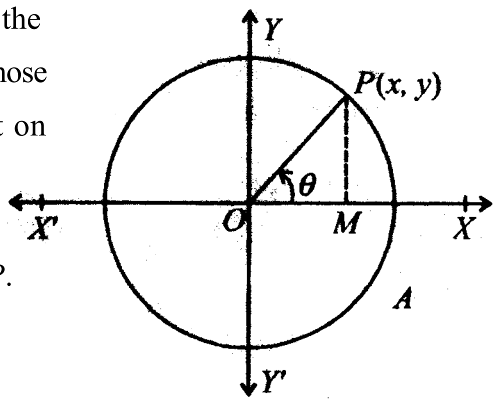

Fig. 11.8

Hence the two equations $x = r\cos\theta$ and $y = r\sin\theta$ taken together represent a circle. These are known as ***parametric form of the equations*** of the circle, where θ is a parameter.

Note : If the centre of the circle is at hand k, then the parametric form of the *equation of the circle is* $x = h + r\cos\theta$ and $y = k + r\sin\theta$ where θ is is a parametre which lies in the interval $[0, 2\pi]$.

Example 11.8 : Find the parametric form of each of the following circles:

(i) $(x-1)^2 + (y+2)^2 = 9$ $\qquad$ (ii) $(x+2)(x-4) + (y-3)(y+1) = 0$

Solution : (i) The equation of the circle is

$$(x-1)^2 + (y+2)^2 = 3^2$$

Comparing it with the' standard equation' of the circle, we get centre $(1, -2)$ and radius = 3 for the given circle.

$\therefore$ Parametric form of the equation of the circle is

$$x = 1 + 3\cos\theta\ ;\ \ y = -2 + 3\sin\theta$$

(ii) Equation of the circle can be written as

$$x^2 + y^2 - 2x - 2y - 11 = 0$$

Comparing it with the general equation of the circle and fmding centre and radius, we get Centre (1, 1) and radius $= \sqrt{13}$

$\therefore$ Parametric form of the equation of the circle is

$$x = 1 + \sqrt{13}\cos\theta,\ \ y = 1 + \sqrt{13}\sin\theta$$

Example 11.9 : Find the equation of the circle whose centre lies on the X-axis and passing through (–2, 3) and (4, 5).

Solution: Equation of the circle whose centre lies on X-axis is

$$x^2 + y^2 + 2gx + c = 0 \qquad ...(1)$$

Since it passes through (–2, 3) we have

$$-4g + c + 13 = 0 \qquad ...(2)$$

Since it passes through (4, 5) we have

$$8g + c + 41 = 0 \qquad ...(3)$$

Solving (2), (3) for g, c we get

$$g = -\frac{7}{3},\ c = -\frac{67}{3}$$

Thus the equation of the circle is $3x^2 + 3y^2 - 14x - 67 = 0$.

Example 11.10 : Find the equation of the circle through (2, –3) and (–4, 5) and having the centre on the line $4x + 3y + 1 = 0$

Solution: Let the equation of the required circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0. \qquad ...(1)$$

Since passes through (2, –3), we obtain

$$4g - 6f + c + 13 = 0 \qquad ...(2)$$

Since it passes through (–4, 5) we obtain

$$-8g + 10f + c + 41 = 0 \qquad ...(3)$$

Since centre $(-g, -f)$ lies on the line $4x + 3y + 1 = 0$

we obtain $-4g - 3f + 1 = 0$ ...(4)

Solving (2), (3) and (4) we get $g = 1$, $f = -1$, $c = -23$

Thus the equation of the required circle is

$$x^2 + y^2 + 2x - 2y - 23 = 0$$

Example 11.11 : If (1, –6) (5, 2) (7, 0), (–1, c) are concyclic, find c.

Solution: Let A(1, –6), B(5, 2), C(7, 0) and D(–1, c)

Let the equation of the circle through A, B, C is

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad ...(1)$$

Since it passes through A(1, –6), B(5, 2), C(7, 0)

We obtain, $2g - 12f + c + 37 = 0$ ...(2)

$10g + 4f + c + 29 = 0$ ...(3)

$14g + c + 49 = 0$ ...(4)

Solving (2), (3) and (4) we get

$g = -3 \qquad f = 2 \qquad c = -7$

thus the circle through A, B, C is

$$x^2 + y^2 - 6x + 4y - 7 = 0$$

If the given points A, B, C, D are concyclic. then D(-1, c) lies on circle through A, B, C.

i.e., $(-1)^2 + c^2 - 6(-1) + 4(1) - 7 = 0$

$c^2 + 4c = 0$

$c = 0, \quad c = -4.$

EXERCISE 11.1

1. Find the equation of the circle whose centre (– 2, 3) and radius is 4.
2. Find the centre and radius of the circle

   (a) $x^2 + y^2 + 3x - y = 6$

   (b) $4x^2 + 4y^2 - 2x + 3y - 6 = 0$

3. Find the equation of the circle which passes through the points (0, 2) (2, 0) and (0, 0).

4. Find the equation of the circle which touches the y-axis and passes through the points (−1, 2)and (−2,1)

5. Find the equation of the circle described on the line joining the points (2, 3) and (−2,6) as diameter.

6. Find the parametric form of the equation of each of the following circles:

   (a) $(x + 1)^2 + (y + 1)^2 = 4$

   (b) $4x^2 + 4y^2 + 2x + 2y - 3 = 0$

   (c) $(x - 1)(x + 1) + (y - 1)(y + 1) = 0$

7. Find the equation of the circle through (4, 1), (6, 5) and having the centre on the line $4x + y - 16 = 0$.

8. If (2, 0), (0, 1), (4, 5) and (0, c) are concyclic, then find c.

11.6 POSITION OF A POINT WITH RESPECT TO A CIRCLE - DEFINITION OF A TANGENT

In this section, we shall define the position of a point with respect to a circle and power of a point. Further we define tangent and its length from a point, and learn the formula for it.

11.6.1 Notation

Now we introduce certain notations that will be used in the rest of this section and subsequently.

(i) The expression $x^2 + y^2 + 2gx + 2fy + c$ is denoted by S

i.e., $S \equiv x^2 + y^2 + 2gx + 2fy + c.$

(ii) The expression $xx_i + yy_i + g(x + x_i) + f(y + y_i) + c$ is denoted by S_i.

i.e., $S_1 \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c,$

$S_2 \equiv xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c.$

(iii) The expression $x_i x_j + y_i y_j + g(x_i + x_j) + f(y_i + y_j) + c$ is denoted by S_{ij} $(i, j = 1, 2, 3, ...)$For example

$$S_{12} = x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c.$$

$$S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

11.6.2 Position of a point with respect to a circle

A circle in a plane divides the plane into three parts namely

(i) the interior of the circle (see fig. 11.1)

(ii) the circumference which is the circular curve (see fig. 11.2)

(iii) the exterior of the circle (see fig. 11.3)

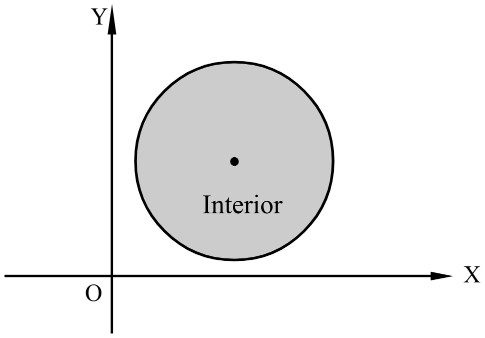

Fig. 11.1

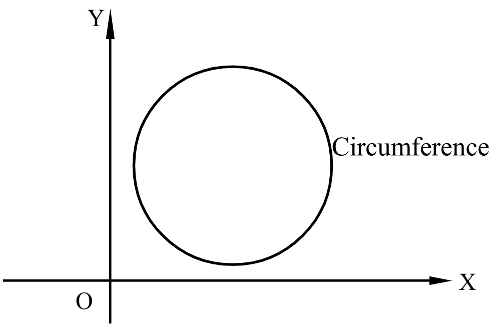

Fig. 11.2

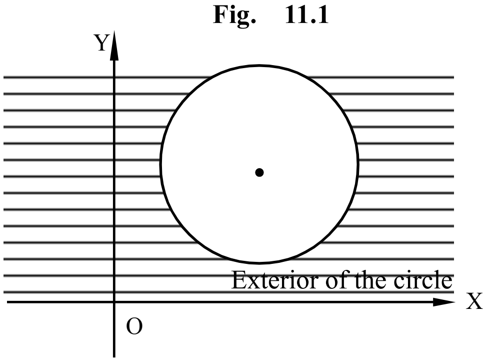

Fig. 11.3

1. Theorem : *Let $S = 0$ be a circle in a plane and $P(x_1, y_1)$ be any point in the same plane. Then*

(i) *P lies in the interior of the circle $\Leftrightarrow S_{11} < 0$.*

(ii) *P lies on the circle $\Leftrightarrow S_{11} = 0$.*

(iii) *P lies in the exterior of the circle $\Leftrightarrow S_{11} > 0$.*

Proof : $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the given circle and $P(x_1, y_1)$, be any point in the plane. Then $C(-g, -f)$, is the centre and $r = \sqrt{g^2 + f^2 - c}$ is the radius of the circle.

(i) P lies in the interior of the circle

$\Leftrightarrow$ $CP < r$ (see fig. 11.4)

$\Leftrightarrow$ $CP^2 < r^2$.

$\Leftrightarrow$ $(x_1 + g)^2 + (y_1 + f)^2 < g^2 + f^2 - c$

$\Leftrightarrow$ $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$

$\Leftrightarrow$ $S_{11} < 0$.

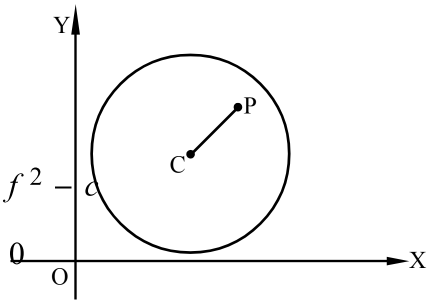

Fig. 11.4

(ii) P lies on the circle

$\Leftrightarrow$ $CP = r$ (see fig. 11.5)

$\Leftrightarrow$ $CP^2 = r^2$

$\Leftrightarrow$ $(x_1 + g)^2 + (y_1 + f)^2 = g^2 + f^2 - c$

$\Leftrightarrow$ $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

$\Leftrightarrow$ $S_{11} = 0$.

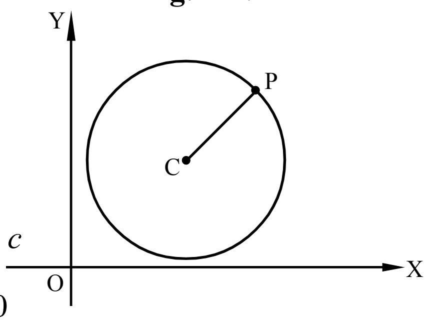

Fig. 11.5

(iii) P lies in the exterior of the circle

$\Leftrightarrow$ $CP > r$ (see Fig. 11.6)

$\Leftrightarrow$ $CP^2 > r^2$

$\Leftrightarrow$ $(x_1 + g)^2 + (y_1 + f)^2 > g^2 + f^2 - c$

$\Leftrightarrow$ $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$

$\Leftrightarrow$ $S_{11} > 0$.

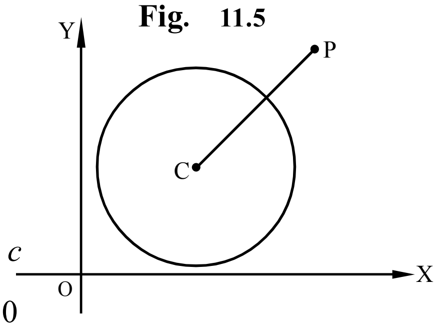

Fig. 11.6

Example 11.12 : Locate the position of the point $(-2, 3)$ with respect to the circle $x^2 + y^2 + 4x + 6y - 3 = 0$.

Sol : Here $(x_1, y_1) = (-2, 3)$, $g = 2$, $f = 3$, $C = -3$

$$\therefore \quad S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + C$$

$$= (-2)^2 + (3)^2 + 4(-2) + 6(3) - 3$$

$$= 20.$$

Since $S_{11} > 0$, by 1. theorem, the point $(-2, 3)$ is an exterior point.

11.6.3 Definition of a tangent

Let P be any point on a given circle and Q be neighbouring point of P lying on the circle. Join P and Q. Then $\overleftrightarrow{PQ}$ is a secant (see fig. 11.7)

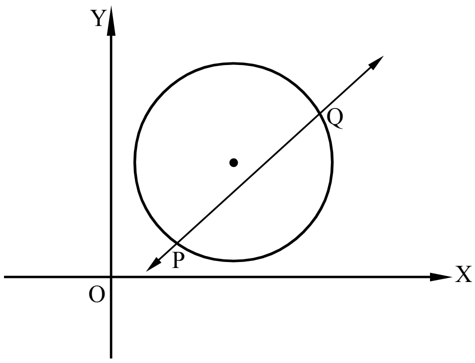

Fig. 11.7

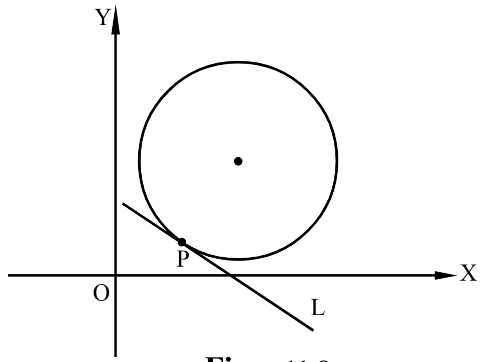

Fig. 11.8

The limiting position of the line (secant) PQ when $Q \to P$ along the circle, is called the tangent at P (see fig. 20.8)

Theorem : The equation of the tangent at the point $P(x_1, y_1)$ on the circle

$S = x^2 + y^2 + 2gx + 2fy + c = 0$ is $S_1 = 0$

Proof: Given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

differentiating on bothsides with respect to 'x'

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x+g}{y+f}$$

$\therefore$ Slope of tangent to the circle S = 0 at $P(x_1, y_1)$ is

$$\left(\frac{dy}{dx}\right)_p = -\frac{x_1+g}{y_1+f}$$

Equation of the tangent to the circle S = at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{x_1+g}{y_1+f}(x - x_1)$$

$\Rightarrow xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx + 2fy_1 + c$

$\Rightarrow S_1 = S_{11}$

$\Rightarrow S_1 = 0$ (since P lies on S = 0 $\Rightarrow S_{11} = 0$)

Note: Equation of the tangent to the circle $x^2 + y^2 = r^2$ at $P(x_1, y_1)$ is

$$xx_1 + yy_1 - r^2 = 0.$$

11.6.4 Definition

If P is an external point to the circle S = 0 and PT is the tangent from P to the circle S = 0 then $\overline{PT}$ is called the length of the tangent from P to the circle (see fig. 11.9)

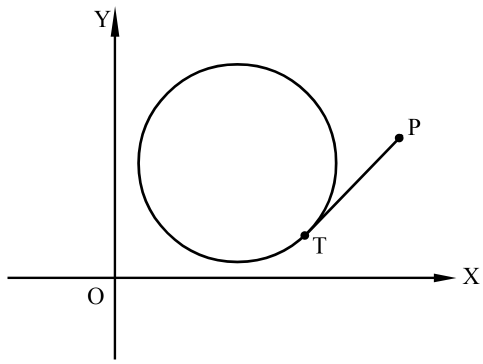

Fig. 11.9

Note : The length of the tangent drawn from $P(x_1, y_1)$ on to the circle S = 0 is $\sqrt{S_{11}}$.

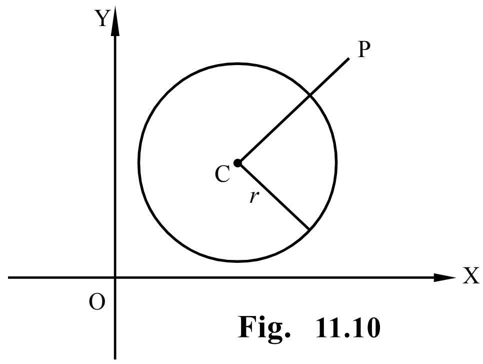

Fig. 11.10

11.6.5 Definition

Supose S = 0 is the equation of a circle with centre C and radius r. Let $P(x_1, y_1)$ be any point in the plane. Then $CP^2 - r^2$ is *defined as the power of P with respect to* S = 0 (see fig. 11.10, 11.11, 11.12).

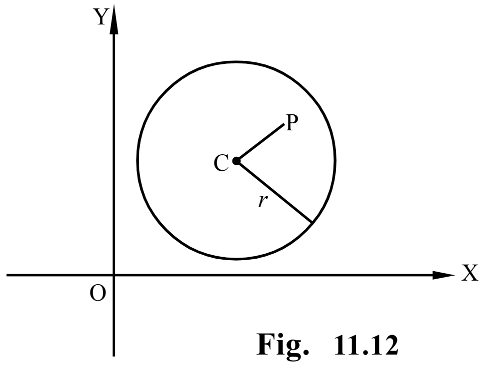

Fig. 11.12

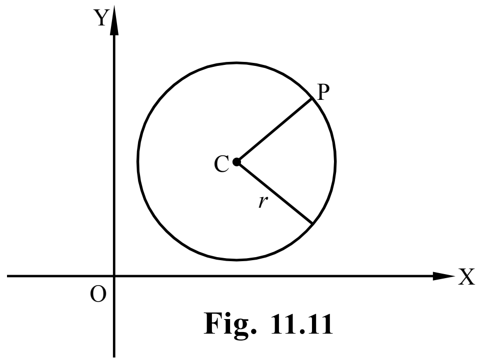

Fig. 11.11

Notes :

1. *A point $P(x_1, y_1)$ lies in the interior of the circle or in the exterior of the circle according as the power of P with respect to the circle is negative, zero or positive respectively.*
2. *The power of $P(x_1, y_1)$ with respect to $S = 0$ is S_{11}.* ($CP^2 - r^2 = S_{11}$)

Example 11.13 : Find the length of the tangent from (1, 1) to the circle $x^2 + y^2 + 8x - 2y + 1 = 0$.

Length of the tangent drawn from (2, –3) to the given circle is $\sqrt{S_{11}}$

$$= \sqrt{(2)^2 + (-3)^2 + 8(2) - 2(-3) + 1}$$

$$= \sqrt{36}$$

$$= 6$$

Example 11.14 : If the Power of P(1, 1) with respect to the circle $x^2 + y^2 - 4x + 3y + K = 0$ is 3, find the value of K. by the given data $S_{11} = 3$

$$\Rightarrow \quad (1)^2 + (1)^2 - 4(1) + 3(1) + K = 3$$

$$\Rightarrow \quad 1 + 1 - 4 + 3 + K = 3$$

$$\Rightarrow \quad 1 + K = 3$$

$$\therefore \quad K = 2.$$

MODULE - II
Coordinate Geometry

Notes

Example 11.15 : Find the equation of the tangent to the circle

$$s = x^2 + y^2 - 8x + 4y - 5 = 0 \text{ at the point } P(1, 2).$$

Solution: Given circle $s = x^2 + y^2 - 8x + 4y - 5 = 0$

Equation of tangent to S = 0 at (1, 2) is $S_1 = 0$

$$xx_1 + yy_1 - 4(x + x_1) + 2(y + y_1) - 5 = 0$$

$$x(1) + y(2) - 4(x + 1) + 2(y + 2) - 5 = 0$$

$$3x - 4y + 5 = 0$$

11.7 POSITION OF A STRAIGHT LINE IN THE PLANE WITH RESPECT TO A CIRCLE

In the earlier section we learnt the position of a point with respect to a circle. In this section we shall learn the position of a straight line in a plane with respect to a circle.

11.7.1 Different cases of position of a straight line with respect to a circle

Given a straight line L = 0 and a circle S = 0 we have three possibilities, namely :

(i) The line meets the circle in two distinct points (see fig. 11.13)

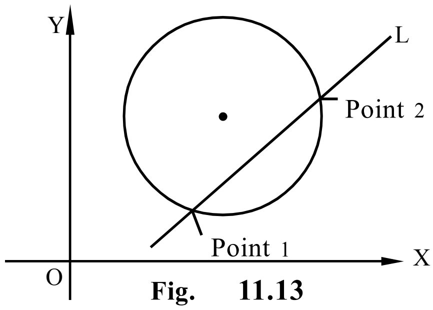

Fig. 11.13

(ii) The line meets the circle in one and only one point (i.e., touching the circle (see fig. 11.14).

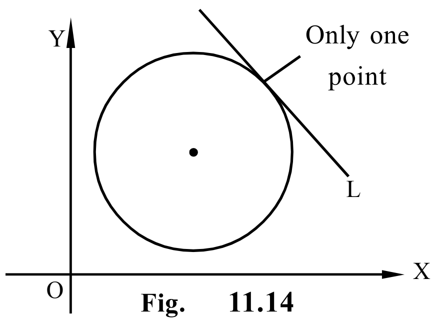

Fig. 11.14

(iii) The line L does not meet the circle i.e., L and the circle have no common points (see fig. 11.15)

Now we examine under what conditions the above three situations arise.

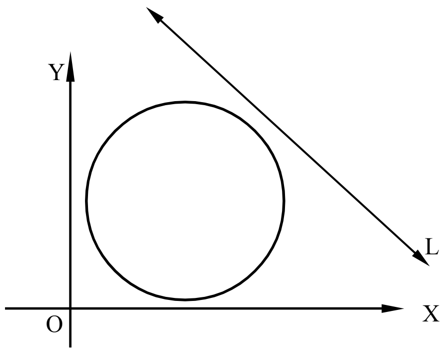

Fig. 11.15

11.7.2 Conditions : A straight line $y = mx + c$

(i) meets the circle $x^2 + y^2 = r^2$ in two distinct points if $\frac{c^2}{1+m^2} < r^2$

(see fig. 11.13)

(ii) touches the circle $x^2 + y^2 = r^2$ if $\frac{c^2}{1+m^2} = r^2$ (see fig. 20.14)

(iii) has no points in common with the circle $x^2 + y^2 = r^2$ if $\frac{c^2}{1+m^2} > r^2$

(see fig. 11.15)

(iv) A straight line $y = mx + c$ touches the circle $x^2 + y^2 = r^2$ then

$c = \pm\, r\sqrt{1+m^2}$ thus the equation of the tangent to the circle $x^2 + y^2$ $= r^2$ is $y = mx \pm r\sqrt{1+m^2}$.

(v) If the straight line $y = mx+c$ touches the circle $x^2+y^2+2gx+2fy+c=0$ then the equation of the tangent to the circle S = 0 is

$$y + f = m(x+g) \pm r\sqrt{1+m^2}$$

11.7.3 Definition

The normal at any point P of the circle is the line which passes through P and is perpendicular to the tangent at P. (see fig. 11.16).

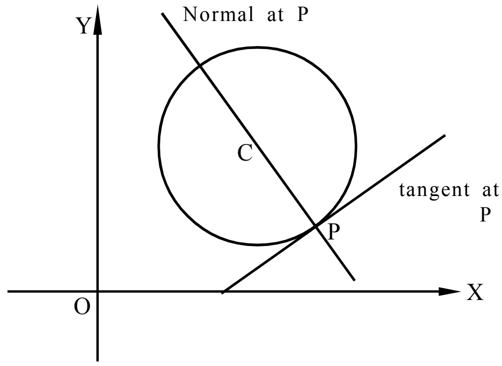

Fig. 11.16

Note : Recall that the normal at any point of the circle passes through the centre of the circle.

2. **Theorem** : The equation of the normal at $P(x_1, y_1)$ of the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

is $$(x - x_1)(y_1 + f) - (y - y_1)(x_1 + g) = 0.$$

Proof : Let C be the centre of the circle given by (1). Then $C = (-g, -f)$. We know that normal at anly point passes through the centre of the circle (see fig.1.40).

The slope of $\overline{CP} = \dfrac{y_1 + f}{x_1 + g}$

Hence the equation of the normal at $P(x_1, y_1)$ is

$$(y - y_1) = \frac{(y_1 + f)}{(x_1 + g)}(x - x_1)$$

i.e., $$(x - x_1)(y_1 + f) - (y - y_1)(x_1 + g) = 0$$

Note :

The equation of the normal to the circle $x^2 + y^2 = r^2$ at $P(x_1, y_1)$ is $xy_1 - yx_1 = 0$.

Example 11.16 : Find the equation of the notmal to $x^2 + y^2 - 4x + 6y - 3 = 0$ at $(1, -2)$.

Sol : By comparing the given equation of circle with its standard form, we have

$$2g = -4 \Rightarrow g = -2$$
$$2f = 6 \Rightarrow f = 3,$$
$$x_1 = 1, \quad y_1 = -2.$$

by Theorem 2. : the equation of normal at (1, –2) is

$$(x - x_1)(y_1 + f) - (y - y_1)(x_1 + g) = 0$$

$$\therefore\ (x - 1)(-2 + 3) - (y + 2)(1 - 2) = 0$$

$$\Rightarrow\quad (x - 1)(1) - (y + 2)(-1) = 0$$

$$\Rightarrow\quad x - 1 + y + 2 = 0$$

$$\Rightarrow\quad x + y + 1 = 0$$

$\therefore$ The equation of normal to $x^2 + y^2 - 4x + 6y - 3 = 0$ at $(1, -2)$ is $x + y + 1 = 0$.

Example 11.17 : Find the equation of normal $x^2 + y^2 = 16$ at $(-5,6)$.

Sol : The equation of normal to $x^2 + y^2 = r^2$ at $P(x_1, y_1)$ is $xy_1 - yx_1 = 0$

$\therefore$ The equation of normal to $x^2 + y^2 = 16$ at $(-5, 6)$ is $x(6) - y(-5) = 0$.

$$\Rightarrow\ 6x + 5y = 0.$$

Example 11.18 : Find the equation of tangents to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ which are parallel to $x + 2y - 8 = 0$.

Solution : Here $g = -2$, $f = 3$, $r = \sqrt{4+9+12} = 5$

Slope of the tangent = $\dfrac{-1}{2}$

$\therefore$ Equation of the tangents to the circle are

$$y + f = m(x+g) \pm r\sqrt{1+m^2}$$

$$y + 3 = \frac{-1}{2}(x-2) \pm 5\sqrt{1+\frac{1}{4}}$$

i.e., $x + 2y + \left(4 \pm 5\sqrt{5}\right) = 0$.

Example 11.19 : Find the length of the chord intercepted by the circle $x^2 + y^2 + 8x - 4y - 16 = 0$ on the line $3x - y + 4 = 0$.

Solution : Centre of the circle C = (–4, 2)

radius $r = \sqrt{16+4+16} = 6$

$d = \perp r$ distance from C to the line $3x - y + 4 = 0$

$$= \frac{|3(-4)-2+4|}{\sqrt{3^2+(-1)^2}} = \frac{10}{\sqrt{10}} = \sqrt{10}$$

$\therefore$ length of the chord $= 2\sqrt{r^2-d^2} = 2\sqrt{36-10}$

$= 2\sqrt{26}.$

11.8 EQUATION OF CHORD OF CONTACT AND POLAR

In this section we shall define chord of contact and polar and learn the equations for it.

11.8.1 Definition

In the tangents drawn through $P(x_1, y_1)$ to a circle S = 0 touch the circle at points A and B then the secant $\overleftrightarrow{AB}$ is called the chord of contact of P with respect to S = 0. (see fig. 11.17).

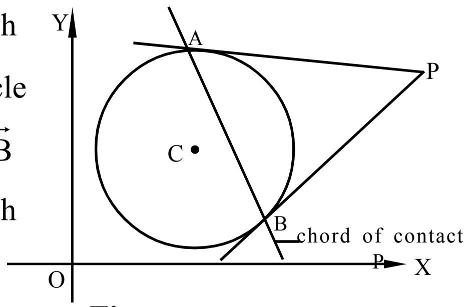

Fig. 11.17

Note : If the point $P(x_1, y_1)$ is on the circle S = 0, then the tangent itself can be defined as the chord of contact.

3.Theorem : If $P(x_1, y_1)$ is an exterior point to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact of P with respect to S = 0 is $S_1 = 0$.

11.8.2 Definition

Let S = 0 be a circle and P be any point in the plane other than the centre of S = 0. If any line drawn through the point P meets the circle in the two points Q and R, then the points of intersection of tangents drawn at Q and R lie on a line called polar of P and P is called pole of the polar (see fig. 11.18).

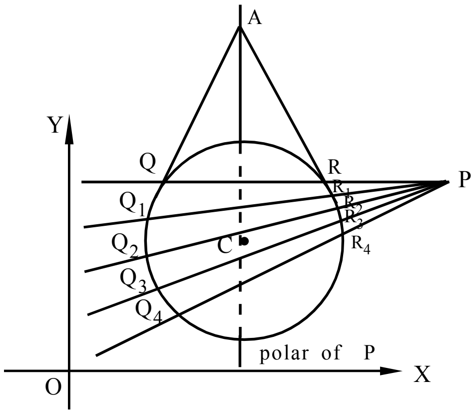

Fig. 11.18

4. Theorem: The equation of the polar of $P(x_1, y_1)$ with respect to S = 0 is $S_1 = 0$.

5. Theorem : The pole of $lx + my + n = 0$ $(n \neq 0)$ with respect to the circle

$$x^2 + y^2 = r^2 \text{ is } \left(-\frac{r^2 l}{n}, \frac{-r^2 m}{n}\right)$$

Proof : Let $P(x_1, y_1)$ be the pole of $lx + my + n = 0$ ...(1)

w.r.to the circle $x^2 + y^2 = r^2$

$\therefore$ The polar of $P(x_1, y_1)$ w.r. to the circle $x^2 + y^2 = r^2$ is

$$xx_1 + yy_1 - r^2 = 0 \qquad ...(2)$$

Equations (1) and (2) represents the same line

hence $\frac{x_1}{l} = \frac{y_1}{m} = \frac{-r^2}{n}$

$\Rightarrow x_1 = \frac{-r^2 l}{n}, \quad y_1 = \frac{-r^2 m}{n}$

Therefore pole of $lx + my + n = 0$ w.r. to the circle

$$x^2 + y^2 = r^2 \text{ is } p(x_1, y_1) = \left(\frac{-r^2 l}{n}, \frac{-r^2 m}{n}\right)$$

Note: The pole of $lx + my + n = 0$ $(n \neq 0)$ with respect to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\left(-g+\frac{lr^2}{lg+mf-n},\ -f+\frac{mr^2}{lg+mf-n}\right)$$

11.8.3 Conjugate points

Two points P and Q are said to be conjugate points with respect to the circle S = 0 if Q lies on the polar of P(or) P lies on the polar of Q.

Note: The condition that the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are conjugate points with respect to the circle S = 0 is $S_{12} = 0$.

11.8.4 Conjugate lines

If P and Q are conjugate points with respect to the circle S = 0 then the polars of P and Q are called conjugate lines w.r.to the circle S = 0.

Note: The straight lines $l_1x + m_1y + n_1 = 0$, $l_2x + m_2y + n_2 = 0$ are conjugate lines with respect to the circle

$$S = x^2 + y^2 + 2gx + 2fy + c = 0 \Leftrightarrow r^2(l_1l_2 + m_1m_2)$$
$$= (l_1g + m_1f - n_1)(l_2g + m_2f - n_2)$$

Example 11.20 : Find the chord of contact of A(4, 2) with respect to the circle $x^2 + y^2 + 4x - 2y - 12 = 0$

Solution : Chord of contact of P with respect to the circle S = 0 is $S_1 = 0$

$\therefore$ chord of contact of A(4, 2) with respect to $x^2 + y^2 + 4x - 2y - 12 = 0$ is A(4, 2) is $S_1 = 0$.

$$\Rightarrow x(4) + y(2) + 2(x + 4) - 1(y + 2) - 12 = 0$$
$$\Rightarrow 4x + 2y + 2x + 8 - y - 2 - 12 = 0$$
$$\Rightarrow 6x + y - 6 = 0.$$

Example 11.21 : Find the chord of contact of P(6, −1) with respect to the circle $x^2 + y^2 - 2x + 6y - 6 = 0$.

Solution : Required equation of the chord of contact is $S_1 = 0$

$$\Rightarrow x(6) + y(-1) - 1(x + 6) + 3(y - 1) - 6 = 0$$

$\Rightarrow \quad 6x - y - x - 6 + 3y - 3 - 6 = 0$

$\Rightarrow \quad 5x + 2y - 15 = 0.$

Example 11.22 : Find the pole of $3x + 4y - 12 = 0$ with respect to the circle $x^2 + y^2 = 24$.

Solution : Pole of the line $3x + 4y - 12 = 0$ w.r.to the circle $x^2 + y^2 = 24$ is

$$P = \left(\frac{-r^2 l}{n}, \frac{-r^2 m}{n}\right)$$

Here $r^2 = 24, \; l = 3, \; m = 4, \; n = -12$

$$\therefore \text{ pole } P = \left(\frac{-24\times 3}{-12}, \frac{-24\times 4}{-12}\right)$$

$$= (6, 8).$$

Example 11.23 : Find the pole of the line $x - 2y + 22 = 0$ w.r.to the circle $x^2 + y^2 - 5x + 8y + 6 = 0$.

Solution: Pole $= \left(-g + \dfrac{lr^2}{lg + mf - n}, -f + \dfrac{mr^2}{lg + mf - n}\right)$

Here $g = -\dfrac{5}{2}, \; f = 4, \quad r = \sqrt{\dfrac{25}{4} + 16 - 6} = \sqrt{\dfrac{65}{4}}$

$l = 1, \; m = -2, \; n = 22$

$$\text{pole} = \left(\frac{5}{2} + \frac{1\times\frac{65}{4}}{1\times\frac{-5}{2} + (-2)\times 4 - 22}, -4 + \frac{(-2)\times\frac{65}{4}}{1\times\frac{-5}{2} + (-2)\times 4 - 22}\right)$$

$= (2, -3).$

Example 11.24 : Show that the points $(4, -2), (3, -6)$ are conjugate points w.r.to the circle $x^2 + y^2 = 24$.

Solution : Let $S = x^2 + y^2 - 24 = 0$

$P(x_1, y_1) = (4, -2), \; Q(x_2, y_2) = (3, -6)$

$\therefore\quad S_{12} = x_1x_2 + y_1y_2 - 24$

$= 4 \times 3 + (-2) \times 6 - 24$

$= 12 + 12 - 24$

$= 0$

$\therefore$ Given points (4, –2), (3, –6) are conjugute points.

Example 11.25: If the points (2, 3), (k, –1) are conjugate points w.r.t to the circle $x^2 + y^2 - 2x + 2y - 1 = 0$ then find k.

Solution: Let $S = x^2 + y^2 - 2x + 2y - 1 = 0$

If (2, 3), (k, –1) are conjugate points w.r.to the circle S = 0 then $S_{12} = 0$

$x_1x_2 + y_1y_2 - (x_1 + x_2) + (y_1 + y_2) - 1 = 0$

$2 \times k + 3 \times -1 - (2 + k) + (3 - 1) - 1 = 0$

$2k - 3 - 2 - k + 2 - 1 = 0$

$\Rightarrow\quad k - 4 = 0$

$\Rightarrow\quad k = 4.$

Example 11.26 : If the lines $2x + y + 12 = 0$, $kx - 3y - 10 = 0$ are conjugate lines w.r.to the circle.

$x^2 + y^2 - 4x + 3y - 1 = 0$ then find k.

Solution : Given circle $x^2 + y^2 - 4x + 3y - 1 = 0$

$$g = -2,\ f = \frac{3}{2},\ r = \sqrt{4 + \frac{9}{4} + 1} = \sqrt{\frac{29}{4}}$$

Given lines $2x + y + 12 = 0,\ kx - 3y - 10 = 0$

Here $l_1 = 2,\ m_1 = 1,\ n_1 = 12,\ l_2 = k,\ m_2 = -3,\ n_2 = -10$

If the lines are conjugate lines then

$$r^2(l_1l_2 + m_1m_2) = (l_1g + m_1f - n_1)(l_2g + m_2f - n_2)$$

$$\frac{29}{4}(2k - 3) = \left(-4 + \frac{3}{2} - 12\right)\left(-2k - \frac{9}{2} + 10\right)$$

$$\frac{29}{4}(2k-3)=\left(\frac{-29}{2}\right)\left(\frac{-4k+11}{2}\right)$$

$$2k - 3 = 4k - 11$$

$$\Rightarrow \quad 2k = 8$$

$$\Rightarrow \; k = 4.$$

11.9 RELATIVE POSITION OF TWO CIRCLES

In this section we shall learn the common tangents that can be drawn to two given circles, and it depends on their relative positions of two circles and the number of common tangents exists in each case.

11.9.1 Definition

A straight line L is said to be a common tangent to the circles S = 0 and S$'$ = 0 if it is tangent to both S = 0 and S$'$ = 0 (see fig. 20.19).

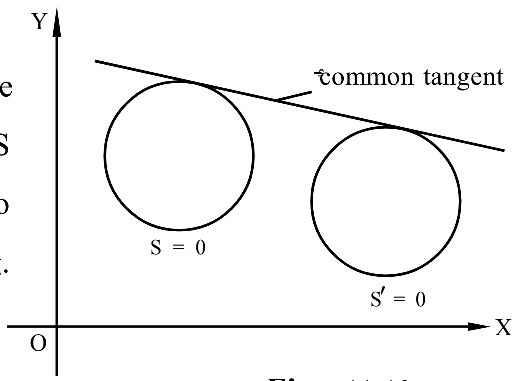

Fig. 11.19

11.9.2 Definition

The circles are said to be touching each other if they have only one common point (see fig. 20.20, 20.21).

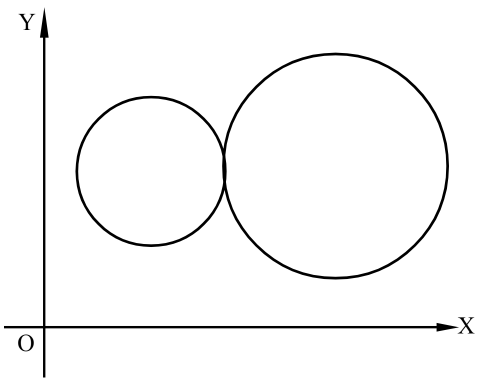

Fig. 11.20

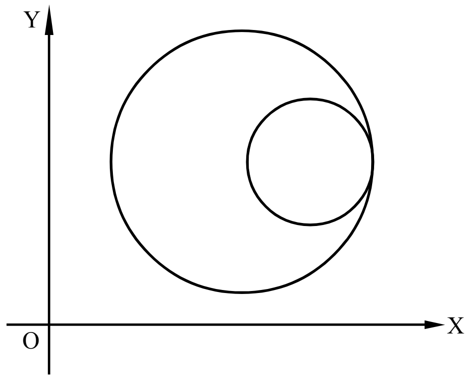

Fig. 11.21

11.9.3 Relative positions of two circles

Let C_1, C_2 be the centres and r_1, r_2 be the radii of two circles $S = 0$ and $S' = 0$ respectively. Further let $\overline{C_1C_2}$ represents the line segment from C_1 to C_2. The following cases arise with regard to the relative position of two circles.

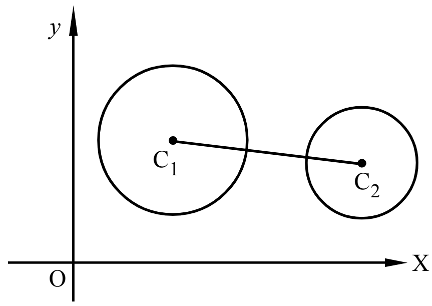

Fig. 11.22

(i) $C_1C_2 > r_1 + r_2$

In this case the two circles do not intersect and one circle will be away from the other circle (see fig 11.22)

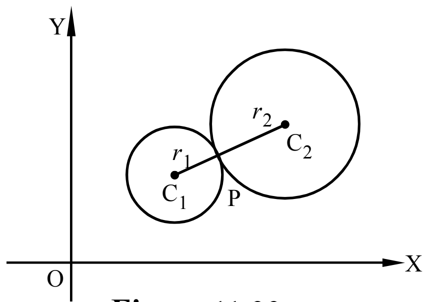

Fig. 11.23

(ii) $C_1C_2 = r_1 + r_2$

In this case the two circles touch each other externally (See fig. 11.23).

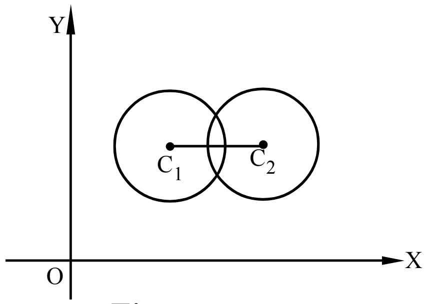

Fig. 11.24

(iii) $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

In this case the two circles intersect in two distinct points (see fig. 11.24).

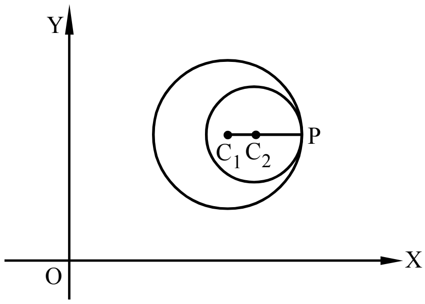

Fig. 11.25

(iv) $C_1C_2 = |r_1 - r_2|$

The two circles touch each other internally (see fig. 11.25) in this case

(v) $C_1C_2 < \quad |r_1 - r_2|$

In this case the two circles do not intersect / touch and one circle will be completely inside the other (see fig. 11.26)

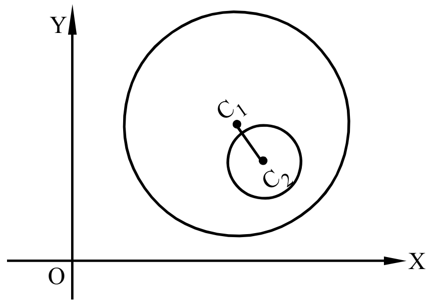

Fig. 11.26

Note

If $C_1C_2 = 0$ then the centres of the two circles co-incide and they are concentric circles (see fig. 11.27).

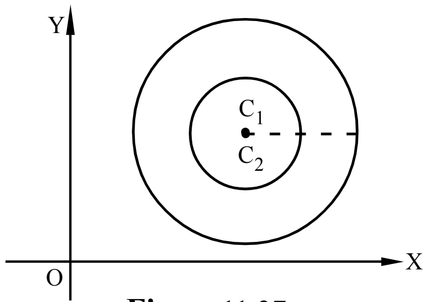

Fig. 11.27

Example 11.27: Prove that the circles $x^2 + y^2 - 8x - 6y + 21 = 0$ and $x^2 + y^2 - 2y - 15 = 0$ have exactly two common tangents.

Sol : Let the given equations of the circles be

$$x^2 + y^2 - 8x - 6y + 21 = 0 \qquad ...(1)$$

and Let $x^2 + y^2 - 2y - 15 = 0 \qquad ...(2)$

Let C_1, C_2 be the centres and r_1, r_2 be the radii of circles given by (1) and (2) respectively. Then,

$C_1 = (4, 3)$, $C_2 = (0, 1)$, $r_1 = 2$, $r_2 = 4$.

$$\therefore \overline{C_1C_2} = \sqrt{(4-0)^2 + (3-1)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}.$$

$|r_1 - r_2| = |2 - 4| = 2,$

$r_1 + r_2 = 6$

$|r_1 - r_2| = < C_1C_2 < r_1 + r_2 \qquad \left(\because \sqrt{4} < \sqrt{20} < \sqrt{36}\right)$

Given circles intersect each other and have exactly two common tangents.

Example 11.28 : Show that the circles $x^2 + y^2 - 14x + 6y + 33 = 0$ and $x^2 + y^2 + 30x - 2y + 1 = 0$ have four common tangents.

Solution : The centres and radii of the given circles are $C_1 = (7, -3)$, $C_2 = (-15, 1)$, $r_1 = 5$, $r_2 = 15$.

$$\overline{C_1C_2} = \sqrt{(7+15)^2 + (-3-1)^2} = 10\sqrt{5}$$

$$r_1 + r_2 = 20$$

$$\overline{C_1C_2} > r_1 + r_2 \qquad \left(\because \overline{C_1C_2} = \sqrt{500}\ r_1 + r_2 = \sqrt{400}\right)$$

Since each of the given pair of circles lie in the exterior of the other the circles have from common tangents.

EXERCISE 11.2

1. Locate the position of thepoint P with respect to the circle S = 0 when

   (i) $S \equiv x^2 + y^2 + 2x - 8y + 4 = 0$; P(1, 3)

   (ii) $S \equiv x^2 + y^2 + 6x - 4y + 9 = 0$; P(1, −2)

   (iii) $S \equiv x^2 + y^2 - 4x + 2y - 11 = 0$; P(−2, 5).

2. Find the power of the point P with respect to the circle S = 0 when

   (i) $S \equiv x^2 + y^2 - 6x + 4y - 12$; P = (−1, 1)

   (ii) $S \equiv x^2 + y^2 + 6x + 8y - 96$; P = (1, 2)

   (iii) $S \equiv x^2 + y^2 - 2x + 6y - 5$; P = (2, 3)

3. If the length of the tangent from (2, 5) to the circle $x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{37}$, then find k.

4. Show that the line $5x + 12y - 4 = 0$ touches the circle $x^2 + y^2 - 6x + 4y + 12 = 0$.

5. Find the equation of the tangent at 'P' of the circle S = 0 where P and S are given by

   (i) $S \equiv x^2 + y^2 - 6x + 4y - 12$; $P = (-1, 1)$

   (ii) $S \equiv x^2 + y^2 - 4x - 6y + 11$; $P = (3, 4)$.

6. Find the equation of the normal at P of the circle S = 0, Where P and S are given by

   (i) $S \equiv x^2 + y^2 - 22x - 4y + 25$; $P = (1, 2)$

   (ii) $S \equiv x^2 + y^2 - 10x - 2y + 6$; $P = (3, 5)$.

7. Find the chord of contact of (0, 5) with respect to the circle $x^2 + y^2 - 5x + 4y - 2 = 0$.

8. Find the equation of the polar of (1, –2) with respect to the circle

   $x^2 + y^2 - 10x - 10y + 25 = 0$.

9. Discuss the relative position of the following pair of circles.

   (i) $x^2 + y^2 - 2x + 4y - 4 = 0$ (ii) $x^2 + y^2 + 6x + 6y + 14 = 0$
   $x^2 + y^2 + 4x - 6y - 3 = 0$ $x^2 + y^2 - 2x - 4y - 4 = 0$

10. Find the length of the chord intercepted by the circle

    $x^2 + y^2 - 8x - 2y - 8 = 0$ on the line $x + y + 1 = 0$.

11. Show that the line $x + y + 1 = 0$ touches the circle.

    $x^2 + y^2 - 3x + 7y + 14 = 0$ and also find its point of contact.

12. Show that the line $2x + 3y + 11 = 0$, $2x - 2y - 1 = 0$ are conjugate with respect to the circle $x^2 + y^2 + 4x + 6y + 12 = 0$.

13. If $(4, k)$ and (2, 3) are conjugate points with respect to the circle

    $x^2 + y^2 = 17$ then find k.

11.10 SYSTEMS OF CIRCLES

In this chapter, we shall learn about the angle between two intersecting circles and obtain a condition for their orthogonality. Also, we shall learn

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about the radical axis of two circles and its equation. Further, we also learn about the coaxial system of circles and a system of circles orthogonal to the given coaxial system of circles.

11.10.1 Angle between two intersecting circles

In the previous chapter 'circle', we have learnt that two circles will intersect with each other if the distance between their centres lies between the absolute value of the distance of their radii and the sum of their radii. In this section, we define the angle between such circles.

11.10.1.1 Definition

The angle between two intersecting circles is defined as the angle between the tangents at the point of intersection of the two circles (see Fig. 11.28)

$\angle T_1 P T_2$ is the angle between the circle at P.

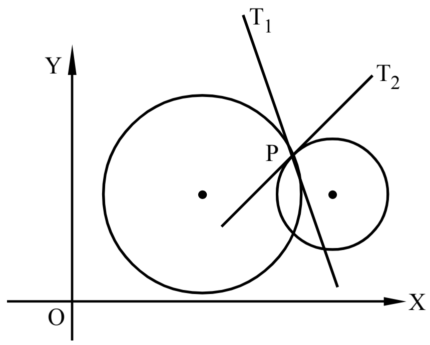

Fig. 11.28

Note

If two circles S = 0, S' = 0 intersect at P and Q then the angle between the two circles at the points P and Q are equal.

1. Theorem

(i) If C_1, C_2 are the centres of two given intersecting circles

(ii) $d = C_1C_2$

(iii) r_1, r_2 are radii of these circles

(iv) θ is the angle between these circles then

$$\cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$$

Proof : Let P be a point of intersection of two given circles. Let the tangents drawn to two circles at P intersect the line joining the centres at T_1 and T_2 (see Fig. 11.29). Then $\angle T_1PT_2 = \theta$.

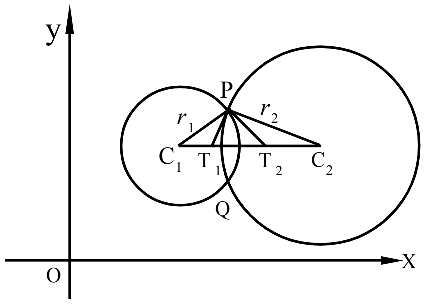

Fig. 11.29

Consider $\angle C_1PC_2 = \angle C_1PT_2 + \angle T_2PC_2$

$$= 90^0 + 90^0 - \theta$$

$$= 180^0 - \theta$$

From ΔC_1PC_2, we have

$$C_1C_2^2 = C_1P^2 + C_2P^2 - 2(C_1P)(C_2P)\cos\angle C_1PC_2$$

i.e., $$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(180^0 - \theta)$$

$$\therefore \cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}.$$

Note : If θ is the angle between the intersecting circles

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots (1)$$

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0 \quad \ldots (2)$$

Then, $$\cos\theta = \frac{c + c' - 2gg' - 2ff'}{2.\sqrt{g^2 + f^2 - c}\;.\;\sqrt{g'^2 + f'^2 - c'}}.$$

11.10.1.2 Definition

Two intersecting circles are said to be orthogonal if the angle between them is right angle (i.e.,90^0).

11.10.1.3 Condition for orthogonality

Let the two intersecting circles be given by

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots (1)$$

and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ ... (2)

These two circles are orthogonal

$$\Leftrightarrow \quad \frac{c+c'-2gg'-2ff'}{2\sqrt{g^2+f^2-c}\,\sqrt{g'^2+f'^2-c'}} = 0$$

$$\Leftrightarrow \quad 2(gg' + ff') = c + c'.$$

$\therefore$ Thus the condition for orthogonatlity of the two intersecting circles (1) and (2) is

$$2(gg' + ff') = c + c'.$$

Example 11.29 : Find the angle between the circles $x^2 + y^2 - 2x = 0$ ... (1)

and $x^2 + y^2 - 4x + 6y + 4 = 0$ ... (2)

Sol: Here $g = -1,\ f = 0,\ c = 0,\ g' = -2,\ f' = 3,\ c' = 4$

Let 'θ' be the angle between the circles (1) and (2).

Then $$\cos\theta = \frac{0+4-(2)(-1)(-2)-(2)(0)(3)}{2.\sqrt{(-1)^2+(0)^2-0}\,.\,\sqrt{(-2)^2+(3)^2-4}}$$

$$= \frac{0+\cancel{4}-\cancel{4}-0}{2.\sqrt{1+0-0}.\sqrt{\cancel{4}+9-\cancel{4}}}$$

$$= \frac{0}{2.\sqrt{1}.\sqrt{9}} = 0$$

$$\therefore \quad \theta = 90^0 \text{ (or) } \frac{\pi}{2}.$$

Example 11.30: If the circles $x^2 + y^2 - 2x + 4y + 3= 0$ ... (1)

and $x^2 + y^2 + 6x - 2y + K = 0$ ...(2) cut orthogonally. Then find the value of 'K'?

Solution : The condition for orthogonality of the two intersecting circles is

$g = -1,\ f = 2,\ c = 3,\ g' = 3,\ f' = -1,\ c' = k.$

$2(gg' + ff') = c + c'.$

$\therefore\ 2[(-1)\ (3) + (2)\ (-1)] = 3 + k.$

$2[-3 - 2] = 3 + k.$

$2(-5) = 3 + k$

$-10 = 3 + k$

$\therefore\ k = -10 - 3$

$\Rightarrow\ k = -13.$

Example 11.31: Find the equation of the circle passes through (1, 1) and cuts orthogonally each of the circle.

$$x^2 + y^2 - 8x - 2y + 16 = 0 \text{ and } x^2 + y^2 - 4x - 4y - 1 = 0$$

Solution: Let the equation of the required circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(1)$$

It passes through (1, 1) we obtain

$$2g + 2f + c + 2 = 0 \qquad ...(2)$$

Required circle is orthogonal to $x^2 + y^2 - 8gx - 2y + 16 = 0$

$$-8g - 2f - c - 16 = 0$$

$$\Rightarrow 8g + 2f + c + 16 = 0 \qquad ...(3)$$

Similarly, circle is orthogonal to $x^2 + y^2 - 4x - 4y - 1 = 0$

$$-4g - 4f = c - 1$$

$$\Rightarrow 4g + 4f + c - 1 = 0 \qquad ...(4)$$

Solving (2), (3) and (4) g, f and c, we get

$$g = -\frac{7}{3},\ f = \frac{23}{6},\ c = -5.$$

Thus the required circle is $3(x^2 + y^2) - 14x + 23y - 15 = 0.$

Example 11.32: Find the equation of the circle which cuts orthogonally the circle $x^2 + y^2 - 4x + 2y - 7 = 0$ and having the centre at (2, 3)

Solution: Let the equation of the required circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(1)$$

centre $(-g, -f) = (2, 3)$

$\Rightarrow g = -2, \quad f = -3.$

Required circle is orghogonal to $x^2 + y^2 - 4x + 2y - 7 = 0$

$\therefore \; 2g(-2) + 2f(1) = c - 7$

$-4g + 2f = c - 7$

$-4(-2) + 2(-3) = c - 7$

$\Rightarrow \; c = 9.$

Thus the required circle is $x^2 + y^2 - 4x - 6y + 9 = 0$

11.11 RADICAL AXIS OF TWO CIRCLES

In this section we shall define the radical axis of two circles and learn about the equation of the radical axis.

11.11.1 Definition

The radical axis of two circles is defined to be the locus of a point which moves so that its powers with respect to the two circles are equal.

Note : (i) If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(1)$

and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0 \qquad ...(2)$

are two non-concentric circles then the radical axis of (1) and (2) is a straight line represented by $S - S' = 0$

i.e., $2(g - g')x + 2(f - f')y + (c - c') = 0$

(ii) While using the formula $S - S' = 0$, to find the equation of the radical axis, first reduce the equations of the circles to general form (if they are not in general form, i.e., the coefficients of x^2 and y^2 must be equal to '1').

Example 11.33 : Find the equation of the radical axis of the circles

$$x^2 + y^2 + 4x - 2y + 5 = 0 \text{ and } x^2 + y^2 - 6x - 4y + 2 = 0.$$

Solution : Since the equations of the given circles are in general form

the equation of the radical axis is $S - S' = 0$

$\Rightarrow (x^2 + y^2 + 4x - 2y + 5) - (x^2 + y^2 - 6x - 4y + 2) = 0$

$\Rightarrow x^2 + y^2 + 4x - 2y + 5y - x^2 - y^2 + 6x + 4y - 2 = 0$

$\Rightarrow 10x + 2y + 3 = 0.$

Example 11.34 : Find the radical axis of the circles $2x^2 + 2y^2 + 6x + 2y - 8 = 0$ and $3x^2 + 3y^2 - 9x + 12y + 6 = 0.$

Solution : Here the given equations are not in general form. Reducing them into general form, we get

$x^2 + y^2 + 3x + y - 4 = 0, \quad x^2 + y^2 - 3x + 4y + 2 = 0$

Now the equation of the radical axis of the above circles is $S - S' = 0$

$\Rightarrow (3 + 3)x + (1 - 4)y + (-4 - 2) = 0 \Rightarrow 6x - 3y - 6 = 0.$

11.11.2 Radical Centre

The point of concurence of the radical axes of each pair of the three circles whose centres are not collinear is called the radical centre.

Note (i) The lengths of tangents from the radical centre to these three circles are equal.

Example 11.35 : Find the radical centre of the circles

$x^2 + y^2 = 4, \quad x^2 + y^2 - 3x = 4, \quad x^2 + y^2 - 4y = 4.$

Solution: Given circles are $x^2 + y^2 - 4 = 0$ ...(1)

$x^2 + y^2 - 3x - 4 = 0$ ...(2)

$x^2 + y^2 - 4y - 4 = 0$ ...(3)

The radical axis of (1) and (2) is

$3x = 0 \Rightarrow x = 0$ ... (4)

The radical axis of (2) and (3) is

$-3x + 4y = 0$ ...(5)

from (4) and (5), we get y = 0

$\therefore$ Radical centre is (0, 0).

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11.12 COAXIAL SYSTEM OF CIRCLES

In this section we shall, define the equation of a coazial system and obtain its equation.

11.12.1 Definitions

(i) A set of circles is called a system of circles.

(ii) A system of circles is said to be coaxial when any two circles of the system have the same radical axis.

Note

(i) Consider a set of circles passing through two fixed points A and B. Then it forms a system of coaxial circles (see fig. 11.30)

For each pair of these circles, the common chord is the radical axis.

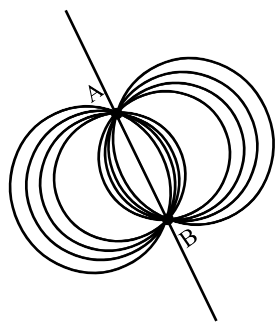


Fig. 11.30

(ii) The centres of circles in a coaxial system are all collinear and lie on a line perpendicular to the common radical axis of the coaxial system.

2. Theorem : Let S = 0 and S' = 0 be two distinct circles. Then $S + \lambda(S - S') = 0$ represents the coaxial system of the circles containing S = 0, S' = 0 as members (the equation of S = 0, S' = 0 are in general form only)

Proof : Let $L \equiv S - S' = 0$ and λ_1, λ_2 are two particular values of λ such that $\lambda_1 \neq \lambda_2$. Consider

$$S + \lambda_1 L = 0$$
$$S + \lambda_2 L = 0$$

$$(S + \lambda_1 L) - (S + \lambda_2 L) = 0$$

$$\Rightarrow \quad (\lambda_1 - \lambda_2) L = 0$$

$$\Rightarrow \quad L = 0 \qquad (\because \lambda_1 \neq \lambda_2)$$

Hence the radical axis of $S = 0$ and $S' = 0$ is the same as the radical axis of any two circles of $S + \lambda(S - S') = 0$. Hence $S + \lambda(S - S') = 0$ represents a coaxial system for which $S = 0, S' = 0$.

Note : If $S = 0, S' = 0$ are two distinct circles then $\lambda S + \mu S' = 0$ ($\lambda + \mu \neq 0$) represents a coaxial system of circles containing $S = 0, S' = 0$ as its members.

Example 11.35 : Find the equation of the circle passing through (–1, 0) and coaxial with the circles

$$x^2 + y^2 + 8x + 4y + 6 = 0 \text{...(1)} \quad x^2 + y^2 + 4x + 2y + 1 = 0 \quad \text{... (2)}$$

Sol: The equation of any circle of the coaxial system of circles for which (1) and (2) are members is

$$(x^2 + y^2 + 8x - 4y + 6) + \lambda(4x + 2y + 5) = 0$$

$$[\because\ S + \lambda(S - S') = 0]$$

Here 'λ' is parameter.

It passes through (–1, 0).

$$[(-1)^2 + (0)^2 + 8(-1) - 4(0) + 6] + \lambda[4(-1) + 2(0) + 5] = 0$$

$$(1 + 0 - 8 - 0 + 6) + \lambda(-4 + 0 + 5) = 0$$

$$-1 + \lambda(1) = 0$$

$$\lambda = 1$$

Thus the required equation of the circle is

$$(x^2 + y^2 + 8x - 4y + 6) + 1(4x + 2y + 5) = 0$$

$$x^2 + y^2 + 12x - 2y + 11 = 0$$

Example 11.36 : Find the equation of the circle passing of through (–1, –2) and coaxial with the circles

$$x^2 + y^2 - 2x + 2y + 1 = 0 \text{ ...(1) and } x^2 + y^2 + 8x - 6y = 0 \text{ ...(2).}$$

Solution : The equation of any circle of the coaxial system of circles for which (1) and (2) are members is

$$(x^2 + y^2 - 2x + 2y + 1) + \lambda\ (-10x + 8y + 1) = 0$$

$$[\because S + \lambda(S - S') = 0]$$

Here 'λ' is parameter. It passes through (−1, −2)

$$[(-1)^2 + (-2)^2 - 2(-1) + 2(-2) + 1] + \lambda[-10(-1) + 8(-2) + 1] = 0$$

$$(1 + 4 + 2 - 4 + 1) + \lambda(10 - 16 + 1) = 0$$

$$4 + \lambda(-5) = 0$$

$$-5\lambda = -4$$

$$\lambda = \frac{4}{5}.$$

Thus the required equation of the circle is

$$(x^2 + y^2 - 2x + 2y + 1) + \frac{4}{5}(-10x + 8y + 1) = 0$$

$$5(x^2 + y^2 - 2x + 2y + 1) + 4(-10x + 8y + 1) = 0$$

$$5x^2 + 5y^2 - 10x + 10y + 5 - 40x + 32y + 4 = 0$$

$$\therefore \quad 5x^2 + 5y^2 - 50x + 42y + 9 = 0$$

11.13 ORTHOGONAL SYSTEM OF A COAXIAL SYSTEM OF CIRCLES

In this section, we shall define and write the equation for the orthogonal system of a coaxial system of circles.

11.13.1 Definition

If two systems of coaxial circles are such that every member of one system cuts every member of the other system orthogonally thus each system is said to be orthogonal to the other system.

3. Theorem : The equation of a coaxial system of circles orthogonal to the coaxial system.

$x^2 + y^2 + 2\lambda x + c = 0$ (λ parameter, c constant) is of the from

$x^2 + y^2 + 2\mu y - c = 0$ (μ parameter)

EXERCISE 11.3

1. Find the angle between the circles given by the equation

   (i) $x^2 + y^2 - 2x + 4y + 4 = 0,\quad x^2 + y^2 + 3x + 4y + 1 = 0$

   (ii) $x^2 + y^2 - 12x - 6y + 41 = 0,\quad x^2 + y^2 + 4x + 6y - 59 = 0$

   (iii) $x^2 + y^2 + 4x - 14y + 28 = 0,\quad x^2 + y^2 + 4x - 5 = 0$

2. Find K if the following pair of circles are orthogonal.

   (i) $x^2 + y^2 - 12x - 6y + 41 = 0,\quad x^2 + y^2 + kx + 6y - 59 = 0$

   (ii) $x^2 + y^2 - 6x - 8y + 12 = 0,\quad x^2 + y^2 - 4x - 6y + k = 0$

   (iii) $x^2 + y^2 + 4x - 8 = 0,\quad x^2 + y^2 - 16y - k = 0$

3. Show that the circles given by the following equations intersect each other orthogonally.

   (i) $x^2 + y^2 - 2x + 4y + 3 = 0,\quad x^2 + y^2 + 6x - 2y - 13 = 0$

   (ii) $x^2 + y^2 - 2x + 4y + 4 = 0,\quad x^2 + y^2 + 3x + 4y + 1 = 0$

   (iii) $x^2 + y^2 + 4x - 2y - 11 = 0,\quad x^2 + y^2 - 4x - 8y + 11 = 0$

4. Find the equation of the radical axis of the following circles.

   (i) $x^2 + y^2 + 8x + 6y + 4 = 0,\quad x^2 + y^2 - 6x + 2y - 2 = 0$

   (ii) $x^2 + y^2 - 2x + 4y - 1 = 0,\quad x^2 + y^2 - 10x - 2y - 6 = 0$

   (iii) $x^2 + y^2 + 4x + 6y - 7 = 0,\quad 4(x^2 + y^2) + 8x + 12y - 9 = 0$

5. Find the equation of the system of circles coaxial with the following circles.

   (i) $x^2 + y^2 - 6x + 4y - 2 = 0, \quad x^2 + y^2 - 8x + 2y + 1 = 0$

   (ii) $x^2 + y^2 - 10x + 2y - 8 = 0, \quad x^2 + y^2 - 6x + 4y - 6 = 0$

   (iii) $x^2 + y^2 + 2x - 8y + 2 = 0, \quad x^2 + y^2 + 4x - 10y - 4 = 0$

6. Find the equation of the circle passing through (–2, 3) and coaxial with the circles

   $x^2 + y^2 - 7x + 12 = 0$ and $x^2 + y^2 + 8x + 12 = 0$.

7. Find the equation of the circle which passes through (0, 0) and coaxial with the circles $x^2 + y^2 - 6x + 4y - 16 = 0$ and $x^2 + y^2 + 2x + y - 4 = 0$.

8. Find the radical centre of the following circles

   (i) $x^2 + y^2 - 2x + 6y = 0, \quad x^2 + y^2 - 4x - 2y + 6 = 0$

   $x^2 + y^2 - 12x + 2y + 3 = 0$

   (ii) $x^2 + y^2 - 4x - 6y + 5 = 0, \quad x^2 + y^2 - 2x - 4y - 1 = 0,$

   $x^2 + y^2 - 6x - 2y = 0.$

9. Find the equation of the circle which passes through (2, 0), (0, 2) and orthogonal to the circle $2x^2 + 2y^2 + 5x - 6y + 4 = 0$

KEY WORDS

- **Standard form of the circle**

  $$(x - h)^2 + (y - k)^2 = a^2$$

  Centre (h, k) and radius is a

  The general form of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

  Its centre: $(-g, -f)$ and radius = $\sqrt{g^2 + f^2 - c}$

- **Diameter form of the circle**

  If the end points of a diameter are (x_1, y_1) and (x_2, y_2) then the equation of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

- **Parametric form of the circle**

- $x = a\cos\theta, y = a\sin\theta$ represents the parametric equation of a circle whose centre is at (0,0) and radius = a

- If the centre of the circle is at (h, k) then the parametric equation of the circle is $x = h + a\cos\theta$ and $y = k + a\sin\theta$.

- A point $P(x_1, y_1)$ is an interior point of a circle $S = 0 \Leftrightarrow S_{11} < 0$.

- A point $P(x_1, y_1)$ is an exterior point of a circle $S = 0 \Leftrightarrow S_{11} > 0$.

- A point $P(x_1, y_1)$ is on the circumference of a circle $S = 0$ $\Leftrightarrow S_{11} = 0$.

- The power of $P(x_1, y_1)$ with respect to the circle $S = 0$ is S_{11}.

- The length of the tangent from $P(x_1, y_1)$ to $S = 0$ is $\sqrt{S_{11}}$.

- The equation of tangent at (x_1, y_1) of the circle $S = 0$ is $S_1 = 0$.

- The equation of normal at (x_1, y_1) of the circle $S = 0$ is
  $(x - x_1)(y_1 + f) - (y - y_1)(x_1 + g) = 0$

- The equation of chord of contact of $P(x_1, y_1)$ with respect to the circle $S = 0$ is $S_1 = 0$.

- The equation of the polar of a point $P(x_1, y_1)$ with respect to $S = 0$ is $S_1 = 0$.

- Let C_1, C_2 be the centres and r_1, r_2 be the radii of two circles $S = 0$ and $S' = 0$ respectively. Further let represents the line segment from C_1 to C_2. The following cases arise with regard to the relative position of two circle.

(i) $C_1C_2 > r_1 + r_2$: In this case the two circles do not intersect and one circle will away from the other circle.

(ii) $C_1C_2 = r_1 + r_2$: In this case, the two circles touch each other externally.

(iii) $|r_1 - r_2| < C_1C_2 < r_1 + r_2$: In this case the two circles intersect in two distinct points.

(iv) $C_1C_2 = |r_1 - r_2|$: The two circles touch each other internally.

(v) $C_1C_2 < |r_1 - r_2|$: In this case the two circles do not intersect /touch and one circle will be completely inside the other.

(vi) $C_1C_2 = 0$: In this case the centres of the two circles coincide and they are concentric circles.

- We denote $x^2 + y^2 + 2gx + 2fy + c$ by S and $x^2 + y^2 + 2g'x + 2f'y + c'$ by S'.
- If C_1, C_2 are the centres and r_1, r_2 are radii of two intersecting circles S = 0, S' = 0; $C_1C_2 = d$ and 'θ' is the angle between them then

$$\cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$$

- If 'θ' is the angle between the two intersecting circles S = 0, S' = 0 then

$$\cos\theta = \frac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2 + f^2 - c}.\sqrt{g'^2 + f'^2 - c'}}$$

- Two circles $S = 0$ and $S' = 0$ are orthogonal iff

$$2(gg' + ff') = c + c'$$

- The radical axis of two circles is defined to be the locus of a point 'P' which moves so that its powers with respect to the two circles are equal.
- The radical axis of $S = 0$ and $S' = 0$ is $S - S' = 0$
- A system of circles is said to be coaxial when they have a common radical axis.
- If $S = 0$ and $S' = 0$ be two distinct circles then $\lambda S + \mu S' = 0 (\lambda + \mu \neq 0)$ represents a coaxial system containing $S = 0$, $S' = 0$ as members.
- The equation of a coaxial system of circles orthogonal to the coaxial system $x^2 + y^2 + 2\lambda x + c = 0$ (λ parameter) is $x^2 + y^2 + 2\mu y - c = 0$ (μ parameter).

SUPPORTIVE WEB SITES

http://www. wikipedia. org

http://mathworld. wolfram. com

PRACTICE EXERCISE

1. Find the equation of a circle with centre (4, –6) and radius 7.
2. Find the centre and radius of the circle $x^2 + y^2 + 4x - 6y = 0$.
3. Find the equation of the circle passes through the point (1,0), (–1,0) and (0,1).
4. Find the parametric form of the equation of circles given below:

   (a) $x^2 + y^2 = 3$

   (b) $x^2 + y^2 - 4x + 6y = 12$

ANSWERS

EXERCISE 11.1

1. (a) $x^2 + y^2 = 9$

   (b) $x^2 + y^2 + 4x - 6y - 3 = 0$

2. (a) $\left(-\frac{3}{2}; 1\right); \frac{\sqrt{37}}{2}$ (b) $\left(\frac{1}{4}, -\frac{3}{8}\right); \frac{\sqrt{109}}{8}$

3. $x^2 + y^2 - 2x - 2y = 0$

4. $x^2 + y^2 + 2x - 2y + 1 = 0$

5. $x^2 + y^2 - 9y + 14 = 0$

6. (a) $x = -1 + 2\cos\theta$ and $y = -1 + 2\sin\theta$

   (b) $x = -\frac{1}{4} + \sqrt{\frac{7}{8}}\cos\theta \quad y = -\frac{1}{4} + \sqrt{\frac{7}{8}}\sin\theta$

   (c) $x = \sqrt{2}\cos\theta$ and $y = \sqrt{2}\sin\theta$

7. $x^2 + y^2 - 6x - 8y + 15 = 0$

8. $c = \frac{14}{3}$

EXERCISE 11.2

1. (i) since $S_{11} < 0$, 'P' lies inside the circle. (P is an interior Point)

   (ii) since $S_{11} < 0$, 'P' lies out side of the circle. (P is an exterior Point)

   (iii) since $S_{11} < 0$, 'P' lies out side of the circle (i.e., P is an exterior Point)

2. (i) 0 (ii) –69 (iii) 22

3. K = –2

5. (i) $4x - 3y + 7 = 0$ (ii) $x + y - 7 = 0$

6. (i) $y = 2$ (ii) $2x + y - 11 = 0$

7. $5x - 14y - 16 = 0$

8. $4x + 7y - 30 = 0$

9. (i) Intersect with each other

   (ii) Each lies on the exterior of the other.

10. $2\sqrt{7}$

11. $(2, -3)$

13. $k = 3$

EXERCISE 11.3

1. (i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{\pi}{3}$

2. (i) ± 4 (ii) -24 (iii) -8

4. (i) $14x + 4y + 6 = 0$

   (ii) $8x + 6y + 5 = 0$

   (iii) $8x + 12y - 19 = 0$

5. (i) $(x^2 + y^2 - 6x + 4y - 2) + \lambda(2x + 2y - 3) = 0.$

   (ii) $(x^2 + y^2 - 10x + 2y - 8) + \lambda(-4x - 2y - 2) = 0.$

   (iii) $(x^2 + y^2 + 2x - 8y + 2) + \lambda(-2x + 2y + 6) = 0.$

6. $2(x^2 + y^2) + 25x + 24 = 0$

7. $3x^2 + 3y^2 + 4x = 0$

8. (i) $\left(0, \frac{3}{4}\right)$ (ii) $\left(\frac{7}{6}, \frac{11}{6}\right)$

9. $7(x^2 + y^2) - 8x - 8y - 12 = 0$

PRACTICE EXERCISE

1. $x^2 + y^2 - 8x + 12y + 3 = 0$

2. Centre $(-2, 3)$; Radius = $\sqrt{3}$

3. $x^2 + y^2 = 1.$

4. (a) $x = \sqrt{3}\cos\theta,\ y = \sqrt{3}\sin\theta$

   (b) $x = 2 + 5\cos\theta,\ y = -2 + 5\sin\theta$

Chapter 12

STRAIGHT LINES

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- recognise a circle, parabola and ellipse as sections of a cone;
- recognise the parabola and ellipse as certain loci;
- identify the concept of eccentricity, directrix, focus and vertex of a conic section;
- identify the standard equations of parabola and ellipse; and
- find the equation of a parabola given its directrix and focus.

PREREQUISITES

- Basic knowledge of coordinate Geometry
- Various forms of equation of a straight line
- Equation of a circle in various forms

INTRODUCTION

While cutting a carrot you might have noticed different shapes shown by the edges of the cut. Analytically you may cut it in three different ways, namely

(i) Cut is parallel to the base (see Fig.12.1)

(ii) Cut is slanting but does not pass through the base (see Fig.12.2)

(iii) Cut is slanting and passes through the base (see Fig.12.3)

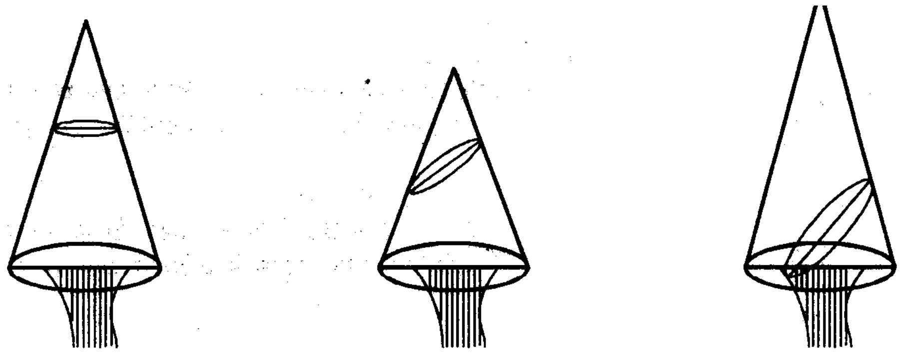


Fig. 12.1 **Fig. 12.2** **Fig. 12.3**

The different ways of cutting, give us slices of different shapes.

In the fIrst case, the slice cut represent a circle which we have studied in previous lesson.

In the second and third cases the slices cut represent different geometrical curves, which we shall study in this lesson.

12.1 CONIC SECTION

In the introduction we have noticed the various shapes of the slice of the carrot. Since the carrot is conical in shape so the section formed are sections of a cone. They are therefore called conic sections.

Mathematically, a conic section is the locus of a point P which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed line.

The fixed point is called the *focus* and is usually denoted by *S*.

The fixed straight line is called the *Directrix*.

The straight line passing through the focus and perpendicular to the directrix is called the *axis*.

The constant ratio is called the *eccentricity* and is denoted by *e*.

What happens when

(i) $e < 1$ (ii) $e = 1$ (iii) $e > 1$

In these cases the conic section obtained are known as ellipse, parabola and hyperbola respectively.

In this lesson we shall study about ellipse and parabola only.

12.2 ELLIPSE

Recall the cutting of slices of a carrot. VVhen we cut it obliquely, slanting without letting the knife pass through the base, what do we observe?

You might have come across such shapes when you cut a boiled egg vertically.

The slice thus obtained represents an ellipse. Let us defme the ellipse mathematically as follows:

"An ellipse is the locus of a point which moves in a plane such that its distance from a fixed point bears a constant ratio to its distance from a fixed line and this ratio is less than unity".

12.2.1 STANDARD EQUATION OF AN ELLIPSE

Let S be the focus, ZK be the directrix and P be a moving point. Draw SK perpendicular from S on the directrix. Let e be the eccentricity.

Divide SK internally and externally at A and A'(on KS produced) repectively in the ratio $e : 1$, as $e < 1$.

$$SA = e.AK \qquad ...(1)$$

and $$SA' = e.A'K \qquad ...(2)$$

Since A and A' are points such that their distances from the focus bears a constant ratio e ($e < 1$) to their respective distances from the directrix and so they lie on the ellipse. These points are called vertices of the ellipse.

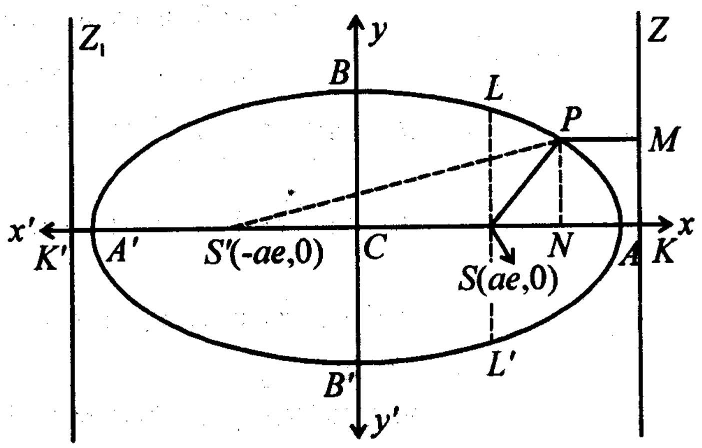


Fig. 12.4

Let AA' be equal to $2a$ and C be its mid point, i.e., $CA = CA' = a$

The point C is called the centre of the ellipse.

Adding (1) and (2), we have

$$SA + SA' = e.AK + e.A'K$$

or $$AA' = e(CK - CA + A'C + CK)$$

or $$2a = e\, 2\, CK$$

or $$CK = \frac{a}{e} \qquad ...(3)$$

Subtracting (1) from (2), we have

$$SA' - SA = e\,(A'K - AK)$$

or $$(SC + CA') - (CA - CS) = e.A'A$$

or $$2\,CS = e.2a$$

or $$CS = ae \qquad ...(4)$$

Let us choose C as origin, CAX as x-axis and CY, a line perpendicular to CX as y-axis.

$\therefore$ Coordinates of S are then $(ae, 0)$ and equation of the directrix is $x = \frac{a}{e}$

Let the coordinates of the moving point P be (x, y).

Join SP, draw $PM \perp ZK$.

By definition $SP = e.PM$

or $\quad SP^2 = e^2 \cdot PM^2$

or $\quad SN^2 + NP^2 = e^2 \cdot (NK)^2$

or $\quad (CN - CS)^2 + NP^2 = e^2 \cdot (CK - CN)^2$

or $\quad (x - ae)^2 + y^2 = e^2 \left(\frac{a}{e} - x\right)^2$

or $\quad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

or $\quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \qquad$ (On dividing by $a^2(1 - e^2)$)

Putting $a^2(1 - e^2) = b^2$ we have the standard form of the ellipse as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Major axis : The line joining the two vertices A' and A, i.e., A' A is called the major axis and its length is $2a$.

Minor axis: The line passing through the centre perpendicular to the major axis, *i.e.,* BB' is called the minor axis and its length is $2b$.

Principal axis : The two axes together (major and minor) are called the principal axes of the ellipse.

Latus rectum : The length of the line segment LL' is called the latus rectum.

Equation of the directrix : $x = \pm \frac{a}{c}$

Eccentricity : e is given by $e^2 = 1 - \frac{b^2}{a^2}$

Example 12.1

Find the equation of the ellipse whose focus is (1,–1), eccentricity $e = \frac{1}{2}$ and the directrix is $x - y = 3$.

Solution: Let P(h, k) be any point on the ellipse then by the definition, its distance from the focus = e. Its distance from directrix

or $\quad SP^2 = e^2 \cdot PM^2$

(Mis the foot of the perpendicular drawn fromP to the directrix).

or $\quad (h - 1)^2 + (k + 1)^2 = \frac{1}{4}\left(\frac{h-k-3}{\sqrt{1+1}}\right)^2$

or $\quad 7(h^2 + k^2) + 2kx - 10h + 10k + 7 = 0$

$\therefore$ The locus of P is

$$7(x^2 + y^2) + 2xy - 10x + 10y + 7 = 0$$

which is the required equation of the ellipse.

Example 12.2: Find the eccentricity, coordinates of the foci and the length of the axis of the ellipse $3x^2 + 4y^2 = 12$.

Solution: The equation of the ellipse can be written in the following form

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

On comparing this equation with that of the standard equation of the ellipse, we have $a^2 = 4$ and $b^2 = 3$ then

(i) $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow e = \frac{1}{2}$

(ii) coordinates of the foci are (1,0) and $(-1,0)$

$[\because$ The coordinate are $(\pm\, ae, 0)]$

(iii) Length of the major axes $2a = 2 \times 2 = 4$ and

length of the minor axis $= 2b = 2 \times \sqrt{3} = 2\sqrt{3}$

EXERCISE 12.1

1. Find the equation of the ellipse referred to its centre

   (a) whose latus rectum is 5 and whose eccentricity is $\frac{2}{3}$

   (b) whose minor axis is equal to the distance between the foci and whose latus rectum is 10.

   (c) whose foci are the points (4,0) and (– 4,0) and whose eccentricity is $\frac{1}{3}$.

2. Find the eccentricity of the ellipse, ifits latus rectum be equal to one half its minor axis.

12.3 PARABOLA

Recall the cutting of slice of a carrot. When we cut obliquely and letting the knife pass through the base, what do we observe?

Also when a batsman hits the ball in air, have you ever noticed the path of the ball?

Is there any property common to the edge of the slice of the carrot and the path traced out by the ball in the example cited above?

Yes, the edge of such a slice and path of the ball have the same shape which is known as a parabola. Let us define parabola mathematically.

"A parabola is the locus of a point which moves in a plane so that its distance from a fixed point in the plane is equal to its distance from a fixed line in the plane.

12.3.1 STANDARD EQUATION OF A PARABOLA

Let S be the fixed point and ZZ' be the directrix of the parabola. Draw SK perpendicular to ZZ. Bisect SK at A.

Since SA = AK, by the definition of the parabola*A* lies on the parabola. *A* is called the vertex of the parabola.

Take *A* as origin, AX as the$x$-axis and AY perpendicular to AX through A as the y-axis.

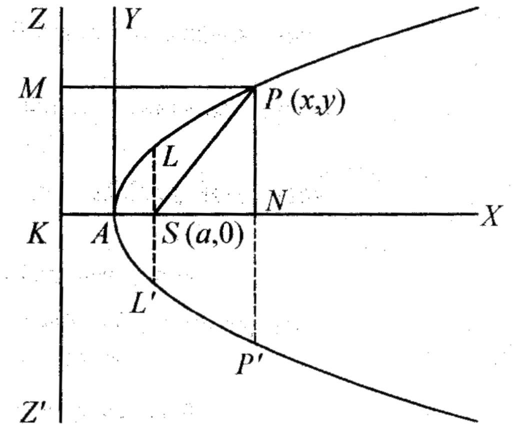


Fig. 12.5

Let KS = $2a$ $\therefore$ AS = AK = a

$\therefore$ The coordinates of A and S are (0,0) and $(a, 0)$ respectively.

Let P(x, y) be any point on the parabola. Draw PN $\perp$ AS produced

$\therefore$ AN = x and NP = y

Join *SP* and draw PM $\perp$ ZZ$'$

$\therefore$ By defmition of the parabola

SP = PM

or SP^2 = PM^2

or $(x - a)^2 + (y - 0)^2 = (x + a)^2$

($\because$ PM = NK = NA + AK = $x + a$)

or $(x - a)^2 - (x + a)^2 = - y^2$

or $y^2 = 4ax$.

which is the *standard equation* of the parabola.

Note: *In this equation of the parabola*

(i) Vertex is (0, 0)

(ii) Focus is (a, 0)

(iii) Equation of the axis is $y = 0$

iv) Equation of the directrix is $x + a = 0$

v) Latus rectum = $4a$

12.3.2 OTHER FORMS OF THE PARABOLA

What will be the equation of the parabola when

(i) focus is $(-a, 0)$ and directrix is $x - a = 0$

(ii) focus is $(0, a)$ and directrix is $y + a = 0$

(iii) focus is $(0, -a)$ and directrix is $y - a = 0$

It can easily be shown that the equation of the parabola with above conditions takes the following forms:

(i) $y^2 = -4ax$

(ii) $x^2 = 4ay$

(iii) $x^2 = -4ay$

The figures are given below for the above equations of the parabolas.

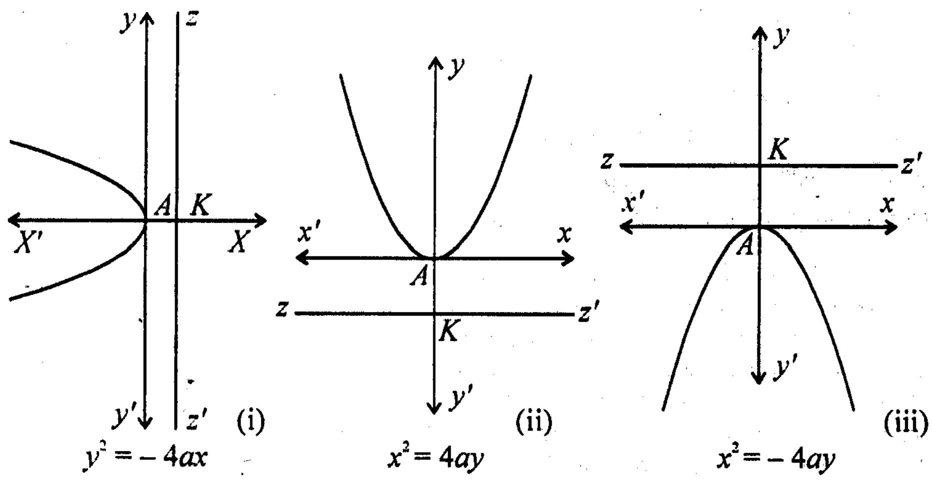


Fig. 12.6

Corresponding results of above forms of parabolas are as follows:

Forms	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Coordinates of vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Coordinates of focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Coordinates of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Coordinates of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
length of Latus rectum	$4a$	$4a$	$4a$	$4a$

Example 12.3 : Find the equation of the parabola whose focus is the origin and whose directrix is the line $2x + y - 1 = 0$.

Solution : Let S(0, 0) be the focus and ZZ' be the directrix whose equation is $2x + y - 1 = 0$.

Let $P(x, y)$ be any point on the parabola.

Let PM be perpendicular to the directrix (See Fig. 12.5)

$\therefore$ By definition SP = PM

or $SP^2 = PM^2$

or $$x^2 + y^2 = \frac{(2x+y-1)^2}{\sqrt{2^2+1^2}}$$

or $5x^2 + 5y^2 = 4x^2 + y^2 + 1 + 4xy - 2y - 4x$

or $x^2 + 4y^2 - 4xy + 2y + 4x - 1 = 0$

Example 12.4 : Find the equation of the parabola, whose focus is the point (2,3) and whose directrix is the line $x - 4y + 3 = 0$.

Solution : Given focus is S(2, 3), and the equation of the directrix $x - 4y + 3 = 0$.

$\therefore$ As in the above example

$$(x-2)^2 + (y-3)^2 = \left\{\frac{x-4y+3}{\sqrt{1^2+4^2}}\right\}^2$$

$\Rightarrow 16x^2 + y^2 + 8xy - 74x - 78y + 212 = 0$

EXERCISE 12.2

1. Find the equation of the parabola whose focus is (a, b) and whose directrix is $\frac{x}{a} + \frac{y}{b} = 1$.
2. Find the equation of the parabola whose focus is (2, 3) and whose directrix is $3x + 4y = 1$.

12.4 EQUATION OF TANGENT AND NORMAL AT A POINT ON THE PARABOLA

In this section the equations of the tangent and normal at a point on the parabola are obtained.

12.4.1 Notation : Hereafter the following notation will be adopted in the present and forthcoming sections.

(i) $S \equiv y^2 - 4ax$

(ii) $S_1 \equiv yy_1 - 2a(x + x_1)$

(iii) $S_{12} \equiv y_1y_2 - 2a(x_1 + x_2)$

(iv) $S_{11} \equiv y_1^2 - 4ax_1$.

1. Theorem : The equation of the chord joining the points (x_1, y_1) and (x_2, y_2) on $S = 0$ is $S_1 + S_2 = S_{12}$.

Proof : Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two point on the parabola $S \equiv y^2 - 4ax = 0$ then $S_{11} = 0$ and $S_{22} = 0$. Consider the first degree equation

$$S_1 + S_2 = S_{12}$$

i.e., $\{y\,y_1 - 2a(x + x_1)\} + \{y\,y_2 - 2a(x + x_2)\} = y_1y_2 - 2a(x_1 + x_2)$

i.e., $4ax - (y_1 + y_2)y + y_1y_2 = 0$ which represents a straight line.

substituting (x_1, y_1) it becomes $S_{11} + S_{12} = S_{12}$.

$$\therefore \quad 0 + S_{12} = S_{12}.$$

$\therefore (x_1, y_1)$ satisfies the equation $S_1 + S_2 = S_{12}$ Similarly (x_2, y_2) satisfies the equation $S_1 + S_2 = S_{12}$.

$\therefore S_1 + S_2 = S_{12}$ is a straight line passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$.

$\therefore$ The equation of the chord PQ is $S_1 + S_2 = S_{12}$.

2. Theorem : The equation of the tangent at $P(x_1, y_1)$ to the parabola $S = 0$ is $S_1 = 0$.

Proof : Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the parabola $S \equiv y^2 - 4ax = 0$ then $S_{11} = 0$ and $S_{22} = 0$.

By 1. Theorem the equation of the chord PQ is

$$S_1 + S_2 = S_{12} \qquad ...(1)$$

The chord PQ becomes the tangent at P when Q approaches P

(i.e., (x_2, y_2) approaches to (x_1, y_1)).

$\therefore$ The equation of the tangent P obtained by taking limits as (x_2, y_2) tends to (x_1, y_1) on either sides of (1).

So the equation of the tangent at P given by

$$\underset{Q \to P}{Lt} (S_1 + S_2) = \underset{Q \to P}{Lt} S_{12}.$$

i.e., $\quad S_1 + S_1 = S_{11}$ $[S_2 \to S_1, S_{12} \to S_{11}$ as $(x_2, y_2) \to (x_1, y_1)$]

$\therefore \quad 2S_1 = 0 \Rightarrow S_1 = 0.$

$\therefore$ The equation of the tagnent to the parabila

$S \equiv y^2 - 4ax = 0$ at $P(x_1, y_1)$ is

$S_1 \equiv yy_1 - 2a(x + x_1) = 0.$

Note : Sum of the slopes of two tangents drawn from an exterior point on the parabola is y_1/x_1

3. Theorem : The equation of the normal at P(x_1, y_1) on the parabola S = 0 is

$$(\boldsymbol{y}-\boldsymbol{y}_1) = -\frac{\boldsymbol{y}_1}{\boldsymbol{2a}}(\boldsymbol{x}-\boldsymbol{x}_1).$$

Proof : By 2. Theorem, the equation of the tangent to the parabola $y^2 - 4ax = 0$ at P(x_1, y_1) is $S_1 \equiv yy_1 - 2a(x+x_1) = 0.$

$\therefore$ Slope of the tangent at P is $\dfrac{2a}{y_1}$

$\therefore$ Slope of the normal at P is $-\dfrac{y_1}{2a}.$

Hence equation of the normal at $P(x_1, y_1)$ is $(y-y_1) = -\dfrac{y_1}{2a}(x-x_1).$

Example 12.5 : Find the equation of the tangent to the parabola $yy_1 = 4(x + x_1)$ at the end of lotus rectum which lies in the forth quadrant.

Sol: The two extreme points of latus rectum in $y^2 = 8x$ forth quadrant are

$$(a, -2a) = (2, -4)$$

$S_1 = 0$ is the tangent

$yy_1 = 4(x + x_1)$

$-4y = 4(x + 2)$

$\Rightarrow x + y + 2 = 0.$ is the required equation of the tangent.

Example 12.6 : Find the sum of the slopes of the tangents drawn from (−2, 3) on to the parabila $y^2 = 6x$.

Sol: Let the slopes be m_1, m_2 .

Then $m_1 + m_2 = \dfrac{y_1}{x_1} = -\dfrac{3}{2}.$

- Find the equation of the tangent at A(1, 3) to the parabola $y^2 = 12x$

$S \equiv y^2 - 12x$

$(x_1, y_1) = (1, 3)$

equation of the tangent is $S_1 = 0$

$$yy_1 - 6(x + x_1) = 0$$

$$y(3) - 6(x + 1) = 0$$

$$3y - 6x - 6 = 0$$

$$y - 2x - 2 = 0$$

- Find the equation of tangent and normal to the ellipse $x^2 + 8y^2 = 33$ at $(-1, 2)$

$$S \equiv x^2 + 8y^2 - 33; \quad (x_1, y_1) = (-1, 2)$$

equation of tangent $\quad S_1 = 0$

$$xx_1 - 8yy_1 - 33 = 0$$

$$x(-1) + 8y(2) - 33 = 0$$

$$-x + 16y - 33 = 0$$

$$x - 16y + 33 = 0$$

Solpe of tangent $\quad m = \dfrac{-1}{-16} = \dfrac{1}{16}$

Slope of normal $\quad m = -16$

$\therefore$ equation of normal at $(-1, 2)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{16}(x + 1)$$

$$16y - 32 = x + 1$$

$$x - 16y + 33 = 0$$

12.5 POLE AND POLAR

The definitions of chord of contact, pole and polar related to parabola are exactly the same as in the case of a circle. In this section we find the equations of chord of contact polar of a point with respect to a given parabola, and also we shall learn the method of finding the pole.

12.5.1 Definition (Chord of contact)

If two tangents are drawn to a conic from an external point, the secant line joining the points of contact is called the chrod of contanct of that point with respect to the conic.

4. Theorem : The equation to the chord of contact of the external point (x_1, y_1) with respect to the parabola S = 0 is $S_1 = 0$.

Proof : Let $S \equiv y^2 - 4ax = 0$ be the equation of the parabila and $P(x_1, y_1)$ be an external point in the plane of the parabila.

Let the tangent from $P(x_1, y_1)$ touch the parabola at $Q(x_2, y_2)$ and $R(x_3, y_3)$. The tangent at $Q(x_2, y_2)$ is $yy_2 - 2a(x + x_2) = 0$ (fig. 12.1)

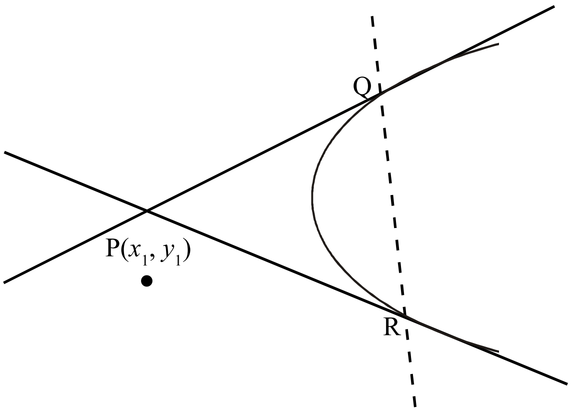


Fig. 12.5

if this passes through $P(x_1, y_1)$ then $y_1y_2 - 2a(x_1 + x_2) = 0$ ... (1)

similarly the tangent at $R(x_3\, y_3)$ is $yy_3 - 2a(x + x_3) = 0$ if this passes through $P(x_1, y_1)$ then

$$y_1y_3 - 2a\ (x_1 + x_3) = 0 \qquad \text{... (2)}$$

From (1) and (2) the points $Q(x_2, y_2)$ and $R(x_3, y_3)$ satisfy the line $S_1 \equiv yy_1 - 2a(x + x_1) = 0$

Hence the equation of $\overleftrightarrow{QR}$, the chord of contact of P with respect to

$$S = 0 \text{ is } S_1 = 0.$$

12.5.2 Definition (Pole and Polar)

The polar of a point with respect to a conic is the straight line containing the points of intersection of the tangents drawn at the extremities of chords passing through that point. The point is called the pole of the polar.

5. Theorem : The equation of the polar of the point $P(x_1, y_1)$ with respect to the parabola $S = 0$ is $S_1 = 0$.

Proof : Let the equation of the parabola be $S \equiv y^2 - 4ax = 0$ and $P(x_1, y_1)$ be a point in the plane of the parabola. Any chord through P meets the parabola in D and E. Let the tangents at D, E to the parabola meet at $Q(h, k)$.

$\therefore$ $\overleftrightarrow{DE}$, the chord of contanct of $Q(h, k)$ with respect to the parabila $S = 0$ whose equation is $yk - 2a(x + h) = 0$ passes through $P(x_1, y_1)$ (fig. 12.6).

$\therefore$ $y_1k - 2a(x_1 + h) = 0$.

Then the point $Q(h, k)$ satisfies the equation $yy_1 - 2a(x + x_1) = 0$

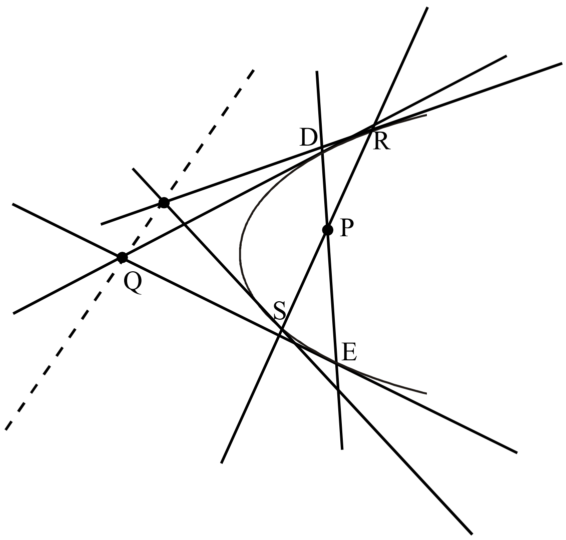


Fig. 12.6

$\therefore$ The equation of polar of $P(x_1, y_1)$ with respect to $S = 0$ is $S_1 = 0$.

Note :

1. If P lies out side the parabola then chord of contact and polar of P with respect to $S = 0$ coincide.

2. If P lies on the parabola then tangent at P and polar of P with resepect to S = 0 coincide.

3. If P lies inside the parabola then polar of P with respect to S = 0 does not meet the parabola.

4. Polar line can never be a horizontal line.

5. The pole of a non horizontal line $lx + my + n = 0$ with respect to the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, \frac{-2am}{l}\right)$.

Proof : Let the pole of the line $lx + my + n = 0 \ (l \neq 0)$ ... (1)

with respect to the parabola $y^2 = 4ax$ ... (2)

is $P(x_1, y_1)$

The equation of the polar with respect to S = 0 is $yy_1 = 2a(x + x_1)$

(or) $2ax - yy_1 + 2ax_1 = 0$ ... (3)

(1), (3) represents same line

$$\frac{l}{2a} = \frac{m}{-y_1} = \frac{n}{2ax_1} \Rightarrow x_1 = \frac{n}{l},\ y_1 = \frac{-2am}{l}$$

$\therefore$ The pole of (1) with respect to S = 0 is $(x_1, y_1) = \left(\frac{n}{l}, \frac{-2am}{l}\right)$.

Example 12.7 : Find the pole of the line $2x + 3y - 4 = 0$ with respect to the parabola $y^2 = 4x$

Sol : Pole $= \left(\frac{+n}{l}, \frac{-2am}{l}\right)$

$$= \left(\frac{-4}{2}, \frac{-2(1)(3)}{2}\right)$$

$$= (-2, -3).$$

EXERCISE 12.3

1. Find the equation of the tangent at A(1, 3) to the parabola $y^2 = 12x$
2. Find the equation of the normal at R(2, 5) on the parabola $y^2 = 6x$.
3. Find the equation of the tangent to the parabola at the end of latus rectum which lies in the fourth quardent.
4. Claculate the sum of the slopes of the tangents drawn from (1, 4) on to the parabola $y^2 = 36x$.
5. Find the pole of the line $4x + 2y + 8 = 0$ with respect to the parabila $y^2 = 8x$

12.6 EQUATION OF TANGENT AND NORMAL

In this section, the relation between an ellipse and a straight line in its plane is discussed. The condition for a straight line to be tangent to a given ellipse is obtained. The cartesian and parametric equations of the tangent and normal at a given point on the ellipse are also derived.

Definition : Any straight line that intersects the ellipse at only one point (touches) is tangent to the ellipse.

(i) The condition for a straight line $y = mx + c$ to be a tangent to the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$c^2 = a^2m^2 + b^2$.

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1 \Rightarrow x^2(a^2m^2 + b^2) + 2a^2m(c^2 - b^2) = 0 \qquad ...(1)$$

The line will touch the ellipse iff the two points are coincident.

$\Leftrightarrow$ discreminent of (1) is zero.

$\Leftrightarrow \quad 4a^4c^2m^2 - 4(a^2m^2 + b^2)\ a^2(c^2 - b^2) = 0$

$\Leftrightarrow \quad c^2 = a^2m^2 + b^2 \Leftrightarrow c = \pm\sqrt{a^2m^2+b^2}$

(ii) The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the ellipse S = 0 is $S_1 + S_2 = S_{12}$

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0;$$

$$S_1 = \frac{x\,x_1}{a^2} + \frac{y\,y_1}{b^2} - 1 = 0; \quad S_2 = \frac{x\,x_2}{a^2} + \frac{y\,y_2}{b^2} - 1 = 0$$

$$S_{12} = \frac{x_1\,x_2}{a^2} + \frac{y_1\,y_2}{b^2} - 1 = 0.$$

(iii) The equation of the tangent $P(x_1, y_1)$ to the ellipse S = 0 is $S_1 = 0$

$$\Rightarrow \quad \frac{x\,x_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

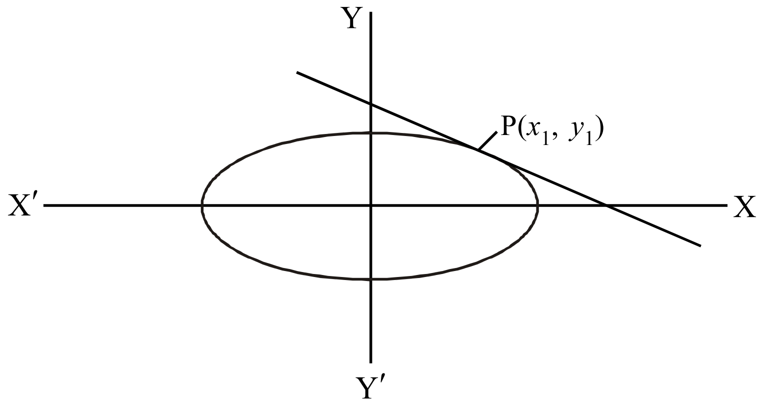


Fig. 12.7

(iv) The equation of the normal at $P(x_1, y_1)$ to the ellipse $S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

is $\dfrac{a^2x}{x_1} - \dfrac{b^2y}{y_1} = a^2 - b^2 \ (x_1 \neq 0,\ y_1 \neq 0)$

Example 12.8 : Find the equation of tangent and normal to the ellipse $9x^2 + 4y^2 = 20$ at $\left(\frac{-4}{3}, 1\right)$.

Solution: Given ellipse $9x^2 + 4y^2 = 20$

$$9x\, x_1 + 4\, y\, y_1 = 20$$

$$(x_1, y_1) = (-4/3, 1)$$

$$9x\left(\frac{-4}{3}\right) + 4y(1) = 20 \Rightarrow 3x - y + 5 = 0$$

Equation of tangent at $\left(\frac{-4}{3}, 1\right)$ is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 - b^2$

$$\frac{(20/9)x}{(-4/3)} - \frac{5y}{1} = \frac{20}{9} - 5 \Rightarrow 3x + 9y = 5$$

Example 12.9 : Find the value of k if $x + ky - 5 = 0$ is tangent to the ellipse $4x^2 + 9y^2 = 20$.

Sol : Given ellipse $\frac{x^2}{5} + \frac{y^2}{(20/9)} = 1$

then $a^2 = 5,\ b^2 = \frac{20}{9}$

$\therefore$ if $x + ky - 5 = 0$ is the tangent, Then $5(1)^2 + \frac{20}{9} k^2 = (-5)^2$

$\Rightarrow 45 + 20k^2 = 225 \Rightarrow 20k^2 = 180 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$

$[\therefore a^2 l^2 + b^2 m^2 = n^2]$

Example 12.10 : Find the equation of the ellipse referred to its centre whose minor axis is equal to the distance between the foci and whose latus rectum is 10.

Sol: Let the required equation of ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (a > b)$$

$$2b = 2ac \qquad \frac{2b^2}{a}=10$$

$$b = ac \qquad b^2 = 5a$$

but $\quad b^2 = a^2\,(1 - c^2)$

$$b^2 = a^2 - (ac)^2$$

$$b^2 = a^2 - b^2$$

$$2b^2 = a^2$$

$$2(5a) = a^2$$

$$a = 10$$

Example 12.11 : Find the equation of ellipse to its major and minor axes as the co-ordinate axes respectively w ith latus rectum of length 4 and distance between foci $4\sqrt{2}$.

Solution: Let the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \; (a > b)$ length of the latus rectum $\frac{2b^2}{a}=4 \Rightarrow b^2 = 2a$

Distance between foci S = $(ac, 0)$, $S^1 = (-ac, 0)$ is

$$2ac = 4\sqrt{2}$$

$$\Rightarrow \quad ac = 2\sqrt{2}$$

but $b^2 = a^2\,(-c^2) \Rightarrow 2a = a^2 - (ac)^2 = a^2 - 8$

$$a^2 - 2a - 8 = 0$$

$$(a - 4)\,(a + 2) = 0$$

$$a = 4 \; (a > 0)$$

$$b^2 = 2a = 2(4) = 8$$

$\therefore$ equation of ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{8} = 1$.

Example 12.12 : If the length of the latus rectum is equal to half of its minor axis of an ellipse in the standard form, then find the eccentricity of the ellipse.

Solution : Let $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ be the ellipse in its standard form.

Given that

The length of latus rectum = $\dfrac{1}{2}$(minor axis)

$$\frac{2b^2}{a} = \frac{1}{2}(ab) \Rightarrow 2b = a$$

$$\therefore \; 4b^2 = a^2 \Rightarrow 4a^2(1-e^2) = a^2$$

$$\therefore \; 1-e^2 = \frac{1}{4} \Rightarrow e^2 = 1-\frac{1}{4} = \frac{3}{4}$$

$$e = \frac{\sqrt{3}}{2}.$$

EXERCISE 12.4

1. Find the equation of the tangent and normal to the ellipse $x^2 + 2y^2 - 4x + 12y + 14 = 0$ at$(2, -1)$

2. Find the value of K if $4x + y + K = 0$ is a tangent to the ellipse $x^2+3y^2 = 3$.

3. Find the equation of tangent and normal to the ellipse $x^2 + 8y^2 = 33$ at $(-1, 2)$.

12.7 POLE AND POLAR

In this section, we shall discuss about chord of contact, pole and polar of a given ellipse.

If $P(x_1, y_1)$ is the external point to the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$.

Then the chord of contact with respect to $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

is $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} - 1 = 0$

12.7.1 If $P(x_1, y_1)$is an external point to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ Then the equation fo the chord of the contact to P with respect to S = 0 is $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} - 1 = 0$.

12.7.2 The polar of the point (x_1, y_1) (other than centre) will respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ is $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} - 1 = 0$.

12.7.3 The pole of the line $lx + my + n = 0$ $(n \neq 0)$ with respect to the ellipse $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} - 1 = 0$ is $\left(\frac{-a^2 l}{n}, \frac{-b^2 m}{n}\right)$

12.7.4 The condition for two lines $l_1x + m_1y + n_1 = 0(n_1 \neq 0)$ and $l_2x + m_2y + n_2 = 0$ $(n_2 \neq 0)$ to be conjugate with respect to the ellipse $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ is $a^2 l_1 l_2 + b^2 m_1 m_2 = n_1 n_2$.

Example 12.13 : Find the pole of the line $x - y + 2 = 0$ with respect to the ellipse $x^2 + 4y^2 - x - 16y - 10$ ...(1)

Sol: Let the pole of the line

The equation of the polar of $P(x_1, y_1)$ w.r.t S = 0 is

$$S_1 = xx_1 + 4yy_1 - (x + x_1) - 8\ (y + y_1) - 10 = 0$$

$$\Rightarrow\ x(x_1 - 1) + 4y\ (y_1 - 2) - (x_1 + 8y_1 + 10) = 0 \qquad ...(2)$$

Equations L = 0 and $S_1 = 0$ represent the same line. Hence

$$\frac{x_1 - 1}{1} = \frac{4(y_1 - 2)}{-1} = \frac{-(x_1 + 8y_1 + 10)}{2} = k$$

$$\therefore\ x_1 = k + 1,\ y_1 = \frac{-k}{4} + 2,\ x_1 + 8y + 10 = -2k \qquad ...(3)$$

put the values x_1 and y_1 in equation (3)

$$k + 1 + 8\left(\frac{-k+8}{4}\right) + 10 = -2k$$

$$\Rightarrow\ k + 1 + 2(-k + 8) + 10 = -2k \Rightarrow k = -27$$

$$\Rightarrow x_1 = -26,\ y_1 = \frac{35}{4}\ \text{ pole } P = \left(-26, \frac{35}{4}\right)$$

EXERCISE 12.5

1. Find the pole of the line $5x + 7y + 8 = 0$ with respect to the ellipse $5x^2 + 7y^2 = 8$.

2. Find the pole of the line $21x - 16y - 12 = 0$ wiht respect to the ellipse $3x^2 + 4y^2 = 12$.

12.8 HYPERBOLA - SOME IMPORTANT PROPERTIES

The concept of latus rectum, tangent, normal at a point on a hyperbola are defined analogously as in the case of an ellipse. In the following we list out certain important properties of a hyperbola. The reader can easily supplement the proofs, which are similar to those of an ellipse.

1. A point $P(x_1, y_1)$ in the plane of the hyperbola S = 0 lies inside the hyperbola if $S_{11} > 0$, lies outside if $S_{11} < 0$ and on the curve if $S_{11} = 0$.

2. The end of the latus recta are $\left(\pm ae, \pm \frac{b^2}{a}\right)$ and the length of the latus rectum is $\frac{2b^2}{a}$

3. The equation of the tangent at $P(x_1, y_1)$ is $S_1 = 0$.

4. The equation of the tangent 'θ' is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta - 1$ $\left(\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}\right)$.

5. The equation of the normal at $P(x_1, y_1)$ is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$ $(y \neq 0)$ which is always the case except at vertices. At verties X-axis is the normal.

6. The equation of the normal at 'θ' is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ $(\theta \neq 0, \pi)$

7. The condition for a straight line $y = mx + c$ to be a tangent to the hyperbola S = 0 is $c^2 = a^2m^2 - b^2$.

Example 12.14 : Find the equation of the hyperbola whose foci is $\frac{9}{2}$ and eccentricity is $\frac{5}{4}$.

Sol : $\frac{2b^2}{a} = \frac{9}{2}, e = \frac{5}{4} \Rightarrow b^2 = \frac{9a}{4} \Rightarrow a^2\left(\frac{25}{16} - 1\right) = \frac{9a}{4}$

$\Rightarrow a = 4,\ b^2 = 9 \quad \therefore$ equation of the hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

Example 12.15 : Find the foci of given equation of hyperbola $\dfrac{x^2}{36} - \dfrac{y^2}{16} = 1$.

Sol : $a^2 = 36;\ b^2 = 16 \ \therefore\ e = \dfrac{\sqrt{a^2+b^2}}{a} = \dfrac{\sqrt{36+16}}{6} = \dfrac{\sqrt{52}}{6} = \dfrac{\sqrt{13}}{3}$

Foci are $(\pm\, ae, 0) = (\pm\, 2\sqrt{13}, 0)$.

12.9 POLE AND POLAR

The concept of pole and polar with respect to a hyperbola are defined analogously as in the case of ellipse

1. If $P(x_1, y_1)$ is any point in the plane of the hyperbola S = 0, then the equation of the polar of P w.r.t. S = 0 is $S_1 = 0$
2. The pole of the line $lx + my + n = 0\ (n \neq 0)$ w.r.t the hyperbola S = 0 is $\left(\dfrac{-a^2 l}{n}, \dfrac{b^2 m}{n}\right)$

Example 12.16: Find the condition that the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ are conjugate with respect to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Sol : Equation of the polar of $P(x_1, y_1)$ w.r.t S = 0 is $S_1 \equiv \dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} - 1 = 0$

Since L = 0 and $S_1 = 0$ represents the polar of $P(x_1, y_1)$ w.r.t. S = 0

$$\frac{x_1}{\frac{a^2}{l_1}} = \frac{y_1}{-\frac{b^2}{m_1}} = \frac{-1}{n_1} \Rightarrow x_1 \frac{-a^2 l_1}{n_1},\ y_1 = \frac{b^2 m_1}{n_1}$$

The pole of L = 0 w.r.t S = 0 is $\left(\frac{-a^2 l_1}{n_1}, \frac{b^2 m_1}{n_1}\right)$ lies on the line L$'$ = 0.

($\therefore$ The lines L = 0 and L$'$ = 0 are conjugate w.r.t S = 0)

$$\frac{-a^2 l_1 l_2}{n_1} + \frac{b^2 m_1 m_2}{n_1} + n_2 = 0$$

$\therefore$ $a^2 l_1 l_2 - b^2 m_1 m_2 = n_1 n_2$ Which is the required condition.

EXERCISE 12.6

1. Prove that the poles of normal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lie on the curve $\frac{a^6}{a^2} + \frac{b^6}{b^2} = (a^2 + b^2)^2$.

KEY WORDS

- **Conic Section**
- "A conic section is the locus of a point *P* which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line".

(i) **Focus:** The fixed point is called the focus.

(ii) **Directrix:** The fixed straight line is called the directrix.

(iii) **Axis:** The straight line passing through the focus and pependicular to the directrix is called the axis.

(iv) **Eccentricity:** The constant ratio is called the eccentricity.

(v) **Latus Rectum:** The double ordinate passing through the focus and parallel to the directrix is known as latus rectum. (In Fig.12. 5 *LSL'* is the latus rectum).

- **Standard Equation of the Ellipse is :** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

  (i) Major axis = $2a$

  (ii) Minor axis = $2b$

  (iii) Equation of directrix is $x = \pm \frac{a}{c}$

  (iv) Foci : $(\pm\, ae, 0)$

  (v) Eccentricity, i.e., e is given by $e^2 = 1-\frac{b^2}{a^2}$

- Standard Equation of the Parabola is: $y^2 = 4ax$

  (i) Vertex is (0,0)

  (ii) Focus is $(a,0)$

  (iii) Axis of the parabola is $y = 0$

  (iv) Directrix of the parabola is $x + a = 0$

  (v) Latus rectum = $4a$

OTHER FORMS OF THE PARABOLA ARE

i) $y^2 = -\,4ax$ (concave to the left).

ii) $x^2 = 4ay$ (concave upwards).

iii) $x^2 = -4ay$ (concave downwards).

- A horizontal line cannot be a tangent to the parabola $y^2 = 4ax$.

- Equation of the tangent at the point (x_1, y_1) on the parabila S = 0 is $S_1 = 0$.
- Equation of the normal at the point (x_1, y_1) on the parabola S = 0 is $(y - y_1) = \frac{-y_1}{2a}(x - x_1)$.
- Equation of the chord of contact of the external point $P(x_1, y_1)$ with respect to the parabola S = 0 is $S_1 = 0$.
- Equation of polar of $P(x_1, y_1)$ with respect to the parabola S = 0 is $S_1 = 0$.
- Pole of the line $lx + my + n = 0$ $(l \neq 0)$, with respect to the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, \frac{-2am}{l}\right)$.
- Standard equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- The condition for a straight line $y = mn + c$ to be a tangent to the ellipse $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} - 1 = 0$ is $c^2 = a^2m^2 + b^2$
- The equation of the tangent $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} - 1 = 0$ is w.r.t the ellipse.
- The equation of the normal $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$ is w.r.t the ellipse.
- The polar equation is $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} - 1 = 0$
- Equation of the tangent of $p(\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

- Equation of the normal at $p(\theta)$ to the ellipse S = 0 is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ when $\theta \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

- Ends of latusrectum is $\left(\pm ae, \pm \frac{b^2}{a}\right)$

- Length of latusrectum is $\frac{2b^2}{a}$

- The equation of the tangent at $P(x_1, y_1)$ is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = 0$

- The pole of the line $lx + my + n = 0 \, (n \neq 0)$ w.r.t the hyperbola S = 0 is $\left(\frac{-a^2 l}{n}, \frac{b^2 m}{n}\right)$

SUPPORTIVE WEB SITES

http://www. wikipedia. org

http://mathworld. wolfram. com

PRACTICE EXERCISE

1. Find the equation of the ellipse in each of the following cases, when

   (a) focus is (0, 1), directrix is $x + y = 0$ and $e = \frac{1}{2}$

   (b) focus is (–1, 1) directrix is $x - y + 3 = 0$ and $e = \frac{1}{2}$.

2. Find the coordinates of the foci and the eccentricity of each of the following ellipses:

   (a) $4x^2 + 9y^2 = 1$

   (b) $25x^2 + 4y^2 = 100$

3. Find the equation of the parabola whose focus is (–8, –2) and directrix is $y - 2x + 9 = 0$.

ANSWERS

EXERCISE 12.1

1. (a) $20x^2 + 36y^2 = 405$

   (b) $x^2 + 2y^2 = 100$

   (c) $8x^2 + 9y^2 = 1152$

2. $\frac{\sqrt{3}}{2}$

EXERCISE 12.2

1. $(ax - by)^2 - 2a^3 x - 2b^3y + a^4 + a^2 b^2 + b^4 = 0$

2. $16x^2 + 9y^2 - 94 x - 142y - 24 xy + 324 = 0$

EXERCISE 12.3

1. $6x - 3y + 6 = 0$

2. $5x + 3y - 25 = 0$

3. $2x + 2y + 2 = 0$

4. 4

5. $(2, -2)$.

EXERCISE 12.4

1. $y + 1 = 0, \ x - 2 = 0$

2. $K = \pm 7$

3. $x - 16y + 33 = 0, \ 16x + y + 44 = 0.$

EXERCISE 12.5

1. $(-1, -1)$
2. $(7, -4)$

PRACTICE EXERCISE

1. (a) $7x^2 + 7y^2 - 2xy - 16y + 8 = 0$

   (b) $7x^2 + 7y^2 + 2xy + 10x - 10y + 7 = 0$

2. (a) $\left(\pm \frac{\sqrt{5}}{6}, 0\right); \frac{\sqrt{5}}{3}$

   (b) $\left(0, \pm \sqrt{21}\right); \frac{\sqrt{21}}{5}$

3. $x^2 + 4y^2 + 4xy + 116x + 2y + 259 = 0$

Chapter 13

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- associate a point, in three dimensional space with given triplet and vice versa;
- find the distance between two points in space;
- find the coordinates of a point which divides the line segment joining twogiven points in a given ratio internally and externally;
- define the direction cosines/ratios of a given line in space;
- find the direction cosines of a line in space;
- find the projection of a line segment on another line; and
- find the condition of prependicularity and parallelism of two lines in space.

PREREQUISITES

- Two dimensional co-ordinate geometry
- Fundamentals of Algebra, Geometry, Trigonometry and vector algebra.

MODULE - III
Dimensional
Geometry Vectors

Notes

INTRODUCTION

You have read in your earlier lessons that given a point in a plane, it is possible to find two numbers, called its co-ordinates in the plane. Conversely, given any ordered pair (x, y) there corresponds a point in the plane whose co-ordinates are (x, y).

Let a rubber ball be dropped vertically in a room The point on the floor, where the ball strikes, can be uniquely determined with reference to axes, taken along the length and breadth of the room. However, when the ball bounces back vertically upward, the position of the ball in space at any moment cannot be determined with reference to two axes considered earlier. At any instant, the position of ball can be uniquely determined if in addition, we also know the height of the ball above the floor.

If the height of the ball above the floor is 2.5cm and the position of the point where it strikes the ground is given by (5, 4), one way of describing the position of ball in space is with the help of these three numbers (5, 4, 2.5).

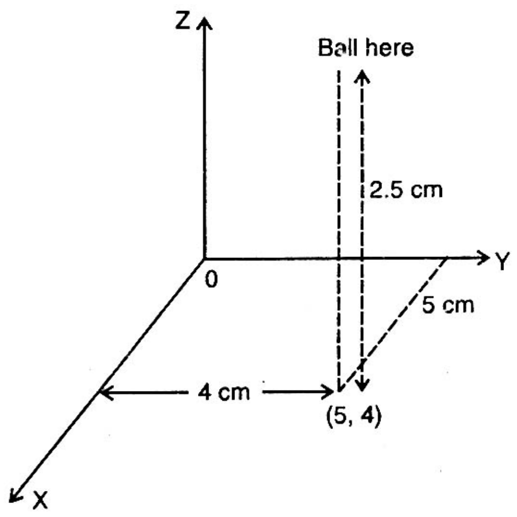


Fig. 14.1

Thus, the position of a point (or an article) in space can be uniquely determined with the help of three numbers. In this lesson, we will discuss in details about the co-ordinate system and co-ordinates of a point in space, distance between two points in space, position of a point dividing the join of two points in a given ratio internally/externally and about the projection of a point/line in space.

13.1 COORDINATE SYSTEM AND COORDINATES OF A POINT IN SPACE

Recall the example of a bouncing ball in a room where one corner of the room was considered as the origin.

It is not necessary to take a particular corner of the room as the origin. We could have taken any corner of the room (for the matter any point of the room) as origin of reference, and relative to that the coordinates of the point change. Thus, the origin can be taken arbitarily at any point of the room.

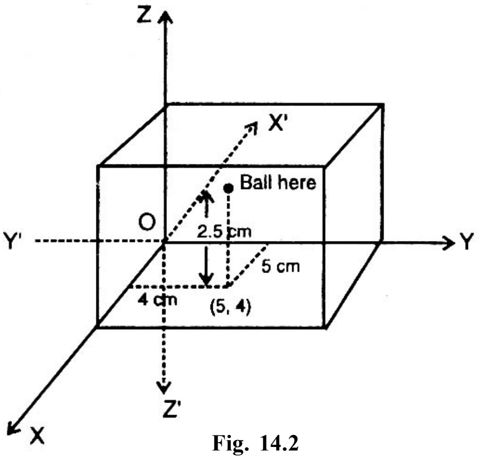


Fig. 14.2

Let us start with an arbitrary point O in space and draw three mutually perpendicular lines X'OX, Y'OY and Z'OZ through O. The point O is called the origin of the co-ordinate system and the lines X'OX, Y'OY and Z'OZ are called the X-axis, the Y-axis and the Z-axis respectively. The positive direction of the axes are indicated by arrows on thick lines in Fig. 13.2. The plane determined by the X-axis and the Y-axis is called XY-plane (XOY plane) and similarly, YZ-plane (YOZ-plane) and ZX-plane (ZOX-plane) can be determined. These three planes

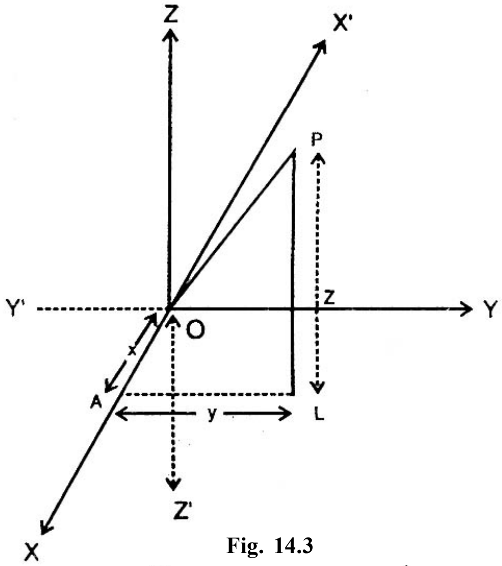


Fig. 14.3

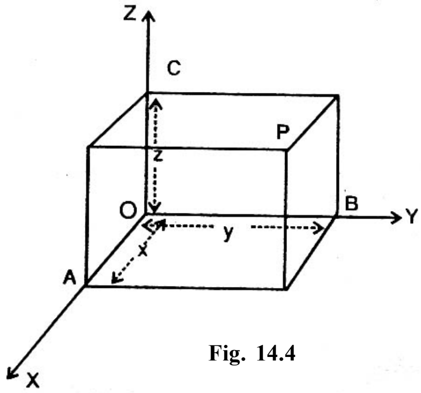


Fig. 14.4

are called co-ordinate planes. The three coordinate planes divide the whole space into eight parts called **octants**.

Let P be any point is space. Through P draw perpendicular PL on XY-plane meeting this plane at L. Through L draw a line LA parallel to OY cutting OX in A. If we write OZ = x, AL = y and LP = z, then (x, y, z) are the co-ordinates of the point P.

Again, if we complete a reactangular parallelopiped through P with its three edges OA, OB and OC meeting each other at O and OP as its main diagonal then the lengths (OA, OB, OC) i.e., (x, y, z) are called the co-ordinates of the point P.

Note : You may note that in Fig. 13.4

(i) The x co-ordinate of P = OA = the length of perpendicular from P on the YZ-plane.

(ii) The y co-ordinate of P = OB = the length of perpendicular from P on the ZX-plane.

(iii) The x co-ordinate of P = OC = the length of perpendicular from P on the XY-plane.

Thus, the co-ordinates x, y, and z of any point are the perpendicular distances of P from the three rectangular co-ordinate planes YZ, ZX and XY respectively.

Thus, given a point P in space, to it corresponds a triplet (x, y, z) called the co-ordinates of the point in space. Conversely, given any triplet (x, y, z), there corresponds a point P in space whose co-ordinates are (x, y, z).

Remarks

1. Just as in plane co-ordinate geometry, the co-ordinate axes divide the plane into four quadrants, in three dimentional geometry, the space is divided into eight octants by the co-ordinate planes, namely OXYZ, OX′YZ, OXY′Z, OXYZ′, OXY′Z′, OX′YZ′, OX′Y′Z and OX′Y′Z′.
2. If P be any point in the first octant, there is a point in each of the other octants whose absolute distances from the co-ordinate planes are

equal to those of P. If P be (a, b, c), the other points are $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$, $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, c)$ and $(-a, -b, -c)$ respectively in order in the octants referred in (i).

3. The co-ordinates of point in XY-plane, YZ-plane and ZX-plane are of the form $(a, b, 0)$, $(0, b, c)$ and $(a, 0, c)$ respectively.
4. The co-ordinates of points on X-axis, Y-axis and Z-axis are of the form $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.
5. You may see that (x, y, z) corresponds to the position vector of the point P with reference to the origin O as the vector $\overline{OP}$.

Example 13.1: Name the octant wherein the given points lies :

(a) $(2, 6, 8)$ (b) $(-1, 2, 3)$ (c) $(-2, -5, 1)$

(d) $(-3, 1, -2)$ (e) $(-6, -1, -2)$

Solution:

(a) Since all the co-ordinates are positive,

$\therefore$ $(2, 6, 8)$ lies in the octant OXYZ.

(b) Since x is negative and y and z are positive,

$\therefore$ $(-1, 2, 3)$ lies in the octant OX'YZ.

(c) Since x and y both are negative and z is positive.

$\therefore$ $(-2, 5, 1)$ lies in the octant OX′Y′Z.

(d) $(-3, 1, -2)$ lies in octant X' Y Z'.

(e) Since x, y and z are all negative.

$\therefore$ $(-6, -1, -2)$lies in the octant OX′Y′Z′.

EXERCISE 13.1

1. Name the octant wherein the given points lies :

(a) $(-4, 2, 5)$ (b) $(4, 3, -6)$ (c) $(-2, 1, -3)$

(d) $(1, -1, 1)$ (e) $(8, 9, -10)$

13.2 DISTANCE BETWEEN TWO POINTS

Suppose there is an electric plug on a wall of a room and an electric iron placed on the top of a table. What is the shortest length of the wire needed to connect the electric iron to the electric plug ? This is an example necessitating the finding of the distance between two points in space.

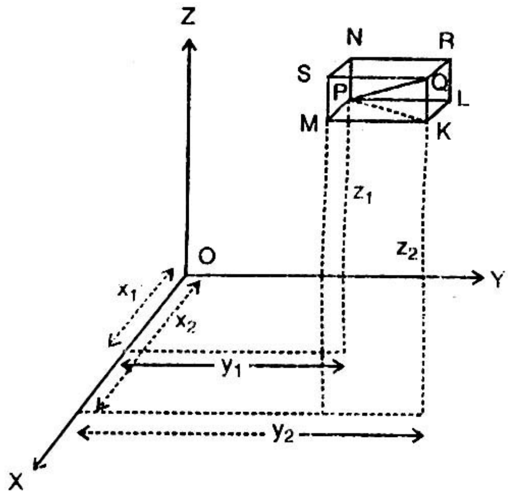


Fig. 14.6

Let the co-ordinates of two points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. With PQ as diagonal, complete the parallopiped PMSNRLKQ.

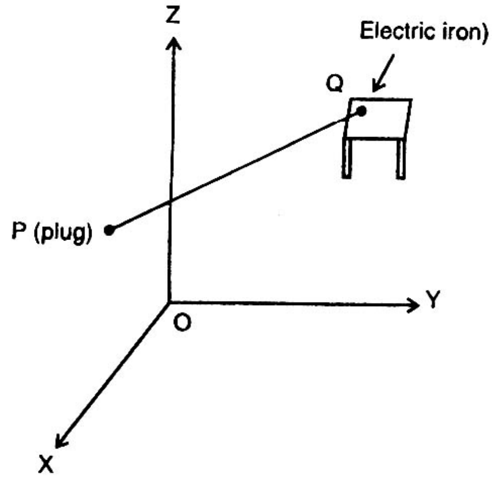


Fig. 14.5

PK is perpendicular to the line KQ.

$\therefore$ From the right-angled triangle PKQ, right angled at K,

We have $PQ^2 = PK^2 + KQ^2$

Again from the right angled triangle PKL right angled at L,

$PK^2 = KL^2 + PL^2$

$= MP^2 + PL^2 \quad (\because KL = MP)$

$\therefore \quad PQ^2 = MP^2 + PL^2 + KQ^2 \qquad$ ...(i)

The edges MP, PL, KQ are parallel to the co-ordinate axes.

Now, the distance of the point P from the plane YOZ $= x_1$ and the distance of Q and from YOZ plane $= x_2$

$\therefore \quad MP = |x_2 - x_1|$

Similarly, PL = $|y_2 - y_1|$ anf KQ = $|z_2 - z_1|$

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad \text{..... [From (i)]}$$

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Corollary : Distance of a Point from the Origin

If the point Q(x_2, y_2, z_2) coincides with the origin (0, 0, 0)

then, $x_2 = y_2 = z_2 = 0$

$\therefore$ The distance of P from the origin is

$$|OP| = \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2 + (z_1 - 0)^2}$$

$$= \sqrt{x_1^2 + y_1^2 + z_1^2}$$

In general, the distance of a point P(x, y, z) from origin O is given by

$$|OP| = \sqrt{x^2 + y^2 + z^2}$$

Example 13.2 : Find the distance between the points (2, 5, –4) and (8, 2, –6).

Solution: Let P(2, 5, –4) and Q(8, 2, –6) be the two given points.

$$|PQ| = \sqrt{(8-2)^2 + (2-5)^2 + (-6+4)^2}$$

$$= \sqrt{36+9+4}$$

$$= \sqrt{49} = 7$$

$\therefore$ The distance between the given points is 7 units.

Example 13.3 : Prove that the points (–2, 4, –3), (4, –3, –2) and (–3, –2, 4) are the vertices of an equilateral triangle.

Solution: Let A(–2, 4, –3), B(4, –3, –2) and C(–3, –2, 4) be the three given points.

Now $|AB| = \sqrt{(4+2)^2 + (-3-4)^2 + (-2+3)^2}$

$$= \sqrt{36+49+1} = \sqrt{86}$$

$$|BC| = \sqrt{(-3-4)^2+(-2+3)^2+(4+2)^2}$$

$$= \sqrt{49+1+36} = \sqrt{86}$$

$$|CA| = \sqrt{(-2+3)^2+(4+2)^2+(-3-4)^2}$$

$$= \sqrt{1+36+49} = \sqrt{86}$$

Since |AB| = |BC| = |CA|, ABC is an equilateral triangle.

Example 13.4 : Verify whether the following points form a triangle or not :

(a) A(−1, 2, 3), B(1, 4, 5) and C(5, 4, 0)

(b) (2, −3, 3), (1, 2, 4) and (3, −8, 2)

Solution: $|AB| = \sqrt{(1+1)^2+(4-2)^2+(5-3)^2}$

$$= \sqrt{2^2+2^2+2^2} = 2\sqrt{3} = 3.464 \text{ (approx.)}$$

$$|BC| = \sqrt{(5-1)^2+(4-4)^2+(0-5)^2}$$

$$= \sqrt{16+0+25} = \sqrt{41} = 6.4 \text{ (approx.)}$$

and $|AC| = \sqrt{(5+1)^2+(4-2)^2+(0-3)^2}$

$$= \sqrt{36+4+9} = 7$$

$\therefore$ |AB| + |BC| = 3.464 + 6.4 = 9.864 > |AC|

|AB| + |AC| > |BC|

and |BC| + |AC| > |AB|, |AB| + |AC| > |BC|

Since sum of any two sides is greater than the third side, therefore the above points form a triangle.

(b) Let the points (2, −3, 3), (1, 2, 4) and (3, −8, 2) be denoted by P, Q and R respectively,

then $|PQ| = \sqrt{(1-2)^2+(2+3)^2+(4-3)^2}$

$$= \sqrt{2+25+1} = 3\sqrt{3}$$

$$|QR| = \sqrt{(3-1)^2 + (-8-2)^2 + (2-4)^2}$$

$$= \sqrt{4+100+4} = 6\sqrt{3}$$

$$|PR| = \sqrt{(3-2)^2 + (-8+3)^2 + (2-3)^2}$$

$$= \sqrt{1+25+1}$$

$$= 3\sqrt{3}$$

In this case $|PQ| + |PR| = 3\sqrt{3} + 3\sqrt{3} = 6\sqrt{3} = |QR|$ Hence the given points do not form a triangle. In fact the points lie on a line.

Example 13.5 : Show that the points A(1, 2, – 2), B(2, 3, –4) C(3, 4, –3) form a right angled triangle.

Solution: $AB^2 = (2 - 1)^2 + (3 - 2)^2 + (-4 + 2)^2 = 6$

$BC^2 = (3 - 2)^2 + (4 - 3)^2 + (-3 + 4)^2 = 3$

and $AC^2 = (3 - 1)^2 + (4 - 2)^2 + (-3 + 2)^2 = 9$

We observe that $AB^2 + BC^2 = 6 + 3 = 9 = AC^2$

$\therefore$ ΔABC is a right angled triangle.

$\therefore$ Hence the given points form a right angled triangle.

Example 13.6 : Prove that the points A(0, 4, 1), B(2, 3, –1), C(4, 5, 0) and D(2, 6, 2) are vertices of a square.

Solution: Here, $AB = \sqrt{(2-0)^2 + (3-4)^2 + (-1-1)^2}$

$$= \sqrt{4+1+4} = 3 \text{ units}$$

$$BC = \sqrt{(4-2)^2 + (5-3)^2 + (0+1)^2}$$

$$= \sqrt{4+4+1} = 3 \text{ units}$$

$$CD = \sqrt{(2-4)^2 + (6-5)^2 + (2-0)^2}$$

$$= \sqrt{4+1+4} = 3 \text{ units}$$

and $\quad DA = \sqrt{(0-2)^2 + (4-6)^2 + (1-2)^2}$

$= \sqrt{4+1+4} = 3$ units

$\therefore$ AB = BC = CD = DA

Now $\quad AC^2 = \sqrt{(4-0)^2 + (5-4)^2 + (0-1)^2}$

$= 16 + 1 + 1 = 18$

$\therefore AB^2 + BC^2 = 9 + 9 = 18 = AC^2$

$\therefore \angle B = 90^0$

$\therefore$ In quadrilateral ABCD, AB = BC = CD = DA; $\angle B = 90^0$

$\therefore$ ABCD is a square.

EXERCISE 13.2

1. Find the distance between the following points :

   (a) (4, 3, −6) and (−2, 1, −3)

   (b) (−3, 1, −2) and (−3, −1, 2)

   (c) (0, 0, 0) and (−1, 1, 1)

2. Show that if the distance between the points (5, −1,7) and (*a*, 5, 1) is 9 units, "a" must be either 2 or 8.

3. Show that the triangle formed by the points (*a*, *b*, *c*) (*b*, *c*, *a*) and (*c*, *a*, *b*) is equilateral.

4. Show that the the points (−1, 0, −4) (0, 1, −6) and (1, 2, −5) form a right angled tringle.

5. Show that the points (0, 7, 10), (−1, 6, 6), (−4, 9, 6)are the vertices of an isosceles right-angled triangle.

6. Show that the points (3, −1, 2), (5, −2, −3), (−2, 4, 1) and (−4, 5, 6) form a parallelogram.

7. Show that the points (2, 2, 2), (−4, 8, 2), (−2, 10, 10) and (4, 4, 10) form a square.

8. Show that in each of the following cases the three points are collinear:

   (a) (−3, 2, 4), (−1, 5, 9), (1, 8, 14)

   (b) (5, 4, 2), (6, 2, −1), (8, −2, −7)

   (c) (2, 5, −4), (1, 4, −3), (4, 7, −6)

13.3 COORDINATES OF A POINT OF DIVISION OF LINE SEGMENT

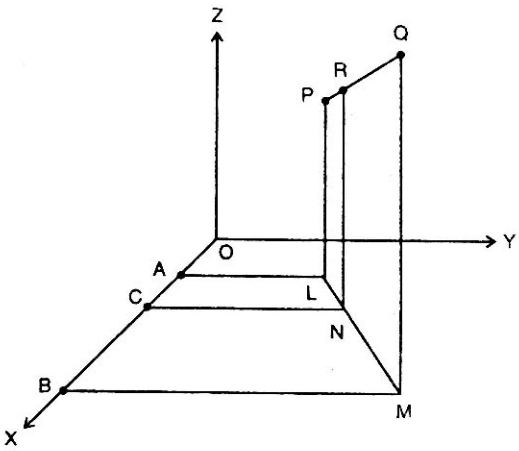


Fig. 14.7

Let the point R (x, y, z) divide PQ in the ratio $l : m$ internally.

Let the co-ordinates of P be (x_1, y_1, z_1) and the co-ordinates of Q be (x_2, y_2, z_2) . From points P, R and Q, draw PL, RN and QM perpendiculars to the XY-plane.

Draw LA, NC and MB perpendiculars to OX

Now, AC = OC − OA = $x - x_1$

and BC = OB − OC = $x_2 - x$

Also we have, $$\frac{AC}{CB} = \frac{LN}{NM} = \frac{PR}{RQ} = \frac{l}{m}$$

i.e., $$\frac{x - x_1}{x_2 - x} = \frac{l}{m}$$

i.e., $mx - mx_1 = lx_2 - lx$

i.e., $(l + m)x = lx_2 + mx_1$

$$\therefore \quad x = \frac{lx_2 + mx_1}{l+m}$$

Similarly, if we draw perpendiculars to OY and OZ respectively,

we get $\quad y = \dfrac{ly_2 + my_1}{l+m}$ and $z = \dfrac{lz_2 + mz_1}{l+m}$

$\therefore$ R is the point $\left(\dfrac{lx_2 + mx_1}{l+m}, \dfrac{ly_2 + my_1}{l+m}, \dfrac{lz_2 + mz_1}{l+m}\right)$

If $\lambda = \dfrac{l}{m}$, then the co-ordinates of the point R which divides PQ in the ratio $\lambda : 1$ are

$$\left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}, \frac{\lambda z_2 + z_1}{\lambda+1}\right); \ \lambda + 1 \neq 0$$

It is clear that to every value of λ, there corresponds a point of the line PQ and to every point R on the line PQ, there corresponds some value of λ. If λ is postive, R lies on the line segment PQ and if λ is negative, R does not lie on line segment PQ.

In the second case you may say the R divides the line segment PQ externally in the ratio $-\lambda$: 1.

Corollary 1 : The co-ordinates of the point dividing PQ externally in the ratio l : m are

$$\left(\frac{lx_2 - mx_1}{l-m}, \frac{ly_2 - my_1}{l-m}, \frac{lz_2 - mz_1}{l-m}\right)$$

Corollary 2 : The co-ordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Example 13.7 : Find the co-ordinates of the point which divides the line segment joining the points (2, −4, 3) and (−4, 5, −6) in the ratio 2 : 1 internally.

Solution : Let A (2, −4, 3), B (−4, 5, −6) be the two points.

Let P(x, y, z) adivides AB in the ratio 2 : 1.

$$\therefore x = \frac{2(-4)+1.2}{2+1} = -2; \; y = \frac{2.5+1(-4)}{2+1} = 2$$

and $$z = \frac{2(-6)+1.3}{2+1} = -3$$

$\therefore$ Thus, the co-ordinates of P are (−2, 2, −3).

Example 13.8 : Find the point which divides the line segment joining the points (−1, −3, 2) and (1, −1, 2) externally in the ratio 2 : 3.

Solution: Let the points (−1, −3, 2) and (1, −1, 2)be denoted by P and Q respectively.

Let R(x, y, z)divide PQ externally in the ratio 2 : 3. Then

$$x = \frac{2.1-3(-1)}{2-3} = -5; \; y = \frac{2(-1)-3(-3)}{2-3} = -7$$

and $$z = \frac{2.2-3.2}{2-3} = 2$$

Thus, the co-ordinates of R are (−5, −7, 2).

Example 13.9 : Find the ratio in which the line segment joining the points (2, −3, 5), (7, 1, 3) is divided by the XY-plane.

Solution: Let the required ratio in which the line segment is divided by l : m.

The co-ordinates of the point are $\left(\frac{7l+2m}{l+m}, \frac{l-3m}{l+m}, \frac{3l+5m}{l+m}\right)$

Since the point lies in the XY-plane, its z-coordinate is zero.

i.e., $\frac{3l+5m}{l+m} = 0$ or

$$\frac{l}{m} = \frac{-5}{3}$$

Hence the XY-plane divides the join of given points in the ratio 5 : 3 externally.

[redacted]

EXERCISE 13.3

1. Find the co-ordinates of the point which divides the line segment joining two points (2, –5, 3) and (–3, 5, –2) internally in the ratio 1 : 4.

2. Find the coordinates of points which divide the join of the points (2, –3, 1) and (3, 4, –5) internally and externally in the ratio 3 : 2.

3. Find the ratio in which the line segment joining the points (2, 4, 5) and (3, 5, –4) is divided by the YZ-plane.

4. Show that the YZ-plane divides the line segment joining the points (3, 5, –7) and (–2, 1, 8) in the ration 3 : 2 at the point $\left(0, \frac{13}{5}, 2\right)$.

5. Show that the ratios in which the co-ordinate planes divide the join of the points (–2, 4, 7), (3, –5, 8) are 2 : 3, 4 : 5 (internally) and 7 : 8 (externally).

6. Find the co-ordinates of a point R which divides the line segment $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio 2 : 1. Verify that Q is the mid-point of PR.

13.4 ANGLE BETWEEN TWO LINES

You are already famililar with the concept of the angle between twc lines in plane geometry. We will extenc this idea to the lines in space.

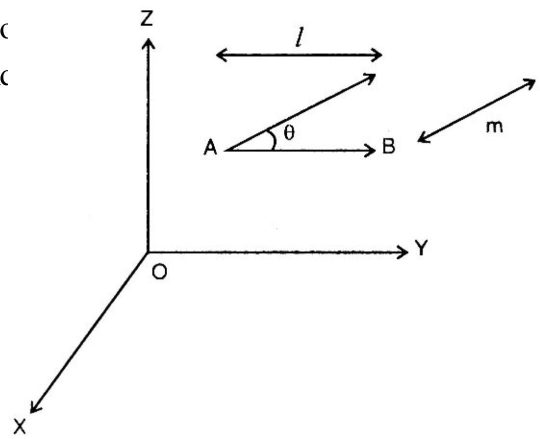


Fig. 13.8

Let there be two lines in space, intersecting or non-intersecting. We consider a point A in space and through it, we draw lines parallel to the given lines in space. The angle between these two lines drawn parallel to the given lines is defined as the angle between the two lines in space.

You may see in the adjointing figure, that T is the angle between the lines l and m.

13.5 DIRECTION COSINES OF A LINE

If α, β and γ are the angles which a line AB makes with the positive directions of X, Y and Z axes respectively, then cos α,cos β and cos γ are called the direction cosines of the line AB and are usually denoted by the letters l, m and n respectively. In other words l = cos α, m = cos β and n = cos γ. You may easily see that the direction cosines of the X-axis are 1, 0, 0, because the line coincides with the X axis and is perpendicular to Y and Z axes since cos $0^0 = 1$, cos$90^0 = 0$. Similarly direction cosines of Y and Z axes are 0, 1, 0 and 0, 0, 1 respectively.

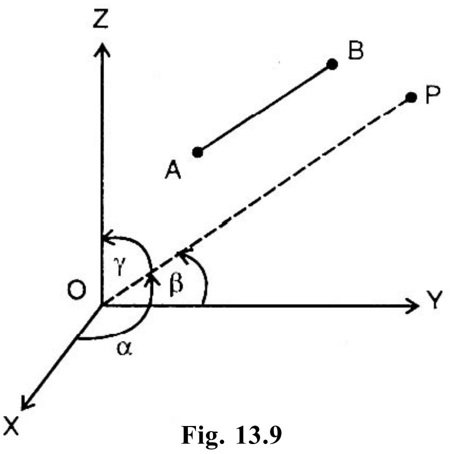


Fig. 13.9

13.5.1 RELATION BETWEEN DRECTION COSINES

Let OP be a line with direction cosines cos α, cos β and cos γ i.e. l, m and n.

Again since each of $\angle$OLP, $\angle$OMP and $\angle$ONP is a right angle.

We have, $\dfrac{OL}{OP} = \cos\alpha = l$

$$\frac{OM}{OP} = \cos\beta = m$$

and $\dfrac{ON}{OP} = \cos\gamma = n$

$\therefore$ l. OP = OL ; m . OP = O

and n . OP = ON

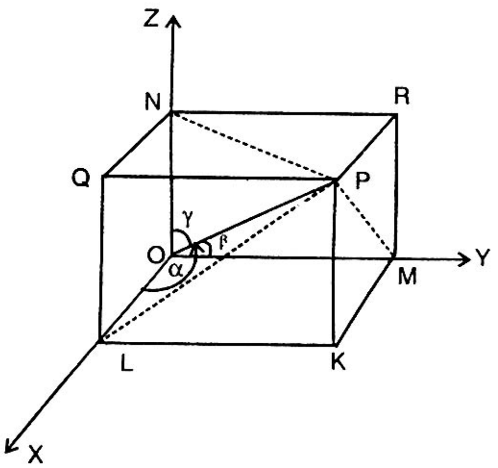


Fig. 13.10

$\therefore \quad OP^2(l^2 + m^2 + n^2)$

$= OL^2 + OM^2 + ON^2 = OP^2$

or $\quad l^2 + m^2 + n^2 = 1$

or $\quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

This is the relation between the direction cosines of a line.

Corollary 1 : Any three numbers *a*, *b* and *c* which are proportional to the direction cosines *l*, *m* and *n* respectively of a given line are called the direction ratios or direction numbers of the given line. If *a*, *b* and *c* are direction numbers and *l*, *m* and *n* are direction cosines of a line, then *l*, *m* and *n* are found in terms of *a*, *b* and *c* as follows :

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}} = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore\ l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}\ ;\ m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}};\ n = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$$

where the same sign either positive or negative is to be taken throughout.

13.6 PROJECTION

Suppose you are standing under the shade of a tree. At a time when the sun is vertically above the tree, its shadow falling on the ground is taken as the projection of the tree on the ground at that instant.

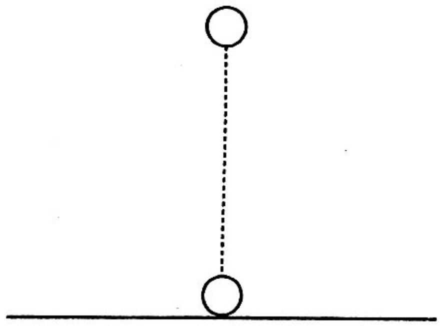


Fig. 14.11

This is called projection because the rays falling vertically on the tree create the image of the each point of the tree constituting its shadow (image). Recall the example of a bouncing ball. When the ball falling freely from a point in space strikes the ground, the point where the ball strikes the ground is the projection of the point in space on the ground.

13.6.1 Projection of a Point and of a Line Segment

The projection of a point on a plane can be taken as the foot of the perpendicular drawn from the point to the plane. Similarly, the line segment obtained by joining the feet of the perpendiculars in the plane drawn from the end points of a line segment is called the projection of the line segment on the plane.

We may similarly define the projection of a point and of a line segment on a given line.

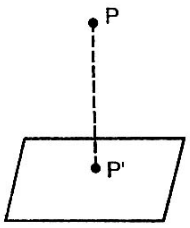


Fig. 13.12

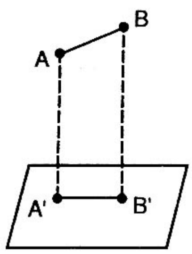


Fig. 13.13

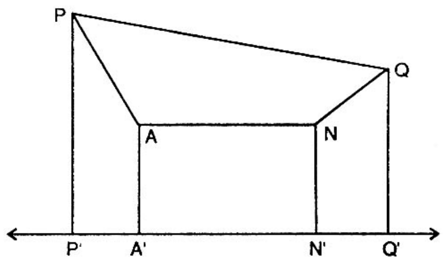


Fig. 13.14

Note : Projection of a line segment PQ on a line is equal to the sum of the projections of the broken line segments i.e., Projections of PQ = Sum of the projections of PA, AN and NQ.

13.6.2 Projection of a Line Segment Joning Two Points on a Line

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points.

To find the projection of PQ on a line with direction cosines l, m and n, through P and Q draw planes parallel to the co-ordinates planes to form a reactangular paralleloppied whose diagonal is PQ.

Now $PA = x_2 - x_1$; $AN = y_2 - y_1$;

$NQ = z_2 - z_1$,

The lines PA, AN and NQ are parallel to X-axis, Y-axis and Z-axis respectively.

Therefore, their respective projections on the line with direction cosines l, m and n are $(x_2 - x_1)l$, $(y_2 - y_1)m$ and $(z_2 - z_1)n$.

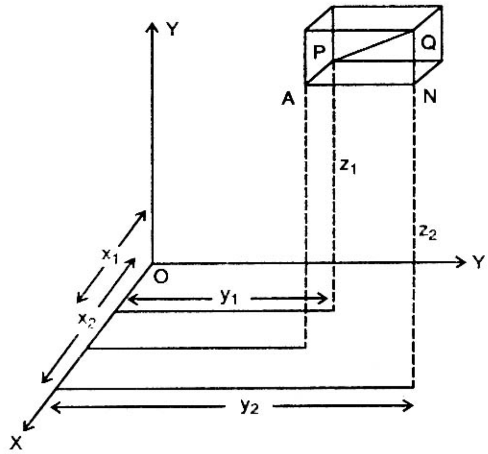


Fig. 13.15

Recall that projection of PQ on any line is equal to the sum of the projections of PA, AN and NQ on the line, therefore the required projection is

$$(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n.$$

13.7 DIRECTION COSINES OF THE LINE JOINING TWO POINTS

Let L and M be the feet of the perpendiculars drawn from $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on the X-axis respectively,

So that $OL = x_1$ and $OM = x_2$

Projection of PQ on X-axis

$$= LM = OM - OL$$

$$= x_2 - x_1$$

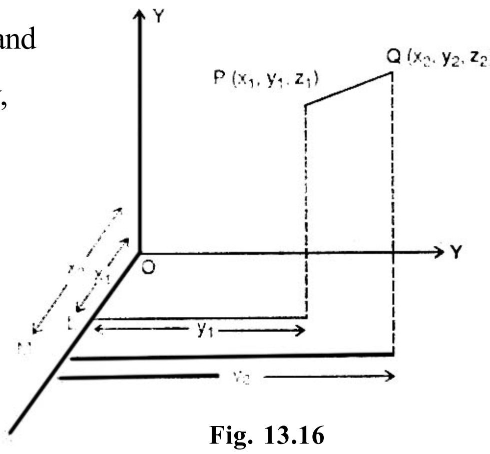


Fig. 13.16

Also, if l, m and n are the direction cosines of PQ, the projection of PQ on X-axis = l. PQ

$$\therefore \quad l \, . \, PQ = x_2 - x_1$$

Similarly by taking projection of PQ on Y-axis and Z-axis respectively,

we get, $m\ .\ PQ = y_2 - y_1$ and $n\ .\ PQ = z_2 - z_1$

$$\therefore\ \frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = PQ$$

Thus, the direction cosines of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$.

Example 13.10 : Find the direction cosines of a line that makes equal angles with the axes.

Solution: Here $\alpha = \beta = \gamma$, We have, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\therefore\ 3\cos^2\alpha = 1 \text{ or } \cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Hence the required direction cosines are

$$\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$$

same sign (positive or negative) to be taken throughout.

Example 13.11 : Verify whether it is possible for a line to make the angles 30^0, 45^0 and 60^0 with the co-ordinate axes or not ?

Solution: Let the line make angles α, β and γ with the co-ordinate axes

$\alpha = 30^0$; $\beta = 45^0$; $\gamma = 60^0$

Since the relation between direction cosines is $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we have $\cos^2 30^0 + \cos^2 45^0 + \cos^2 60^0$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{4} + \frac{1}{2} + \frac{1}{4} = \frac{6}{4} > 1$$

In view of the above identity, it is not possible for a line to make the given angles with the coordinate axes.

Example 13.12 : If 6, 2 and 3 are direction ratios of a line, find its direction cosines.

Solution : Let l, m and n be the direction cosines of the line.

$$\therefore \quad l = \pm\frac{6}{\sqrt{6^2+2^2+3^2}} = \pm\frac{6}{7}; \quad m = \pm\frac{2}{\sqrt{6^2+2^2+3^2}} = \pm\frac{2}{7};$$

and $$n = \pm\frac{3}{\sqrt{6^2+2^2+3^2}} = \pm\frac{3}{7}$$

Hence the required direction cosines are $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$ or $\frac{-6}{7}, \frac{-2}{7}, \frac{-3}{7}$

Example 13.13 : Find the projections (feet of the perpendiculars) of the point (2, 1, −3) on the (a) Co-ordinate planes (b) Co-ordinate axes.

Solution : (a) The projections of the point on the co-ordinate planes YZ, ZX and XY are (0, 1, −3), (2, 0, −3) and (2, 1, 0) respectively.

(b) The projections on the co-ordinate axes are (2, 0, 0), (0, 1, 0) and (0, 0, −3) respectively.

Example 13.14: Find the direction cosines of the line-segment joining the points (2, 5, −4) and (8, 2, −6).

Solution: Let l, m and n be the direction cosines of the line joining the two given points (2, 5, −4) and (8, 2, −6).

Then the direction cosines are proportional to 8 −2, 2 −5 and −6 + 4

i.e., 6, −3, −2 are direction ratios of the line.

$\therefore$ The required direction cosines of the line are $\frac{6}{7}, \frac{-3}{7}, \frac{-2}{7}$ or $\frac{-6}{7}, \frac{3}{7}, \frac{2}{7}$.

Since $$\sqrt{6^2+(-3)^2+(-2)^2} = \pm 7$$

Example 13.15 : Find the projection of the line segment joining the points (3, 3, 5) and (5, 4, 3) on the line joining the points (2, −1, 4) and (0, 1, 5).

Solution: The direction cosines of the line joining the points (2, −1, 4) and (0, 1, 5) are

$$\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}$$

because $\sqrt{(-2)^2+2^2+1^2} = 3$

Thus, the projection of this line segment on the given line is

$$(5-3)\left(-\frac{2}{3}\right)+(4-3)\left(\frac{2}{3}\right)+(3-5)\left(\frac{1}{3}\right)=-\frac{4}{3}$$

Hence the required projection is $\frac{4}{3}$ because the projection is the length of a line segment which is always taken as positive.

EXERCISE 13.4

1. Find the direction cosines of the line having direction ratios

   (a) 3, −1, 2 (b) 1, 1, 1

2. Find the projections of the point (–3, 5, 6) on the

   (a) Co-ordinate palnes (b) Co-ordinate axes

3. Find the direction cosines of the line segment joining the points.

   (i) (5, −3, 8) and (6, −1, 6) (ii) (4, 3, −5) and (−2, 1, −8)

4. Find the projection of a line segment joining the points P (4,–2, 5) and Q (2, 1, −3) on the line with direction ratios 6, 2 and 3.

5. Find the projection of a line segment joining the points (2, 1, 3) and (1, 0, −4) on the line joining the points (2, −1, 4) and (0, 1, 5).

13.8 ANGLE BETWEEN TWO LINES WITH GIVEN DIRECTION COSINES

Let OP and OQ be the two lines through the origin O parallel to two lines in space whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively.

Let θ be the angle between OP an OQ and let the co-ordinates of P be (x_1, y_1, z_1).

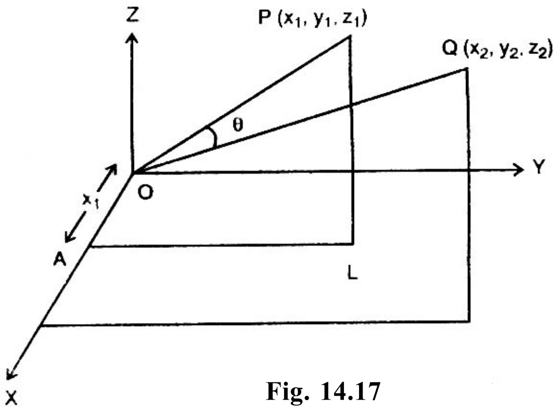


Fig. 14.17

Draw PL perpendicular to XY-plane and LA perpendicular to X-axis. Then the projection of OP on OQ = Sum of projections of OA, AL and LP on OQ.

i.e., $\quad OP \cos\theta = x_1 l_2 + y_1 m_2 + z_1 n_2$ ...(i)

But x_1 = projection of OP om X-axis = OP. l_1

Similarly, $y_1 = OP \, . \, m_1$ and $z_1 = OP \, . \, n_1$ ...(ii)

Thus, we get, $OP \cos\theta = OP\,(l_1 l_2 + m_1 m_2 + n_1 n_2)$ ...[From (i) and (ii)]

giving $\quad l_1 l_2 + m_1 m_2 + n_1 n_2 = \cos\theta$ ...(iii)

Corollary 1 : If the direction ratio of the lines are (a_1, b_1, c_1) and (a_2, b_2, c_2) then the angle θ between the two lines is given by

$$\cos\theta = \pm\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,.\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Here positive or negative sign is to be taken depending upon θ being acute or obtuse.

Corollary 2 : If OP an OQ are perpendicular to each other, (i.e., if $\theta = 90^0$ then

$$l_1l_2 + m_1m_2 + n_1n_2 = \cos 90^0$$

$$= 0$$

Corollary 3 : If OP and OQ are parallel, then $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

Since OP || OQ and O is a common point, OP lies on OQ.

Hence $\sin\theta = 0$

Now $\sin^2\theta = 1 - \cos^2\theta = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2$

$$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$$

$$= (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - l_1n_2)^2$$

and hence $l_1m_2 - l_2m_1 = 0; m_1n_2 - m_2n_1 = 0; n_1l_2 - n_1l_2 = 0$

These gives $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$.

Remarks

(a) Two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are

(i) perpendicular if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(ii) parallel if $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

(b) The condition of perpendicularty of two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(Hint: $l_1 = \dfrac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, m_1 = \dfrac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, n_1 = \dfrac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$).

(c) The condition of parallelism of two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Example 13.16 : Find the angle between the two lines whose direction ratios are $-1, 0, 1$ and $0, -1, 1$.

Solution: Let θ be the angle between the given lines.

$$\cos\theta = \pm\frac{(-1).0+0(-1)+1.1}{\sqrt{(-1)^2+0^2+1^2}.\sqrt{0^2+(-1)^2+1^2}} = \left(\pm\frac{1}{2}\right)$$

$\therefore \quad \theta = 60^0$ or 120^0.

Example 13.17 : Find the acute angle between the lines whose direction ratios are $5, -12, 13$ and $-3, 4, 5$.

Solution: Let θ be the angle between the two given lines, then

$$\cos\theta = \pm\frac{5(-3)+(-12).4+(13).5}{\sqrt{5^2+(-12)^2+13^3}.\sqrt{(-3)^2+4^2+5^2}}$$

$$= \pm\frac{-15-48+65}{\sqrt{338}\ \sqrt{50}}$$

$$= \pm\frac{2}{13\sqrt{2}.5\sqrt{2}} = \pm\frac{1}{65}$$

Since, θ is acute it is given by $\cos\theta = \frac{1}{65}$

$$\therefore\ \theta = \cos^{-1}\left(\frac{1}{65}\right).$$

EXERCISE 13.5

1. Find the angle between the lines whose direction ratios are $1, 1, 2$ and $\sqrt{3}-1, -\sqrt{3}-1, 4$.

2. Show that the points A(7, 6, 3), B(4, 10, 1), C(–2, 6, 2) and D(1, 2, 4) vertices of a reactangle.

3. By calculating the angle of the triangle with vertices (4, 5, 0), (2, 6, 2) and (2, 3, −1) show that it is an isosceles triangle.

4. Find whether the pair of lines with given direction cosines are parallel or perpendicular

(a) $\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}; \frac{-2}{3}, \frac{2}{3}, \frac{-1}{3}$

(b) $\frac{3}{5}, \frac{4}{5}, \frac{0}{5}; \frac{-4}{5}, \frac{3}{5}, \frac{0}{5}$.

KEY WORDS

- For a given point P(x, y, z) in space with reference to reactangular co-ordinate axes, if we draw three planes parallel to the three co-ordinate planes to meet the axes (in A, B and C say), then

  OA = x, OB = y and OC = z where O is the origin.

  Conversely, given any three numbers, x, y and z we can find a unique point in space whose co-ordinates are (x, y, z) .

- P(x_1, y_1, z_1), Q(x_2, y_2, z_2) is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

  In particular the distance of P from the origin O is $\sqrt{x_1^2 + y_1^2 + z_1^2}$

- The co-ordinates of the point which divides the line segment joining two points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) in the ratio $l : m$.

  (a) internally are $\left(\frac{lx_2 + mx_1}{l+m}, \frac{ly_2 + my_1}{l+m}, \frac{lz_2 + mz_1}{l+m}\right)$

(b) externally are $\left(\frac{lx_2 - mx_1}{l-m}, \frac{ly_2 - my_1}{l-m}, \frac{lz_2 - mz_1}{l-m}\right)$

In particular, the co-ordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

- If l, m and n are the direction cosines of the line, then

  $l^2 + m^2 + n^2 = 1$

- The three numbers which are proportional to the direction cosines of a given line are called its direction ratios.

- Direction cosines of the line joining two points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$.

- The projection of the line segment joining the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) on a line with direction cosines l, m and n is $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

- The direction cosines l, m and n of the line joining the points P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) are given by

$$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = \text{PQ}$$

- The angle θ between two lines whose direction cosines are l_1, m_1, n_1; l_2, m_2, n_2 is given by $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

  If the lines are

  (a) perpindicular to each other then,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

(b) parallel to each other then $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines, then the angle θ between them is given by

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ .\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

- The lines will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

SUPPORTIVE WEBSITES

http : //www.wikipedia.org

http ://www. mathworld.wolfram.com

PRACTICE EXERCISE

1. Show that the points (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) form an isosceles right-angled triangle.

2. Prove that the points P, Q and R, whose co-ordinates are respectively (3, 2, −4), (5, 4, −6) and (9, 8, −10) are collinear and find the ratio in which Q divides PR.

3. A(3, 2, 0), B(5, 3, 2), C(−9, 6, −3) are three points forming a triangle. AD, the bisector of the angle $\angle BAC$ meets BC at D. Find the co-ordinates of D.

   (Hint : D divides BC in the ratio AB : AC)

4. Find the direction cosines of the line joining the point (7, −5, 4) and (5, −3, 8).

5. What are the direction cosines of a line equally inclined to the axes ? How many such lines are there ?

6. Determine whether it is possible for a line to make the angle 45^{0}, 60^{0} and 120° with the co-ordinate axes.

7. Show that the points (0, 4, 1), (2, 3, −1), (4, 5, 0) and (2, 6, 2) are the vertices of a square.

8. Show that the points (4, 7, 8), (2, 3, 4), (−1, −2, 1) and (1, 2, 5) are the vertices of a parallelogram.

9. A(6, 3, 2), B(5, 1, 4), C(3, −4, 7) and D(0, 2, 5) are four points. Find the projections of (i) AB on CD, and (ii) CD on AB.

10. Three vertices of a parallelogram ABCD are A(3, −4, 7), B(5, 3, −2), C(1, 2, −3). Find the fourth vertex D.

ANSWERS

EXERCISE 13.1

1. (a) OX'YZ (b) OXYZ'

   (c) OX'YZ' (d) OXY'Z

   (e) OXYZ'

EXERCISE 13.2

1. (a) 7 (b) $2\sqrt{5}$ (c) $\sqrt{3}$

EXERCISE 13.3

1. $(1, -3, 2)$

2. $\left(\frac{13}{5}, \frac{6}{5}, \frac{-13}{5}\right)$; $(5, 18, -17)$

3. $-2 : 3$

6. $(2x_2 - x_1, 2y_2 - y_1, 2z_2 - z_1)$

EXERCISE 13.4

1. (a) $\pm\frac{3}{\sqrt{14}}, \pm\frac{1}{\sqrt{14}}, \pm\frac{2}{\sqrt{14}}$

   (b) $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$

2. (a) $(0, 5, 6), (-3, 0, 6), (-3, 5, 0)$

   (b) $(-3, 0, 0), (0, 5, 0), (0, 0, 6)$

3. (a) $\pm\frac{1}{3}, \pm\frac{2}{3}, \mp\frac{2}{3}$ (b) $\pm\frac{6}{7}, \pm\frac{2}{7}, \pm\frac{3}{7}$

4. $\frac{30}{7}$ 5. $\frac{7}{3}$

EXERCISE 13.5

1. $\frac{\pi}{3}$

4. (a) Parallel

   (b) Perpendicular

PRACTICE EXERCISE

1. 1 : 2

3. $\left(\frac{38}{16}, \frac{57}{16}, \frac{17}{16}\right)$

4. $\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$ or $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

5. $\pm\frac{1}{3}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$; 4

6. Yes.

9. (i) $\frac{13}{7}$ (ii) $\frac{13}{3}$

10. $(-1, -5, 6)$

Chapter 14

THE PLANE

OBJECTIVES

After studying this lesson, you will be able to :

- identify a plane;
- establish the general equation of a plane;
- find the general equation of a plane passing through a given point;
- find the equation of a plane passing through three given points;
- find the equation of a plane in the intercept form and normal form;
- find the angle between two given planes;
- find the equation of a plane bisecting the angle between two given planes; and
- show that the homogeneous equation of second degree in three variables represents a pair of planes.

PREREQUISITES

- Basic knowledge of three dimensional geometry.
- Direction cosines and direction ratio of a line.
- Projection of a line segment on another line.
- Condition of perpendicularity and parallelism of two lines in space.

INTRODUCTION

Look closely at a room in your house. It has four walls, a roof and a floor. The floor and roof are parts of two parallel planes extending infinitely beyond the boundary. You will also see two pairs of parallel walls which are also parts of parallel planes. Similarly, the tops of tables, doors of rooms etc. are examples of parts of planes.

If we consider any two points in a plane, the line joining these points will lie entirely in the same plane. This is the characteristic of a plane.

Look at Fig.14.1.You know that it is a representation of a rectangular box. This has six faces, eight vertices and twelve edges.

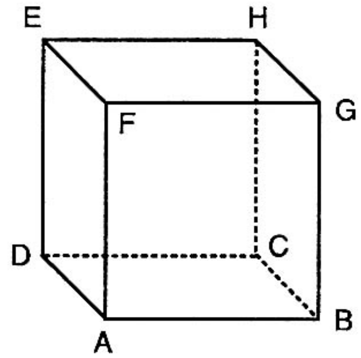


Fig. 14.1

The pairs of opposite and parallel faces are

(i) ABCD; FGHE

(ii) AFED; BGHC

(iii) ABGF; DCHE

and the sets of parallel edges are given below

(i) AB, DC, EH, FG

(ii) AD, BC, GH, FE

(iii) AF, BG, CH, DE

Each of the six faces given above forms a part of the plane, and there are three pairs of parallel planes, denoted by the opposite faces.

In this lesson, we shall establish the general equation of a plane, the equation of a plane passing through three given points, the intercept form of the equation of a plane and the normal form of the equation of a plane. We shall show that a homogeneous equation of second degree in three variables x,y and z represents a pair of planes. We shall also find the equation of a plane bisecting the angle between two planes and area of a triangle in space.

14.1 GENERAL EQUATION OF A PLANE

Recall that a plane is a surface such that if any two points be taken on it, the line joining these two points lies wholly in the plane

Consider the general equation of first degree in x, y and z

$$ax + by + cz + d = 0 \qquad \text{... (i)}$$

If points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ satisfy the equation (i), then

$$ax_1 + by_1 + cz_1 + d = 0 \qquad \text{.... (ii)}$$

$$ax_2 + by_2 + cz_2 + d = 0 \qquad \text{.... (iii)}$$

Multiplying (ii) by n and (iii) by m and adding the results,

we get, $a(mx_2 + nx_1) + b(my_2 + ny_1) + c(mz_2 + nz_1) + d(m + n) = 0$

$$\Leftrightarrow \qquad a\left(\frac{mx_2 + nx_1}{m+n}\right) + b\left(\frac{my_2 + ny_1}{m+n}\right) + c\left(\frac{mz_2 + nz_1}{m+n}\right) + d = 0 \quad \text{.... (iv)}$$

The result (iv) shows that the co-ordinates of the point dividing PQ in the ratio $m : n$ satisfy the equation (i).

$\therefore$ All points lying on the line through P and Q lie on the surface represented by (i)

$\therefore$ The equation (i) represents the general equation of a plane.

14.2 GENERAL EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT

Let $P(x_1, y_1, z_1)$ be a given point and ax + by + cz + d = 0 be the given equation of the plane.

Since the plane $ax + by + cz + d = 0 \qquad \text{...(i)}$

passes through the point (x_1, y_1, z_1), it implies that

$$ax_1 + by_1 + cz_1 + d = 0 \qquad \text{.... (ii)}$$

Subtracting (ii) from (i), we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \qquad \text{.... (iii)}$$

which is the required equation of a plane passing through the point (x_1, y_1, z_1).

Remark : Can you say, how many planes are represented by (iii) ?

As a, b and c can take any real value, there are infinitely many planes passing through a given point.

Note :In equation (i), there are four constants a, b, c and d. When $d \neq 0$, we can divide (i) by d to get

$$\left(\frac{a}{d}\right)x + \left(\frac{b}{d}\right)y + \left(\frac{c}{d}\right)z + 1 = 0$$

or, $$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z + 1 = 0$$

where $\mathrm{A} = \left(\frac{a}{d}\right) \mathrm{B} = \left(\frac{b}{d}\right)$ and $\mathrm{C} = \left(\frac{c}{d}\right)$

You may note that in (iv), A,B and C are three independent constants.

Example 14.1 : Find the equation of a plane passing through the point (1, −3, 2).

Solution: Here $x_1 = 1$ $y_1 = -3$ and $z_1 = 2$

$\therefore$ The required equation of the plane is

$$a(x - 1) + b(y + 3) + c(z - 2) = 0$$

or $$ax + by + cz + (-a + 3b - 2c) = 0$$

Example 14.2 : Find the equation of a plane passing through the point (3, 0, −2).

Solution : Here $x_1 = 3$ $y_1 = 0$ and $z_1 = -2$

$\therefore$ The required equation of the plane is

$$a(x-3)+b(y-0)+c(z+2)=0$$

or $$ax+by+cz+(-3a+2c)=0$$

Example 14.3 : Find the equation of a plane passing through a point which divides the line joining the points (2, 2, 4) and (5, 2, 1) in the ratio of 1 : 2 internally.

Solution :

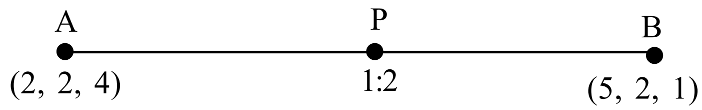


Fig. 14.2

The co-ordinates of P, which divides the join of A and B in the ratio of 1:2 internally are

$$\left(\frac{2\times2+1\times5}{2+1},\frac{2\times2+1\times2}{2+1},\frac{2\times4+1\times1}{2+1}\right)$$

or (3, 2, 3)

$\therefore$ The required equation of the plane is

$$a(x-3)+b(y-2)+c(z-3)=0$$

or $$ax+by+cz+(-3a-2b-3c)=0$$

EXERCISE 14.1

1. Find the equation of a plane passing through the origin.
2. Find the equation of a plane passing through the point (0, 0, –2).
3. Find the equation of a plane passing through the point (5, –7, 3).
4. Find the equation of a plane which bisects the line segement joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) .
5. Find the equation of a plane which divides the line joining the points (1, 2, –3), (4, 2, –3) in the ratio of 1:2 internally.

14.3 EQUATION OF A PLANE PASSING THROUGH THREE GIVEN POINTS

`You may recall that the general equation of a plane contains only three independent constants. Hence, a plane can be uniquely determined if it is given to pass through three given non-collinear points.

Let $ax + by + cz + d = 0$ be the equation of the plane and (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be three given points.

The equation of the plane passing through the point (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \qquad \text{.... (i)}$$

If (i) passes through the points (x_2, y_2, z_2) and (x_3, y_3, z_3) then

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \qquad \text{.... (ii)}$$

and $$a(x_3 - x_1) + b(y_3 - y_1) + c(z_3 - z_1) = 0 \qquad \text{.... (iii)}$$

Eliminating a,b and c from (i), (ii) and (iii), we get the required equation of the plane as

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \qquad \text{...(A)}$$

14.4 EQUATION OF A PLANE IN THE INTERCEPT FORM

Let a, b, c be the lengths of the intercepts made by the plane on the x, y and z axes respectively.

It implies that the plane passes through the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$

Putting $x_1 = a$ $\quad y_1 = 0$ $\quad z_1 = 0$

$x_2 = 0$ $\quad y_2 = b$ $\quad z_2 = 0$

and $x_3 = 0$ $\quad y_3 = 0$ $\quad z_3 = c$ in (A),

we get the required equation of the plane as

$$\begin{vmatrix} x-a & y-0 & z-0 \\ 0-a & b-0 & 0-0 \\ 0-a & 0-0 & c-0 \end{vmatrix} = 0$$

which on expanding gives $bcx + acy + abz - abc = 0$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Equation (B) is called the ***Intercept form*** of the equation of the plane.

Example 14.4 : Find the equation of the plane passing through the points (0, 2, 3), (2, 0, 3) and (2, 3, 0).

Solution: Using (A), we can write the equation of the plane as

$$\begin{vmatrix} x-0 & y-2 & z-3 \\ 2-0 & 0-2 & 3-3 \\ 2-0 & 3-2 & 0-3 \end{vmatrix} = 0$$

or $$\begin{vmatrix} x & y-2 & z-3 \\ 2 & -2 & 0 \\ 2 & 1 & -3 \end{vmatrix} = 0$$

or $x(6 - 0) - (y - 2)(-6 - 0) + (z - 3)(2 + 4) = 0$

or $6x + 6(y - 2) + 6(z - 3) = 0$

i.e., $x + y - 2 + z - 3 = 0$

or $x + y + z = 5.$

Example 14.5 : Show that the equation of the plane passing through the points (2, 2, 0), (2,0, 2) and (4, 3, 1) is $x = y + z$.

Solution: Equation of the plane passing through the point (2,2,0) is

$a(x - 2) + b(y - 2) + c(z - 0) = 0$ ... (i)

$\because$ (i) passes through the point (2, 0, 2)

$\therefore$ $a(2 - 2) + b(0 - 2) + 2c = 0$

or $c = b$ ... (ii)

Again (i) passes through the point (4, 3, 1)

$\therefore \quad a(4-2)+b(3-2)+c=0$

or $\quad 2a+b+c=0 \quad$ ... (iii)

From (ii) and (iii), we get $\quad 2a+b+c=0$ or $a=-b$

$\therefore$ (i) becomes

$$-b(x-2)+b(y-2)+bz=0$$

or $\quad -(x-2)+y-2+z=0$

or $\quad y+z-x=0$

or $\quad x=y+z.$

Example 14.6 : Reduce the equation of the plane $4x-5y+6z-60=0$ to the intercept form. Find its intercepts on the co-ordinate axes.

Solution : The equation of the plane is

$$4x-5y+6z-60=0 \quad \text{or} \quad 4x-5y+6z=60 \qquad \text{...(i)}$$

The equation (i) can be written as

$$\frac{4x}{60}-\frac{5y}{60}+\frac{6z}{60}=1 \qquad \text{or} \qquad \frac{x}{15}+\frac{y}{(-12)}+\frac{z}{10}=1$$

which is the interecept form of the equation of the plane and the intercepts on the co-ordinate axes are 15, -12 and 10 respectively..

EXERCISE 14.2

1. Find the equation of the plane passing through the points

   (a) $(2, 2, -1)$, $(3, 4, 2)$ and $(7, 0, 6)$

   (b) $(2, 3, -3)$, $(1, 1, -2)$ and $(-1, 1, 4)$

   (c) $(2, 2, 2)$, $(3, 1, 1)$ and $(6, -4, -6)$

2. Show that the equation of the plane passing through the points $(3, 3, 1)$, $(-3, 2, -1)$ and $(8, 6, 3)$ is $4x+2y-13z=5$.

3. Find the equation of a plane whose intercepts on the coordinate axes are 2,3 and 4 respectively.

4. Find the intercepts made by the plane $2x + 3y + 4z = 24$ on the co-ordinate axes.

5. Show that the points $(-1, 4, -3)$, $(3, 2, -5)$, $(-3, 8, -5)$ and $(-3, 2, 1)$ are coplanar.

14.5 EQUATION OF A PLANE IN THE NORMAL FORM

Let ON be perpendicular from the origin O to the plane ABC and let P (x, y, z) be any point on the plane. Let PL be drawn perpendicular to XY plane and LM be perpendicular to OX so that OM = x, ML = y and PL = z. The sum of the projections of OM, ML and LP on ON is equal to the projection of OP on ON.

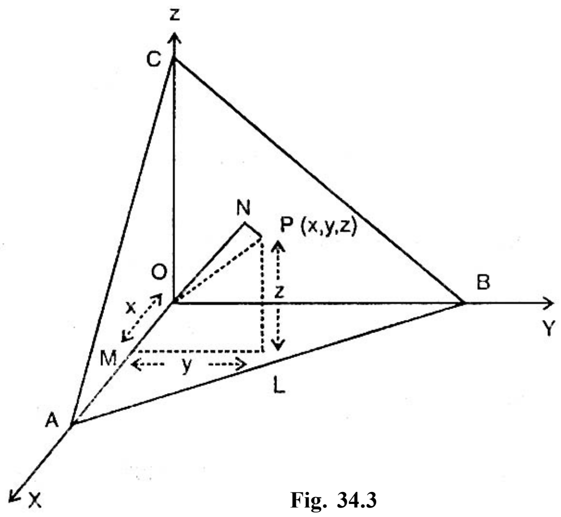


Fig. 34.3

If l, m and n are the direction cosines of ON and p is the length of the perpendicular ON, then $lx + my + nz = p$ ....(i)

(i) is called the normal form of the equation of the plane.

We know that $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$, where α, β, and γ are angles made by ON with the positive directions of x, y and z axes respectively.

$\therefore$ (i) can alternatively be written as

$$x \cos\alpha + y \cos\beta + z \cos\gamma = p$$

Corollary 1 : By comparing the general equation of first degree in x, y, z, i.e., $ax + by + cz + d = 0$ with $lx + my + nz = p$, we see the direction cosines of normal to the plane $ax + by + cz + d = 0$ are proportional to a, b and c and are equal to

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{b}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

the positive or negative sign to be taken according as d is positive or negative as p, by convention, is taken to be positive.

Also, the length of the perpendicular from the origin to the plane is

$$\frac{|-d|}{\sqrt{a^2+b^2+c^2}}.$$

Corollary 2 : The general equation of a plane ax + by + cz + d = 0 is reduced to normal form by dividing each term by $-\sqrt{a^2+b^2+c^2}$ or, $\sqrt{a^2+b^2+c^2}$ according as d is positive or negative.

Example 14.7 : Reduce each of the following equations of the plane to the normal form :

(i) $2x - 3y + 4z - 5 = 0$, (ii) $2x + 6y - 3z + 5 = 0$

Find the length of perpendicular from origin upon the plane in both the cases.

Solution: (i) The equation of the plane is $2x - 3y + 4z - 5 = 0$

Dividing (A) by $\sqrt{2^2+(-3)^2+4^2}$ or, by $\sqrt{29}$

we get, $$\frac{2x}{\sqrt{29}}-\frac{3y}{\sqrt{29}}+\frac{4z}{\sqrt{29}}-\frac{5}{\sqrt{29}}=0$$

or $$\frac{2x}{\sqrt{29}}-\frac{3y}{\sqrt{29}}+\frac{4z}{\sqrt{29}}=\frac{5}{\sqrt{29}}$$

which is the equation of the plane in the normal form.

$\therefore$ Length of the perpendicular is $\dfrac{5}{\sqrt{29}}$

(ii) The equation of the plane is $2x + 6y - 3z + 5 = 0$ ...(B)

Dividing (B) by $\sqrt{2^2+6^2+(-3)^2}$

or by -7 we get, [refer to corollary 2]

$$-\frac{2x}{7}-\frac{6y}{7}+\frac{3z}{7}-\frac{5}{7}=0 \quad \text{or} \quad -\frac{2x}{7}-\frac{6y}{7}+\frac{3z}{7}=\frac{5}{7}$$

which is the required equation of the plane in the normal form.

$\therefore$ Length of the perpendicular from the origin upon the plane is $\frac{5}{7}$.

Example 14.8 : The foot of the perpendicular drawn from the origin to the plane is (4, –2, –5) Find the equation of the plane.

Solution : Let P be the foot of perpendicular drawn from origin O to the plane.

Then P is the point (4, –2, –5)

The equation of a plane through the point P(4, –2, –5) is

$$a(x-4)+b(y+2)+c(z+5)=0$$

Now OP $\perp$ plane and direction cosines of OP are proportional to

$$4-0,\ -2-0,\ -5-0$$

i.e., $\quad 4, -2, -5$

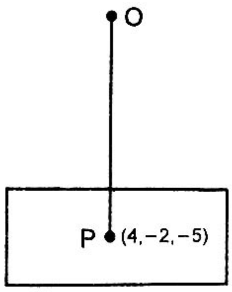


Fig. 14.4

Substituting 4, 2 and 5 for a, b and c in (i), we get

$$4(x-4)-2(y+2)-5(z+5)=0$$

i.e., $\quad 4x-16-2y-4-5z-25=0$

i.e., $\quad 4x-2y-5z=45$

which is the required equation of the plane.

EXERCISE 14.3

1. Reduce each of the following equations of the plane to the normal form:

(i) $4x+12y-6z-28=0$ (ii) $3y+4z+3=0$

2. The foot of the perpendicular drawn from the origin to a plane is the point (1, –3,1) . Find the equation of the plane.

3. The foot of the perpendicular drawn from the origin to a plane is the point (1, –2,1). Find the equation of the plane.

4. A plane meets the co-ordinate axes in points A,B and C such that the centroid of the triangle ABC is the point (a, b, c). Prove that the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$.

5. plane meets the coordinate axes at the points P, Q and R and the centroid of ΔPQR is the point $-3,4,6$. Find the equation of the plane.

14.6 ANGLE BETWEEN TWO PLANES

Let the two planes p_1 and p_2 be given by

$$a_1x + b_1y + cz_1 + d_1 = 0 \qquad ...(i)$$

and $$a_2x + b_2y + c_2z + d_2 = 0 \qquad ...(ii)$$

Let the two planes intersect in the line l and let n_1 and n_2 be normals to the two planes. Let θ be the angle between two planes.

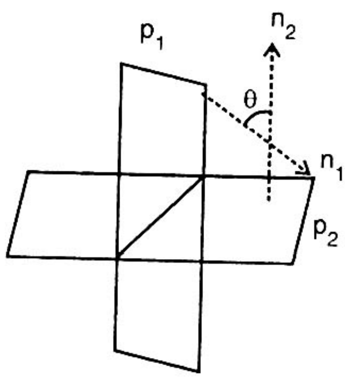


Fig. 14.5

∴ The direction cosines of normals to the two planes are

$$\pm\frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}},\ \pm\frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}},\ \pm\frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$$

and $$\pm\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}},\ \pm\frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}},\ \pm\frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}}.$$

∴ cos θ is given by $$\cos\theta=\pm\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$$

where the sign is so chosen that cos θ is positive.

Corollary 1 :

When the two planes are perpendicular to each other then

$$\theta = 90^\circ, \quad \cos\theta = \cos 90^\circ = 0$$

$\therefore$ The condition for two planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$ to be perpendicular to each other is

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Corollary 2 :

If the two planes are parallel, then the normals to the two planes are also parallel

$$\therefore \qquad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$\therefore$ The condition of parallelism of two planes $a_1x + b_1y + c_1z + d_1 = 0$

$$a_2x + b_2y + c_2z + d_2 = 0 \text{ is } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

This implies that the equations of two parallel planes differ only by a constant. Therefore, any plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is a constant.

Example 14.9 : Find the angle between the planes

$$3x + 2y - 6z + 7 = 0 \qquad \text{...(i)}$$

and $$2x + 3y + 2z - 5 = 0 \qquad \text{...(ii)}$$

Solution: Here $a_1 = 3,\ b_1 = 2,\quad c_1 = -6$

and $a_2 = 2,\quad b_2 = 3,\quad c_2 = 2$

$\therefore$ If θ is the angle between the planes (i) and (ii), then

$$\cos\theta = \frac{3.2 + 2.3 + (-6).2}{\sqrt{3^2 + 2^2 + (-6)^2} \cdot \sqrt{2^2 + 3^2 + 2^2}} = 0$$

$\therefore \quad \theta = 90^o$

Thus the two planes given by (i) and (ii) are perpendicular to each other.

Example 14.10 : Find the equation of the plane parallel to the plane $x - 3y + 4z - 1 = 0$ and (3, 1, −2).

Solution : Let the equation of the plane parallel to the plane

$$x - 3y + 4z - 1 = 0 \text{ be } x - 3y + 4z + k = 0 \qquad \text{...(i)}$$

Since (i) passes through the point (3, 1, −2) , it should satisfy it

$$3 - 3 - 8 + k = 0, \text{ or } k = 8$$

$\therefore$ The required equation of the plane is $x - 3y + 4z + 8 = 0$

Example 14.11 : Find the equation of the plane passing through the points (−1, 2, 3) and (2, −3, 4) and which is perpendicular to the plane $3x + y - z + 5 = 0$

Solution : The equation of any plane passing through the point (−1, 2, 3) is

$$a(x + 1) + b(y - 2) + c(z - 3) = 0 \qquad \text{.... (i)}$$

Since the point (2, −3, 4) lies on the plane (i)

$$3a - 5b + c = 0 \qquad \text{.... (ii)}$$

Again the plane (i) is perpendicular to the plane $3x + y - z + 5 = 0$

$$\therefore 3a + b - c = 0 \qquad \text{.... (iii)}$$

From (ii) and (iii), by cross multiplication method, we get

$$\frac{a}{4} = \frac{b}{6} = \frac{c}{18} \Rightarrow \frac{a}{2} = \frac{b}{3} = \frac{c}{9}$$

Hence the required equation of the plane is

$$2(x + 1) + 3(y - 2) + 9(z - 3) = 0 \qquad \text{...[From (i)]}$$

i.e., $\quad 2x + 3y + 9z = 31$

Example 14.12 : Find the equation of the plane passing through the point (2, –1, 5) and perpendicular to each of the planes.

$$x + 2y - z = 1, \quad 3x - 4y + z = 5$$

Solution : Equation of a plane passing through the point (2, 1, –5) is

$$a(x - 2) + b(y + 1) + c(z - 5) = 0 \qquad \text{.... (i)}$$

As this plane is perpendicular to each of the planes

$$x + 2y - z = 1 \quad \text{and} \quad 3x - 4y + z = 5$$

we have $\quad a.1 + b.2 + c(-1) = 0$

and $\quad a.3 + b(-4) + c.1 = 0$

or $\quad a + 2b - c = 0 \qquad \text{... (ii)}$

$\quad 3a - 4b + c = 0 \qquad \text{... (iii)}$

From (ii) and (iii), we get

$$\frac{a}{2-4} = \frac{b}{-3-1} = \frac{c}{-4-6}$$

or $\quad \dfrac{a}{-2} = \dfrac{b}{-4} = \dfrac{c}{-10}$ or $\dfrac{a}{1} = \dfrac{b}{2} = \dfrac{c}{5} = \lambda(\text{say})$

for a, b and c in (i), we get

$$\lambda(x - 2) + 2\lambda(y + 1) + 5\lambda(z - 5) = 0$$

or $\quad x - 2 + 2y + 2 + 5z - 25 = 0$

or $\quad x + 2y + 5z \ \lambda\ 25 = 0$

which is the required equation of the plane.

EXERCISE 14.4

1. Find the angle between the planes

   (i) $2x - y + z = 6$ and $x + y + 2z = 3$

   (ii) $3x - 2y + z + 17 = 0$ and $4x + 3y - 6z + 25 = 0$

2. Prove that the following planes are perpendicular to each other.

   (i) $x + 2y + 2z = 0$ and $2x + y - 2z = 0$

   (ii) $3x + 4y - 5z = 9$ and $2x + 6y + 6z = 7$

3. Find the equation of the plane passing through the point $(2, 3, -1)$ and parallel to the plane $2x + 3y + 6z + 7 = 0$.

4. Find the equation of the plane through the points $(-1, 1, 1)$ and $(1, -1, 1)$ and perpendicular to the plane $x + 2y + 2z = 5$.

5. Find the equation of the plane which passes through the origin and is perpendicular to each of the planes $x+2y+2z=0$ and $2x+y-2z=0$.

14.7 DISTANCE OF A POINT FROM A PLANE

Let the equation of the plane in normal form be

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p \quad \text{where } p > 0 \quad ...(i)$$

Case 1: Let the point $P(x', y', z')$ lie on the same side of the plane in which the origin lies.

Let us draw a plane through point P parallel to plane (i). Its equation is

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p' \quad ... (ii)$$

where p' is the length of the perpendicular drawn from origin upon the plane given by (ii). Hence the perpendicular distance of P from plane (i) is $p - p'$

As the plane (ii) passes through the point (x', y', z'),

$$x' \cos \alpha + y' \cos \beta + z' \cos \gamma = p'$$

$\therefore$ The distance of P from the given plane is

$$p' - p = p - (x' \cos \alpha + y' \cos \beta + z' \cos \gamma)$$

Case II : If the point P lies on the other side of the plane in which the origin lies, then the distance of P from the plane (i) is,

$$p' - p = x_1 \cos\alpha + y_1 \cos\beta + z_1 \cos\gamma - p$$

Note : If the equation of the plane be given as $ax + by + cz + d = 0$, we have to first convert it into the normal form, as discussed before, and then use the above formula.

14.8 EQUATION OF A PLANE BISECTING THE ANGLE BETWEEN TWO PLANES

Let the equations of the planes be $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

If $P(x_1, y_1, z_1)$ be any point on the plane bisecting the angle between two given planes, then the lengths of the perpendiculars drawn from P to them should be equal.

$$\therefore \qquad \frac{a_1x_1 + b_1y_1 + c_1z_1 + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x_2 + b_2y_2 + c_2z_2 + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$\therefore$ according as the two perpendiculars are of the same sign or of different sign.

$\therefore$ The equation

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \qquad \text{... (i)}$$

is satisfied by co-ordinates of any point on the planes bisecting the angles between the two given planes.

$\therefore$ (i) represents the equations of the planes which bisect the angles between the two given planes.

Note : You may note that we get two planes bisecting the angles between the given planes.

Example 14.13 : Find the distance of the point (1,2,3) from the plane $3x - 2y + 5z + 17 = 0$

Solution : Required distance $\dfrac{3.1-2.2+5.3+17}{\sqrt{3^2+(-2)^2+5^2}} = \dfrac{31}{\sqrt{38}}$ units.

Example 14.14 : Find the distance between the planes

$$x - 2y + 3z - 6 = 0$$

and $$2x - 4y + 6z + 17 = 0$$

Solution: The equations of the planes are

$$x - 2y + 3z - 6 = 0 \qquad \text{.... (i)}$$

$$2x - 4y + 6z + 17 = 0 \qquad \text{.... (ii)}$$

Here $\dfrac{1}{2} = \dfrac{(-2)}{(-4)} = \dfrac{3}{6}$

$\therefore$ Planes (i) and (ii) are parallel

Any point on plane (i) is (6, 0, 0)

$\therefore$ Distance between planes (i) and (ii) = Distance of point (6, 0, 0) from (ii)

$$= \frac{2\times6-4\times0+6\times0+17}{\sqrt{2^2+(-4)^2+6^2}}$$

$$= \frac{29}{\sqrt{56}} \text{ units } = \frac{29}{2\sqrt{14}} \text{ units}$$

Example 14.15 : Find the equation of the planes which bisect the angles between the planes $3x - 2y + 6z + 8 = 0$ and $2x - y + 2z + 3 = 0$.

Solution : The required equations of the bisector planes are

$$\frac{3x-2y+6z+8}{\sqrt{3^2+(-2)^2+6^2}} = \pm\frac{2x-y+2z+3}{\sqrt{2^2+(-1)^2+2^2}}$$

or $\dfrac{3x-2y+6z+8}{7} = \pm\dfrac{2x-y+2z+3}{3}$

or $9x - 6y + 18z + 24 = +(14x - 7y + 14z + 21)$

$9x - 6y + 18z + 24 = -(14x - 7y + 14z + 21)$

or $5x - y - 4z = 3;\ \ 23x - 13y + 32z + 45 = 0$

EXERCISE 14.5

1. Find the distance of the point

   (i) (2, –3, 1) from the plane $5x - 2y + 3z + 11 = 0$.

   (ii) (3, 4, –5) from the plane $2x - 3y + 3z + 27 = 0$.

2. Find the distance between planes

   $3x + y - z - 7 = 0$ and $6x + 2y - 2z + 11 = 0$.

3. Find the equations of the planes bisecting the angles between the planes

   (i) $2x + y - 2z = 4$ and $2x - 3y + 6z + 2 = 0$.

   (ii) $3x - 4y + 12z = 26$ and $x + 2y - 2z = 9$.

   (iii) $x + 2y + 2z - 9 = 0$ and $4x - 3y + 12z + 13 = 0$.

14.9 HOMOGENEOUS EQUATION OF SECOND DE GREE REPRESENTING TWO PLANES

Let the equations of two planes be

$$A = a_1x + b_1y + c_1z + d_1 = 0 \quad \text{.... (i)}$$

and $$B = a_2x + b_2y + c_2z + d_2 = 0 \quad \text{.... (ii)}$$

Consider the equation AB = 0

i.e., $$(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0 \quad \text{.... (iii)}$$

The co-ordinates of all points satisfying (i) or (ii) satisfy the equation (iii)

i.e., all points of A = 0 or, B = 0 lie on the surface represented by the equation AB = 0 Again AB = 0 is a second degree equation in x, y and z.

$\therefore$ A second degree equation in x, y and z represents a pair of planes only if it can be resolved into two linear factors (each factor equated to zero representing a plane)

Note : We give below an important result regarding the factorisation of a general homogeneous equation of second degree in x, y and z.

viz. $$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0 \qquad \text{... (A)}$$

(A) is factorisable into two linear factors if and only if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Also, the angle θ between the two planes is given by

$$\tan\theta = \frac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}$$

The proofs of these results are beyond the scope of this lesson.

Example 14.16 : Prove that the equation

$$8x^2 - 3y^2 - 10z^2 + 10xy + 17yz + 2xz = 0$$

represents a pair of planes through the origin. Also find the angle between the two planes.

Solution : Here $a = 8$; $b = -3$; $c = -10$; $h = 5$, $f = \frac{17}{2}$ and $g = 1$

$$\therefore \quad abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= 8(-3)(-10) + 2\left(\frac{17}{2}\right)(1)(5) - 8\left(\frac{17}{2}\right)^2 + (3)(1)^2 + (10)(5)^2$$

$$= 240 + 85 - 578 + 3 + 250$$

$$= 578 - 578 = 0$$

$\therefore$ The given equation

$8x^2 - 3y^2 - 10z^2 + 10xy + 17yz + 2xz = 0$ represent a pair of planes through the origin.

Also, angle θ between the planes is given by

$$\tan\theta = \frac{2\sqrt{f^2+g^2+h^2-bc-ca-ab}}{a+b+c}$$

$$= \frac{2\sqrt{\frac{289}{4}+1+25+24-30+80}}{-5} = -\frac{2}{5}\sqrt{\frac{689}{4}}$$

$$= -\frac{2}{5}\sqrt{\frac{689}{4}}$$

$$\therefore \qquad \theta = \tan^{-1}\left[-\frac{2}{5}\frac{\sqrt{689}}{4}\right]$$

EXERCISE 14.6

1. Prove that the equation $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ represents a pair of planes. Find also the angle between the planes.

2. Show that the equation $6x^2 + 4y^2 - 10z^2 + 3yz + 4zx - 11xy = 0$ represents a pair of planes and find the angle between the planes.

APPENDIX

A special method for factorising a general homogeneous polynomial of second degree in x, y and z.

For example, let us consider the polynomial

$$8x^2 - 3y^2 - 10z^2 + 10xy + 17yz + 2xz \qquad .... \text{(i)}$$

Putting $z = 0$ in (i), we have $8x^2 - 3y^2 + 10xy$

or $\quad 8x^2 + 12xy - 2xy - 3y^2$

or $\quad 4x(2x + 3y) - y(2x + 3y)$

or $\quad (4x - y)\ (2x + 3y) \qquad$ ... (A)

Putting y = 0 in (i)

we have $\quad 8x^2 + 2xz - 10z^2$

or $\quad 8x^2 + 10xz - 8xz - 10z^2$

or $\quad 2x(4x + 5z) - 2z(4x + 5z)$

or $\quad (2x - 2z)\ (4x + 5z)$ ... (B)

Let $x = 0$ in (i), we get

$- 3y^2 - 10z^2 + 17yz$

or $\quad 3y^2 - 17yz + 10z^2$

or $\quad 3y^2 - 15yz - 2yz + 10z^2$

or $\quad 3y(y - 5z) - 2z(y - 5z)$

or $\quad (3y - 2z) \qquad (y - 5z)$ ... (C)

Combining (A), (B), (C), we get the factors of (i) as

$$(4x - y + 5z); \qquad (2x + 3y - 2z)$$

$$\therefore \quad 8x^2 - 3y^2 - 10z^2 + 10xy + 17yz + 2xz$$

$$\equiv (4x - y + 5z)\ (2x + 3y - 2z)$$

This method is applicable for all such homogeneous polynomials of second degree in x, y and z.

Note :

(i) Look at the factor $(2x + 3y)$in (A) and the factor $(2x - 2z)$ in (B). As 2x is common between the two, the complete factor is $(2x + 3y - 2z)$. Similarly, looking at $(4x - y)$in (A) and $(4x + 5z)$ in (B), we get the complete factor as $(4x - y + 5z)$.

(ii) The equations of the two planes represented by the equation

$$8x^2 - 3y^2 - 10z^2 + 10xy + 17yz + 2xz = 0 \quad \text{are}$$

$$4x - y + 5z = 0 \text{ and } 2x + 3y - 2z = 0$$

(iii) You are advised to find the equations of the pairs of planes in **Check Your Progress 34.6** using the method as shown in the appendix.

14.10 AREA OF A TRIANGLE

Let cos α, cos β, cos β and cos γ be the direction cosines of the normal to the plane whose area is A. Then the

(i) projection of the area A, say A1, on yz-plane is A cos α

i.e., $A_1 = A\cos\alpha$

(ii) Similarly A_2, the projection of the area A on xz-plane and A_3, the projection of A on xy-plane are given by

$$A^2 = A\cos\beta \text{ and } A_3 = A\cos\gamma$$

$$\therefore\ A_1^2+A_2^2+A_3^2 = A^2\cos^2\alpha + A^2\cos^2\beta + A^2\cos^2\gamma$$

$$= A^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

$$= A^2 \text{ (since } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 0)$$

We shall use this result in finding the area of a triangle.

Let the vertices of the triangle ABC be $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ respectively and let the area of the triangle be Δ. Let the projections of Δ on the co-ordinates planes be Δ_{xy} and Δ_{yz}, Δ_{zx} respectively..

The projection of A, B and C on xy-plane are

$$(x_1, y_1, 0),\quad (x_2, y_2, 0) \text{ and } (x_3, y_3, 0)$$

$$\therefore\quad \Delta_{xy} = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

[Recall that you have found the area of a triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) as given in (A) in two dimensional co-ordinate geometry]

Similarly, $\Delta_{yz} = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$ and $\Delta_{zx} = \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}$

$$\therefore\ \Delta^2 = \left(\Delta_{xy}\right)^2 + \left(\Delta_{yz}\right)^2 + \left(\Delta_{zx}\right)^2$$

$$= \frac{1}{4}\left[\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2\right]$$

which gives required area of the triangle in space.

Example 14.17 : Find the area of the triangle with vertices (1, 5, 2), (1, 2, –3), and (3, 2, –1) .

Solution: Here

$$\Delta_{xy}^2 = \frac{1}{4}\begin{vmatrix} 1 & 5 & 1 \\ 1 & 2 & 1 \\ 3 & 2 & 1 \end{vmatrix}^2 = \frac{1}{4}[1(2-2) - 5(1-3) + 1(2-6)]^2$$

$$= \frac{1}{4}(10-4)^2 = 9$$

$$\Delta_{yz}^2 = \frac{1}{4}\begin{vmatrix} 5 & 2 & 1 \\ 2 & -3 & 1 \\ 2 & -1 & 1 \end{vmatrix}^2 = \frac{1}{4}[5(-3+1) - 2(2-2) + 1(-2+6)]^2$$

$$= \frac{1}{4}(-10+4)^2 = 9$$

$$\Delta_{zx}^2 = \frac{1}{4}\begin{vmatrix} 2 & 1 & 1 \\ -3 & 1 & 1 \\ -1 & 3 & 1 \end{vmatrix} = \frac{1}{4}[2(1-3) - 1(-3+1) + 1(-9+1)]^2$$

$$= \frac{1}{4}(-4+2-8)^2 = 25$$

$$\Delta^2 = 9 + 9 + 25 = 43$$

$$\Delta = \sqrt{43} \text{ sq.units.}$$

EXERCISE 14.7

1. Find the area of the triangles with vertices

   (i) (2, –5, 3), (1, –7, 4), (2, –3, 5)

   (ii) (3, 0, 1), (4, –1, –1), (3, –2, –2)

   (iii) (a, 0, 0), (0, b, 0), (0, 0, c)

KEY WORDS

- A plane is a surface such that if any two points are taken on it, the line joining these two points lies wholly in the plane.
- `The general equation of a plane is $ax + by + cz + d = 0$
- The equation of a plane passing through a given point (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

- There are infinite number of planes passing through a given point.
- The equation Ax + By + Cz + 1 = 0of the plane, contains three independent constants.
- The equation of a plane passing through three points

  (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

- Equation of a plane in the intercept from is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

  where a, b and c are intercepts made by the plane on x, y and z axes respectively.
- Equation of a plane in the normal form is $lx + my + nz = p$; where l, m, n are the direction cosines of normal to the plane and p is the length of the perpendicular from the origin to the plane.

- Angle θ between two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

- Two planes are perpendicular to each other if and only if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

- Two planes are parallel if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

- Distance of a point (x', y', z') from a plane

$x \cos \alpha + y \cos \beta + z \cos \gamma = p$ is

$|x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p|$ where the point (x', y', z') lies on the same side of the plane in which the origin lies.

- Equations of the planes bisecting the angles between two planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are given by

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

- A homogeneous equation of second degree

$ax^2 + by^2 + cz^2 + zhxy + 2fyz + 2gzx = 0$ represents a pair of planes if and only if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

- The square of the area of a triangle with vertices (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$= \frac{1}{4}\left[\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}^2\right]$$

SUPPORTIVE WEBSITES

http : //www.wikipedia.org

http ://www. mathworld.wolfram.com

PRACTICE EXERCISE

1. Find the equation of a plane passing through the point $(-2, 5, 4)$.
2. Find the equation of a plane which divides the line segment joining the points $(2, 1, 4)$ and $(2, 6, 4)$internally in the ratio of $2 : 3$.
3. Find the equation of the plane through the points $(1, 1, 0)$, $(1, 2, 1)$ and $(-2, 2, -1)$.
4. Show that the four points $(0,-1, -1)$, $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4)$ are coplanar. Also find the equation of the palne in which they lie.
5. The foot of the perpendicular drawn from $(1, -2, -3)$to a plane is $(3, 2, -1)$ Find the equation of the plane.
6. Find the angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$.
7. Prove that the planes $3x - 5y + 8z - 2 = 0$ and $12x - 20y + 32z + 9 = 0$ are parallel.
8. Determine the value of k for which the planes $3x - 2y + kz - 1 = 0$ and $x + ky + 5z + 2 = 0$ may be perpendicular to each other.
9. Find the distance of the point $(3, 2, -5)$ from the plane $2x - 3y - 5z = 7$.
10. Find the equation of the planes that bisect angles between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$
11. Prove that the equation $6x^2 - 12y^2 + 4z^2 + xy + 13yz - 14xz = 0$ represents a pair of planes through the origin. Find also the angle between the two planes and the equations of the two planes.
12. Find the area of the triangle with vertices $(1, 2, 1)$, $(3, 4, -2)$ and $(4, 2, -1)$.

ANSWERS

EXERCISE 14.1

1. $ax + by + cz = 0$

2. $ax + by + c(z + 2) = 0$

3. $a(x - 5) + b(y + 7) + c(z - 3) = 0$

4. $a\left(x - \frac{x_1 + x_2}{2}\right) + b\left(y - \frac{y_1 + y_2}{2}\right) + c\left(z - \frac{z_1 + z_2}{2}\right) = 0$

5. $a(x - 2) + b(y - 2) + c(z + 3) = 0$

EXERCISE 14.2

1. (a) $5x + 2y - 3z - 17 = 0$; (b) $3x - y + z = 0$

   (c) $x + 2y - z = 4$

3. $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$

4. Intercepts are 12, 8 and 6 on the co-ordinate axes *x, y* and *z* respectively.

EXERCISE 14.3

1. (i) $\frac{4x}{14} + \frac{12y}{14} - \frac{6z}{14} = 2$ (ii) $-\frac{3}{5}y - \frac{4}{5}z = \frac{3}{5}$

2. $x - 3y + z - 11 = 0$

3. $x - 2y + z - 6 = 0$

5. $4x + 3y - 2z = 36$

EXERCISE 14.4

1. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{2}$

3. $2x + 3y + 6z = 7$

4. $2x + 2y - 3z + 3 = 0$

5. $2x - 2y + z = 0$

EXERCISE 14.5

1. (i) $\frac{30}{\sqrt{38}}$ units. (ii) $\frac{6}{\sqrt{22}}$ units.

2, $\frac{25}{2\sqrt{11}}$ units.

3. (i) $10x - y + 2z - 11 = 0$; $4x + 8y - 16z - 17 = 0$

   (ii) $4x + 38y - 62z - 39 = 0$; $22x + 14y + 10z - 195 = 0$

   (iii) $25x + 17y + 62z - 78 = 0$; $x + 35y - 10z - 156 = 0$

EXERCISE 14.6

1. $\cos^{-1}\left(\frac{16}{21}\right)$ 3. $\frac{\pi}{2}$

EXERCISE 14.7

1. (i) $\sqrt{11}$ sq. units (ii) $\frac{\sqrt{14}}{2}$ sq. units

   (iii) $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$ sq. units

PRACTICE EXERCISE

1. $a(x+2)+b(y-5)+c(z-4)=0$

2. $a(x-2)+b(y-3)+c(z-4)=0$

3. $2x+3y-3z-5=0$

4. $5x-7y+11z+4=0$

5. $x+2y+z=6$

6. $\frac{\pi}{3}$

8. $k=-1$

9. $\frac{18}{\sqrt{38}}$

10. $5x-y-4z-3=0$; $23x-13y+32z+45=0$

11. $3x-4y-z=0$; $2x+3y-4z=0$; $\cos^{-1}\left(\frac{-2}{\sqrt{26}.\sqrt{29}}\right)$

12. $\frac{1}{2}\sqrt{77}$ sq. units

Chapter 15

VECTORS

LEARNING OUTCOMES

After studying this lesson, you will be able to :

- explain the need of mentioning direction;
- define a scalar and a vector;
- distinguish between scalar and vector;
- represent vectors as directed line segment;
- determine the magnitude and direction of a vector;
- classify different types of vectors-null and unit vectors;
- define equality of two vectors;
- define the position vector of a point. DC's and Dr's
- add and subtract vectors;
- multiply a given vector by a scalar;
- state and use the properties of various operations on vectors;
- comprehend the three dimensional space;
- resolve a vector along two or three mutually prependicular axes;
- derive and use section formula; and
- define scalar (dot) and vector (cross) product of two vectors.

PREREQUISITES

- Knowledge of plane and coordinate geometry.
- Knowledge of Trigonometry.

INTRODUCTION

In our regular life we cope with many Questions like what is the temperature of the room? How Should the boat Crossing the river? Observe that one possible answer to the first Question is 29°C, a quantity that specifies a value (magnitude) which is a real number. Such Quantities are called scalars. Though the answer for the second question is quantity (called velocity) which involves speed of the object (magnitude) and also direction (in which posistioned an angle by boats man verses the speed of river) such quantities are called vectors. In physics, mathematics and Engenering we oftenly deal with both type of quantities. Like scalar quanties such as length, mass, volume, temperature, density, area, work resistence etc... and vector quantities namely displacement, velocity, acceleration, force, weight, momentum, etc.....

Consider another example of a moving ball. If we wish to predict the position of the ball at any time what are the basics we must know to make such a prediction?

Let the ball be initially at a certain point A. If it were known that the ball travels in a straight line at a speed of 5cm/sec, can we predict its position after 3 seconds? Obviously not. Perhaps we may conclude that the ball would be 15 cm away from the point A and therefore it will be at some point on the circle with A as its centre and radius 15 cms. So, the mere knowledge of speed and time taken are not sufficient to predict the position of the ball. However, if we know that the ball moves in a direction due east from A at a speed of 5cm/sec., then we shall be able to say that after 3 seconds, the ball must be precisely at the point P which is 15 cms in the direction east of A.

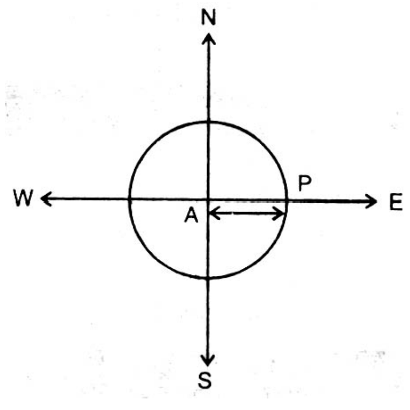


Fig. 15.1

Thus, to study the displacement of a ball after time t (3 seconds), we need to know the magnitude of its speed (i.e. 5 cm/sec) and also its direction (east of A) In this lesson we will be dealing with quantities which have magnitude only, called scalars and the quantities which have both magnitude and direction, called vectors. We will represent vectors as directed line segments and determine their magnitudes and directions. We will study about various types of vectors and perform operations on vectors with properties thereof. We will also acquaint ourselves with position vector of a point w.r.t. some origin of reference. We will find out the resolved parts of a vector, in two and three dimensions, along two and three mutually perpendicular directions respectively. We will also derive section formula and apply that to problems. We will also define scalar and vector products of two vectors.

15.1 SCALARS AND VECTORS

A physical quantity which can be represented by a number only is known as a scalar i.e, quantities which have only magnitude. Time, mass, length, speed, temperature, volume, quantity of heat, work done etc. are all ***scalars***.

The physical quantities which have magnitude as well as direction are known as vectors. Displacement, velocity, acceleration, force, weight etc. are all examples of ***vectors.***

Vectors as a traid of real numbers (Vector as a directed line)

Let l be any straight line in a plane or three dimensional space. This line can be given two directions by means of arrow heads. A line with one of these directions prescribed is called a directed line.

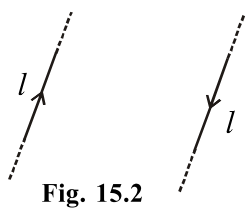


Fig. 15.2

15.2 VECTOR AS A DIRECTED LINE SEGMENT

If A and B are two distinct points in the space, the ordered pair (A B) denoted by AB is called a directed line segment with intial point A and terminal point B.

MODULE - III
Dimensional
Geometry Vectors

Notes

The magnitude of AB denoted by |AB| = a is the length of AB

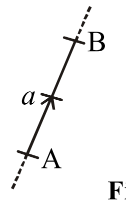


Fig. 15.3

Definition : A line segment with a specified magnitude and direction is called a vector.

15.3 CLASSIFICATION OF VECTORS

15.3.1 Zero Vector (Null Vector)

A vector whose magnitude is zero is called a zero vector or ***null vector***. Zero vector has not definite direction. $\overrightarrow{AA}$, $\overrightarrow{BB}$ are zero vectors. Zero vectors is also denoted by $\vec{0}$ to distinguish it from the scalar 0.

15.3.2 Unit Vector

A vector whose magnitude is unity is called a ***unit vector***. So for a unit vector $\vec{a}$, $|\vec{a}| = 1$. A unit vector is usually denoted by $\hat{a}$. Thus, $\vec{a} = |\vec{a}|\ \hat{a}$.

15.3.3 Equal Vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude. i.e., $|\vec{a}| = |\vec{b}|$ and the same direction as shown in Fig. 32.4. Symbolically, it is denoted by a $\vec{a} = \vec{b}$.

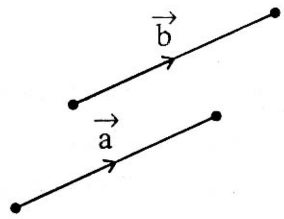


Fig. 15.4

Remark : Two vectors may be equal even if they have different parallel lines of support.

15.3.4 Like Vectors

Vectors are said to be like if they have same direction whatever be their magnitudes. In the adjoining Fig. 15.5, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are like vectors, although their magnitudes are not same.

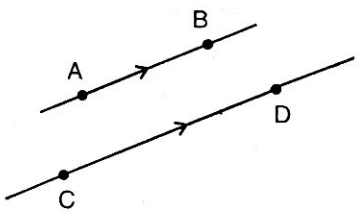


Fig. 15.5

15.3.5 Negative of a Vector

$\overrightarrow{BA}$ is called the ***negative of the vector*** $\overrightarrow{AB}$, when they have the same magnitude but opposite directions.

i.e., $\overrightarrow{BA} = -\overrightarrow{AB}$

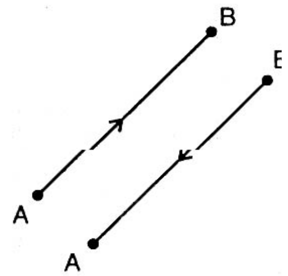


Fig. 15.6

15.3.6 Co-initial Vectors

Two or more vectors having the same initial point are called ***Co-initial vectors***

In the adjoining figure, $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AC}$ are co-initial vectors with the same initial point A.

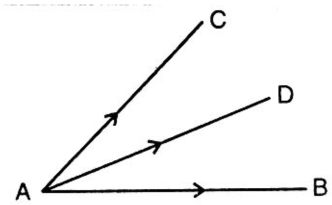


Fig. 15.7

15.3.7 Collinear Vectors

Two or more vectors are said to be collinear if they are parrellel to the same line, irrespective of their magnitudes and direction. Such vectors have the same support or
parrellel support.

In the adjoining figure, $\overrightarrow{CD}$ and $\overrightarrow{EF}$ are collinear vectors. $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are also collinear.

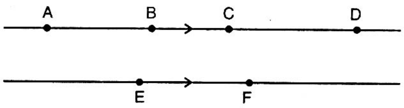


Fig. 15.8

15.3.8 Coplanar Vectors

Vectors whose supports are in the same plane or parellel to the same plane are called Co-polanor vectors. Vectors which are not Coplanar are called non-coplanar vectors.

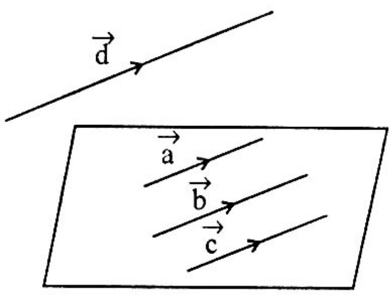


Fig. 15.9

Example 15.1: State which of the are scalars and which are vectors. Give reason.

(a) Mass (b) Weight (c) Momentum
(d) Temperature (e) Force (f) Density
(g) Force (h) length (i) Velocity
(j) area

Solution : (a), (d), (f), (g), (i) are scalars because they have only magnitude.

(b), (c) , (e) (h), (j) are vectors because they have magnitude and direction as well.

Example 15.2 : Represent graphically

(a) a force 40N in a direction 60° north of east.

(b) Displacement of 30 km, making an angle 60° South of west.

Solution :

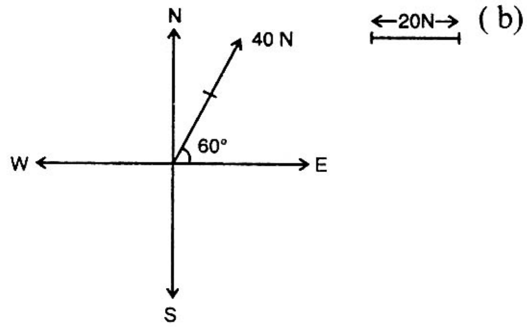


Fig. 15.10

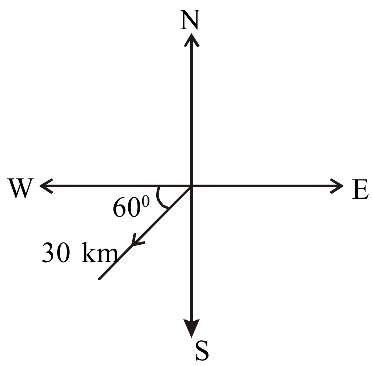


Fig. 15.11

EXERCISE 15.1

1. Which of the following is a scalar quantity ?
   (1) Displacement (2) Velocity (3) Force (4) Length.
2. Which of the following is a vector quantity ?
   (1) Mass (2) force (3) time
   (4) tempertaure
3. You are given a displacement vector of 5 cm due east. Show by a diagram the corresponding negative vector.

4. Distinguish between like and equal vectors.
5. Represent graphically
   (a) a force 60 Newton is a direction 60° west of north.
   (b) Displacement of a vector 70 km making an angle 45° South of East.

15.4 ADDITION OF VECTORS

Recall that you have learnt four fundamental operations viz. addition, subtraction, multiplication and division on numbers. The addition (subtraction) of vectors is different from that of numbers (scalars).

In fact, there is the concept of resultant of two vectors (these could be two velocities, two forces etc.). We illustrate this with the help of the following example:

Let us take the case of a boat-man trying to cross a river in a boat and reach a place directly in the line of start. Even if he starts in a direction perpendicular to the bank, the water current carries him to a place different from the place he desired., which is an example of the effect of two velocities resulting in a third one called the resultant velocity.

Thus, two vectors with magnitudes 3 and 4 may not result, on addition, in a vector with magnitude 7. It will depend on the direction of the two vectors i.e., on the angle between them. The addition of vectors is done in accordance with the triangle law of addition of vectors.

15.4.1 Triangle Law of Addition of Vectors

A vector whose effect is equal to the resultant (or combined) effect of two vectors is defined as the resultant or sum of these vectors. This is done by the triangle law of addition of vectors. In the adjoining Fig. 15.12 vector $\overrightarrow{OB}$ is the resultant or sum of vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ and is written as

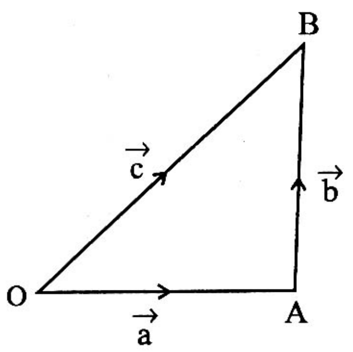


Fig. 15.12

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

i.e., $\vec{a} + \vec{b} = \overrightarrow{OB} = \vec{c}$

You may note that the terminal point of vector $\vec{a}$ is the initial point of vector $\vec{b}$ and the initial point of $\vec{a} + \vec{b}$ is the initial point of $\vec{a}$ and its terminal point is the terminal point of $\vec{b}$.

15.4.2 Addition of more than two Vectors

Addition of more then two vectors is shown in the adjoining figure

$\vec{a} + \vec{b} + \vec{c} + \vec{d}$

$= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$

$= \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DE}$

$= \overrightarrow{AD} + \overrightarrow{DE}$

$= \overrightarrow{AE}$

The vector $\overrightarrow{AE}$ is called the sum or the resultant vector of the given vectors.

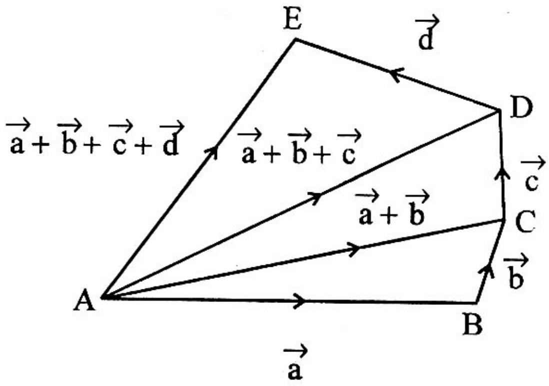


Fig. 15.13

15.4.3 Parallelogram Law of Addition of Vectors

Recall that two vectors are equal when their magnitude and direction are the same. But they could be parallel [refer to Fig. 15.14].

See the parallelogram OABC in the adjoining figure :

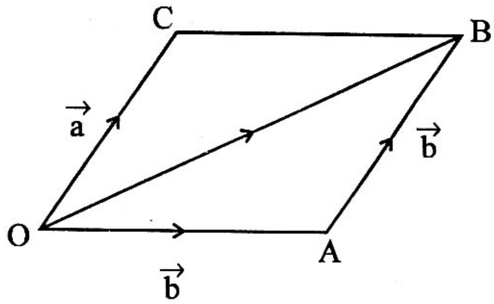


Fig. 15.14

We have $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

But $\overrightarrow{AB} = \overrightarrow{OC}$

$\therefore \quad \overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}$

which is the parallelogram law of addition of vectors**If two vectors are represented by the** two adjacent sides of a parallelogram, then their resultant is represented by the diagonal through the common point of the adjacent sides.

15.4.4 Negative of a Vector

For any vector $\vec{a} = \overrightarrow{OA}$, the negative of $\vec{a}$ is represented by $\overrightarrow{AO}$. The negative of $\overrightarrow{AO}$ is the same as $\overrightarrow{OA}$ Thus, $\left|\overrightarrow{OA}\right| = \left|\overrightarrow{AO}\right| = \vec{a}$ and $\overrightarrow{OA} = -\overrightarrow{AO}$. It follows from definition that for any vector $\vec{a} + (-\vec{a}) = \vec{0}$.

15.4.5 The Difference of Two Given Vectors

For two given vectors $\vec{a}$ and $\vec{b}$, the difference $\vec{a} - \vec{b}$ is defined as the sum of $\vec{a}$ and the negative of the vector $\vec{b}$. i.e., $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

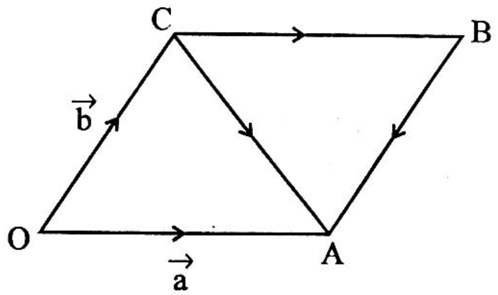


Fig. 15.15

In the adjoining figure if $\overrightarrow{OA} = \vec{a}$ then, in the parallelogram

OABC, $\overrightarrow{CB} = \vec{a}$

and $\overrightarrow{BA} = -\vec{b}$

$\therefore \quad \overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA} = \vec{a} - \vec{b}$

15.4.6 Properties of Vector addition

- For any two vectors a and b,

  $a + b = b + a$ commutative property

- For any three vectors a, b and c

  $(a + b) + c = a + (b + c)$ Associative property

- For any vector a, $a + \vec{0} = 0 + a = a$

  Here the zero vector is called additive identity for the vector addition.

Example 15.3 : When is the sum of two non-zero vectors zero ?

Solution : The sum of two non-zero vectors is zero when they have the same magnitude but opposite direction.

EXERCISE 15.2

1. The diagonals of the parallelogram ABCD intersect at the point O. Find the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$.

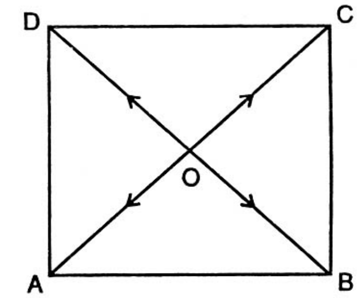


Fig. 15.16

2. The medians of the triangle ABC intersect at the point O. Find the sum of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

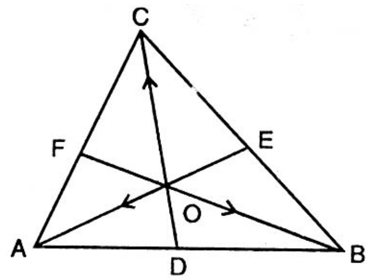


Fig. 15.17

15.5 POSITION VECTOR OF A POINT

Cosider a three dimensional rectangular Co-ordinate System OX, OY, OZ and a point P in the space having Co-ordinates (x, y, z) w.r.to the origion O(0, 0, 0)

Then the vector OP having O and P as its intial and terminal points respectually, is called the position vectors of the points respectually, is called the position vectors of the point P. This is denoted by r. Then the magnitude of

$$\left|\overline{OP}\right| = |r| = \sqrt{x^2 + y^2 + z^2}$$

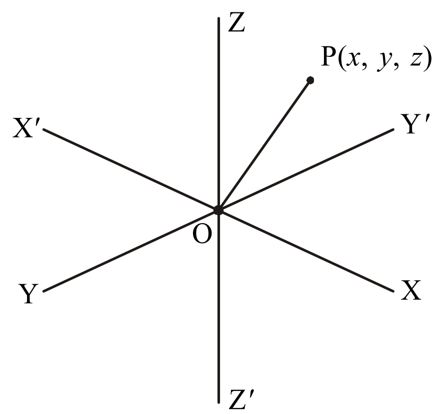


Fig. 15.18

Direction Cosines and Direction ratios

Consider the position vector OP = r of a point $p(x, y, z)$. Let α, β, γ be the angles made by the vector r with the positive direction (Counter clock wise direction) of X, Y, Z axes respectivelly. Then cos α, cos β and cos γ are called the direction cosines of the vector r. These direction. Cosines are usually denoted by l, m, n respectively.

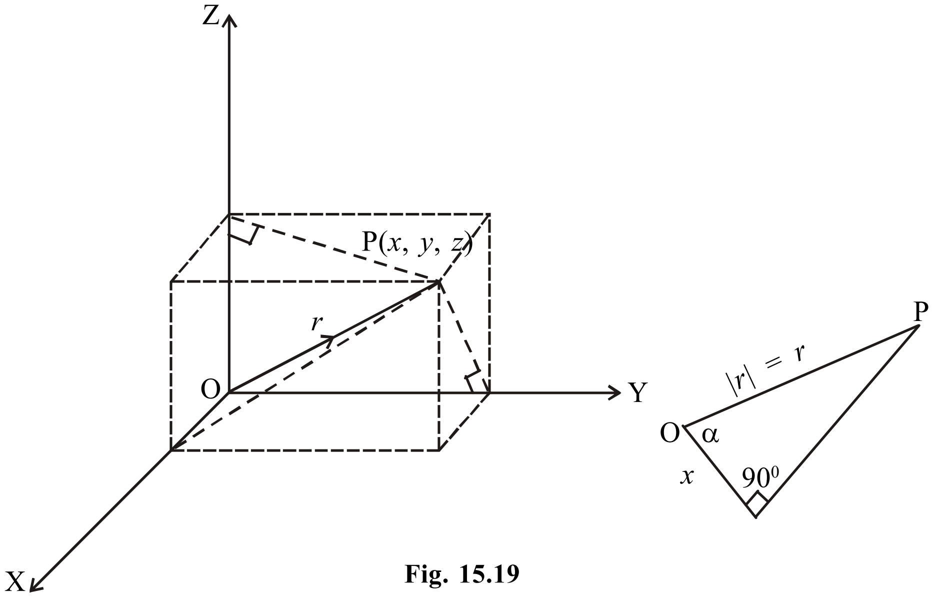


Fig. 15.19

Draw perpendiculars from to the X, Y and Z axes and let A, B, C be the feet of the perpendiculars respectively.

$$\therefore\ \cos\alpha = \frac{x}{r},\ \cos\beta = \frac{y}{r} \text{ and } \cos\gamma = \frac{z}{r}$$

Thus the co-ordinates x, y, z of the point p may also be expressed as (lr, mr, nr). The numbers lr, mr, nr which are proportional to the direction cosines l, m, n are called the direction ratios of the vector "r".

These are usually denoted by a, b, c respectively

We observe here that

$$r^2 = x^2 + y^2 + z^2$$
$$= l^2r^2 + m^2r^2 + n^2r^2$$
$$= r^2(l^2 + m^2 + n^2)$$

So that $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$ in general.

15.6 MULTIPLICATION OF A VECTOR BY A SCALAR

The product of a non-zero vector $\vec{a}$ by the scalar $x \neq 0$ is a vector whose length is equal to $|x||\vec{a}|$ and whose direction is the same as that of $\vec{a}$ if $x > 0$ and opposite to that of $\vec{a}$ if $x < 0$.

The product of vector $\vec{a}$ by the scalar 0 is the vectoro $\vec{0}$.

By the definition it follows that the product of a zero vector by any non-zero scalar is the zero vector i.e., $x\,\vec{0} = \vec{0}$; also $0\,\vec{a} = \vec{0}$.

Laws of multiplication of vectors : If $\vec{a}$ and $\vec{b}$ are vectors and x, y are scalars, then

(i) $x(y\vec{a}) = (xy)\,\vec{a}$

(ii) $x\vec{a} + y\,\vec{a} = (x+y)\,\vec{a}$

(iii) $x\vec{a} + x\,\vec{b} = x\,(\vec{a} + \vec{b})$

(iv) $0\,\vec{a} + x\,\vec{0} = \vec{0}$

(v) $(-x)\,\vec{a} = x(-\vec{a}) = -(x\vec{a})$

Recall that two collinear vectors have the same direction but may have different magnitudes.

This implies that $\vec{a}$ is collinear with a non-zero vector $\vec{b}$ if and only if there exists a number (scalar) x such that

$$\vec{a} = x\,\vec{b}$$

Theorem 15.1 : A necessary and sufficient condition for two vectors $\vec{a}$ and $\vec{b}$ to be collinear is that there exist scalars x and y (not both zero simultaneously) such that $x\vec{a} + y\,\vec{b} = \vec{0}$.

The Condition is necessary

Proof : Let $\vec{a}$ and $\vec{b}$ be collinear. Then there exists a scalar l such that $\vec{a} = l\vec{b}$

i.e., $\vec{a} + (-l)\vec{b} = \vec{0}$

$\therefore$ We are able to find scalars x (= 1) and y (= $-l$) such that

$x\,\vec{a} + y\,\vec{b} = \vec{0}$. Note that the scalar 1 is non-zero.

The Condition is sufficient

It is now given that $x\,\vec{a} + y\,\vec{b} = \vec{0}$ and $x \neq 0$ and $y \neq 0$ simultaneously.

We may assume that $y \neq 0$

The product of the vector by the scalar x is denoted by $x\vec{a}$

$\therefore\; y = \vec{b} = -x\vec{a} \Rightarrow \vec{b} = -\frac{x}{y}\vec{a}$ i.e., $\vec{b}$ and $\vec{a}$ are collinear.

Corollary : Two vectors $\vec{a}$ and $\vec{b}$ are non-collinear if and only if every relation of the form $x\,\vec{a} + y\,\vec{b}$ given as $x = 0$ and $y = 0$.

[**Hint :** If any one of x and y is non-zero say y, then we get $\vec{b} = -\frac{x}{y}\vec{a}$ which is a contradiction]

Example 15.5 : Find the number x by which the non-zero vector $\vec{a}$ be multiplied to get

(i) $\hat{a}$ (ii) $-\hat{a}$

Solution : (i) $x\,\vec{a} = \hat{a}$ i.e., $x|\vec{a}|\,\hat{a} = \hat{a}$

$$\Rightarrow \quad x = \frac{1}{|\hat{a}|}$$

(ii) $x\,\vec{a} = -\hat{a}$ i.e., $x\,|\vec{a}|\,\hat{a} = -\hat{a}$

$$\Rightarrow \quad x = -\frac{1}{|\hat{a}|}$$

Example 15.6: The vectors $\vec{a}$ and $\vec{b}$ are not collinear. Find x such that the vector $\vec{c} = (x-2)\,\vec{a} + \vec{b}$ and $\vec{d} = (2x+1)\,\vec{a} - \vec{b}$.

Solution: $\vec{c}$ is non-zero since the co-efficient of $\vec{b}$ is non-zero.

$\therefore$ There exists a number y such that $\vec{d} = y\vec{c}$.

i.e., $(2x+1)\,\vec{a} - \vec{b} = y(x-2)\,\vec{a} + y\vec{b}$

$\therefore \quad (yx - 2y - 2x - 1)\,\vec{a} + (y + 1)\,\vec{b} = 0.$

As $\vec{a}$ and $\vec{b}$ are non-collinear.

$\therefore \quad yx - 2y - 2x - 1 = 0$ and $y + 1 = 0$

Solving these we get $y = -1$ and $x = \frac{1}{3}$

Thus $\vec{c} = -\frac{5}{3}\vec{a} + \vec{b}$ and $\vec{d} = \frac{5}{3}\vec{a} - \vec{b}$

We can see that $\vec{c}$ and $\vec{d}$ are opposite vectors and hence are collinear.

Example 15.7 : The position vectors of two points A and B are $2\vec{a} + 3\vec{b}$ and $3\vec{a} + \vec{b}$ respectively. Find $\overrightarrow{AB}$.

Solution : Let O be the origin of reference.

Then $\overrightarrow{AB} =$ Position vector of B $-$ Position vector of A

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (3\vec{a} + \vec{b}) - (2\vec{a} + 3\vec{b})$$

$$= (3-2)\,\vec{a} + (1-3)\,\vec{b} = \vec{a} - 2\vec{b}$$

Example 15.8 : Show that the points P, Q and R with position vectors $\vec{a} - 2\vec{b}$, $2\vec{a} + 3\vec{b}$ and $-7\vec{b}$ respectively are collinear.

Solution: $\overrightarrow{PQ} =$ Position vector of Q $-$ Position vector of P

$$= (2\vec{a} + 3\vec{b}) - (\vec{a} - 2\vec{b})$$

$$= \vec{a} + 5\vec{b} \qquad \text{...(i)}$$

and $\overrightarrow{QR} =$ Position vector of R $-$ Position vector of Q

$$= -7\vec{b} - (2\vec{a} + 3\vec{b})$$

$$= -7\vec{b} - 2\vec{a} - 3\vec{b}$$

$$= -2\vec{a} - 10\vec{b}$$

$$= -2[\vec{a} + 5\vec{b}] \qquad \text{...(ii)}$$

From (i) and (ii) we get $\overrightarrow{PQ} = -2\overrightarrow{QR}$, a scalar multiple of $\overrightarrow{QR}$

$\therefore \quad \overrightarrow{PQ} \parallel \overrightarrow{QR}$

But Q is a common point

$\therefore$ $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are collinear. Hence points P, Q and R are collinear.

EXERCISE 15.3

1. The position vectors of the points A and B are $\vec{a}$ and $\vec{b}$ respectively with respect to a given origin of reference. Find $\overrightarrow{AB}$.
2. Interpret each of the following :

   (i) $3\vec{a}$ (ii) $-5\vec{b}$
3. The position vectors of points A, B, C and D are respectively $2\vec{a}$, $3\vec{b}$, $4\vec{a}+3\vec{b}$ and $\vec{a}+2\vec{b}$. Find $\overrightarrow{DB}$ and $\overrightarrow{AC}$.
4. Find the magnitude of the product of a vector $\vec{n}$ by a scalar y.
5. State whether the product of a vector by a scalar is a scalar or a vector.
6. State the condition of collinearity of two vectors $\vec{p}$ and $\vec{q}$.
7. Show that the points with position vectors $5\vec{a}+6\vec{b}$, $7\vec{a}-8\vec{b}$ and $3\vec{a}+20\vec{b}$ are collinear.

15.7 CO-PLANARITY OF VECTORS

Given any two non-collinear vectors $\vec{a}$ and $\vec{b}$, they can be made to lie in one plane. There (in the plane), the vectors will be intersecting. We take their common point as O and let the two vectors be $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Given a third vector $\vec{c}$, coplanar with $\vec{a}$ and $\vec{b}$, we can choose its initial point also as O. Let C be its terminal point. With $\overrightarrow{OC}$ as diagonal complete the parallelogram with $\vec{a}$ and , $\vec{b}$ as adjacent sides.

$\therefore$ $\vec{c} = l\vec{a} + m\vec{b}$

Thus, any $\vec{c}$, coplanar with $\vec{a}$ and $\vec{b}$, is expressible as a linear combination of $\vec{a}$ and $\vec{b}$.

i.e., $\vec{c} = l\vec{a} + m\vec{b}$.

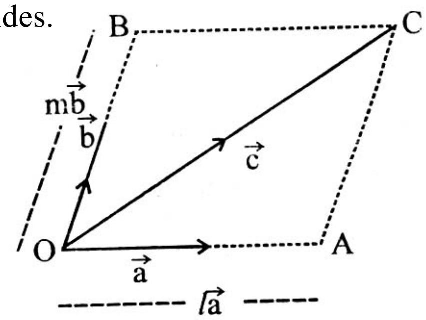


Fig. 15.20

15.8 RESOLUTION OF A VECTOR ALONG TWO PERPENDICULAR AXES

Consider two mutually perpendicular unit vectors $\hat{i}$ and $\hat{j}$ along two mutually perpendicular axes OX and OY. We have seen above that any vector $\vec{r}$ in the plane of $\hat{i}$ and $\hat{j}$, can be written in the form $\vec{r} = x\hat{i} + y\hat{j}$.

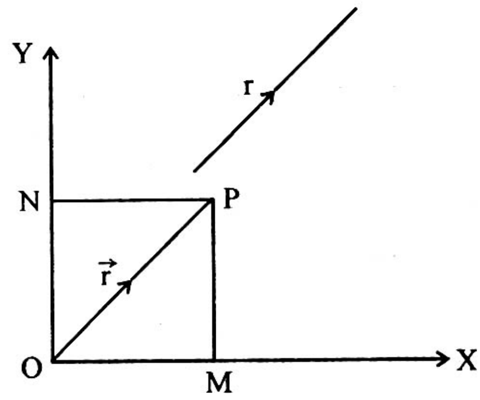


Fig. 15.21

If O is the initial point of $\vec{r}$, then OM = x and ON = y and $\overrightarrow{OM}$ and $\overrightarrow{ON}$, are called the component vectors of along x-axis and y-axis. $\overrightarrow{OM}$ and $\overrightarrow{ON}$ in this special case, are also called the ***resolved parts*** of $\vec{r}$.

15.9 RESOLUTION OF A VECTOR IN THREE DIMENSIONS ALONG THREE MUTUALLY PERPENDICULAR AXES

The concept of resolution of a vector in three dimensions along three mutually perpendicular axes is an extension of the resolution of a vector in a plane along two mutually perpendicular axes.

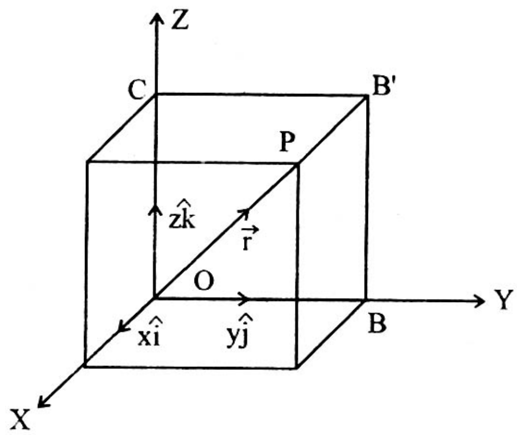


Fig. 15.22

Any vector $\vec{r}$ in space can be expressed as a linear combination of three mutually perpendicular unit vectors $\bar{i}$, $\bar{j}$ and $\bar{k}$ as is shown in the adjoining Fig. 15.22. We complete the rectangular parallelopiped with $\overrightarrow{OP} = \vec{r}$ as its diagonal :

then $\vec{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$x\hat{i}, y\hat{j}$ and $z\hat{k}$ are called the resolved parts of $\vec{r}$ along three mutually perpendicular axes.

Thus any vector $\vec{r}$ in space is expressible as a linear combination of three mutually perpendicular unit vectors $\bar{i}$, $\bar{j}$ and $\bar{k}$.

Refer to Fig. 15.21 in which $OP^2 = OM^2 + ON^2$ (Two dimensions)

or $\overrightarrow{r^2} = x^2 + y^2$ ...(i)

and in Fig. 15.22

$OP^2 = OA^2 + OB^2 + OC^2$

$\overrightarrow{r^2} = x^2 + y^2 + z^2$

Magnitude of $\vec{r} = |\vec{r}|$ in case of (i) is $\sqrt{x^2 + y^2}$

and (ii) is $\sqrt{x^2 + y^2 + z^2}$

Note : Given any three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (not necessarily mutually perpendicular unit vectors) any vector $\vec{d}$ is expressible as a linear combination of $\vec{a}$, $\vec{b}$ and $\vec{c}$ i.e., $\vec{d} = x\vec{a} + y\vec{b} + z\vec{c}$

Linear Combination of Vectors

Let $a_1, a_2, a_3, \ldots a_n$ be vectors and $x_1, x_2, \ldots x_n$ be scalars. Then the vectors $x_1a_1 + x_2a_2 + x_3a_3 + \ldots + x_na_n$ is called a linear combination of vectors $a_1, a_2, a_3 \ldots a_n$.

Components of a vector in three dimensions

Consider the ordered traid (a, b, c) of non-coplanar vectors a, b, c. If r is any vector then it is proved that there exists unique traid (x, y, z) of scalars such that $r = xa + yb + zc$. These scalars x, y, z are called the components of r with respect to the ordered traid (a, b, c).

Example 15.9 : A vector of 10 Newton is 30° north of east. Find its components along east and north directions.

Solution : Let $\bar{i}$ and $\bar{j}$ be the unit vectors along $\overrightarrow{OX}$ and $\overrightarrow{OY}$ (East and North respectively). Resolve OP in the direction OX and OY.

$$\therefore \quad \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{ON}$$

$$= 10\cos 30^0\ \vec{i} + 10\sin 30^0 . \vec{j}$$

$$= 10\ \frac{\sqrt{3}}{2}\ i + 10\ \frac{1}{2}\ j$$

$$= 5\sqrt{3}\ i + 5j$$

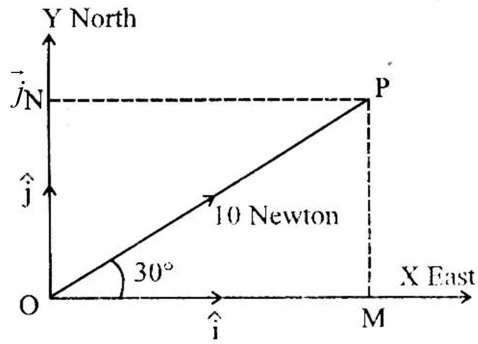


Fig. 15.23

$\therefore$ Component along (i) East $= 5\sqrt{3}$ Newton

(ii) North = 5 Newton

Example 15.10 : Show that the following vectors are coplanar :

$$\vec{a} - \overrightarrow{2b},\ 3\vec{a} + \vec{b} \text{ and } \vec{a} + 4\vec{b}$$

Solution : The vectors will be coplanar if there exists scalars x and y such that

$$\vec{a} + 4\vec{b} = x\,(\vec{a} - 2\vec{b}) + y\,(3\vec{a} + \vec{b})$$

$$= (x + 3y)\ \vec{a} + (-2x + y)\vec{b} \qquad \text{...(i)}$$

Comparing the co-efficients of $\vec{a}$ and $\vec{b}$ on both sides of (i), we get

$$x + 3y = 1 \text{ and } -2x + y = 4$$

which on solving, gives $x = \frac{-11}{7}$ and $y = \frac{6}{7}$

As $\vec{a} + 4\vec{b}$ is expressible in terms of $\vec{a} - 2\,\vec{b}$ and $3\vec{a} + b$ hence the three vectors are coplanar.

Example 15.11 : Given $\vec{r_1} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{r_2} = 2\vec{i} - 4\vec{j} - 3\vec{k}$ find the magnitudes of (a) $\vec{r_1}$ (b) $\vec{r_2}$ (c) $\vec{r_1} + \vec{r_2}$ (d) $\vec{r_1} - \vec{r_2}$

Solution :

(a) $|\vec{r_1}| = |\vec{i} - \vec{j} + \vec{k}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

(b) $|\vec{r_2}| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{29}$

(c) $\vec{r_1} + \vec{r_2} = (\hat{i} - \hat{j} + \hat{k}) + (2\hat{i} - 4\hat{j} - 3\hat{k}) = 3\hat{i} - 5\hat{j} - 2\hat{k}$

$\therefore \quad |\vec{r_1} + \vec{r_2}| = |3\hat{i} - 5\hat{j} - 2\hat{k}| = \sqrt{3^2 + (-5)^2 + (-2)^2} = \sqrt{38}$

(d) $\vec{r_1} - \vec{r_2} = (\hat{i} - \hat{j} + \hat{k}) - (2\hat{i} - 4\hat{j} - 3\hat{k}) = -\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore \ |\vec{r_1} - \vec{r_2}| = |-\hat{i} + 3\hat{j} + 4\hat{k}| = \sqrt{(-1)^2 + 3^2 + 4^2} = \sqrt{26}$

Example 15.12: Determine the unit vector parallel to the resultant of two vectors

$$\vec{a} = 3\hat{i} + 2\hat{j} - 4\hat{k} \text{ and } \vec{b} = \hat{i} + \hat{j} + 2\hat{k}$$

Solution: The resultant vector $\vec{R} = \vec{a} + \vec{b} = (3\hat{i} + 2\hat{j} - 4\hat{k}) = (\hat{i} + \hat{j} + 2\hat{k})$

$$= 4\hat{i} + 3\hat{j} - 2\hat{k}$$

Magnitude of the resultant vector $\vec{R}$ is $|\vec{R}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$

$\therefore$ The unit vector parallel to the resultant vector

$$= \frac{\vec{R}}{|\vec{R}|} = \frac{1}{\sqrt{29}}(4\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$

Example 15.13: Find a unit vector in the direction of $\vec{r} - \vec{s}$

where $\vec{r} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{s} = 2\hat{i} - \hat{j} + 2\hat{k}$

Solution: $\vec{r} - \vec{s} = (\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} - \hat{j} + 2\hat{k})$

$$= -\hat{i} + 3\hat{j} - 5\hat{k}$$

$\therefore \ |\vec{r} - \vec{s}| = \sqrt{(-1)^2 + (3)^2 + (-5)^2} = \sqrt{35}$

$\therefore$ Unit vector in the direction of $(\vec{r}-\vec{s})$

$$= \frac{1}{\sqrt{35}}(-\hat{i}+3\hat{j}-5\hat{k}) = \frac{-1}{\sqrt{35}}\hat{i}+\frac{3}{\sqrt{35}}\hat{j}-\frac{5}{\sqrt{35}}\hat{k}$$

Example 15.14: Find a unit vector in the direction of $2\vec{a}+3\vec{b}$ where $\vec{a}=\hat{i}+3\hat{j}+\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}-\hat{k}$.

Solution : $2\vec{a}+3\vec{b} = 2(\hat{i}+3\hat{j}+\hat{k})+3(3\hat{i}-2\hat{j}-\hat{k})$

$$= (2\hat{i}+6\hat{j}+2\hat{k})+(9\hat{i}-6\hat{j}-3\hat{k})$$

$$= 11\hat{i}-\hat{k}$$

$$\therefore \quad |2\vec{a}+3\vec{b}| = \sqrt{(11)^2+(-1)^2} = \sqrt{122}$$

$\therefore$ Unit vector in the direction of $(2\vec{a}+3\vec{b})$ is $\frac{11}{\sqrt{122}}\hat{i}-\frac{1}{\sqrt{122}}\hat{k}$.

Example 15.15 : Show that the following vectors are coplanar :

$4\vec{a}-2\vec{b}-2\vec{c}$, $-2\vec{a}+4\vec{b}-2\vec{c}$ and $-2\vec{a}-2\vec{b}+4\vec{c}$ and $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors.

Solution : If these vectors be co-planar, it will be possible to express one of them as a linear combination of other two.

Let $\quad -2\vec{a}-2\vec{b}+4\vec{c} = x(4\vec{a}-2\vec{b}-2\vec{c})+y\,(-2\vec{a}+4\vec{b}-2\vec{c})$

where x and y are scalars,

Comparing the co-efficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ from both sides, we get

$$4x-2y=-2,\ -2x+4y=-2 \text{ and } -2x-2y=4$$

These three equations are satisfied by $x=-1, y=-1$ Thus,

$$-2\vec{a}-2\vec{b}+4\vec{c} = (-1)(4\vec{a}-2\vec{b}-2\vec{c})+(-1)(-2\vec{a}+4b-2\vec{c})$$

$\therefore$ Hence the three given vectors are co-planar.

Example 15.16 : Show that points A($2i - j + k$), B($i - 3j - 5k$), C($3i - 4j - 4k$) are the vertices of right angled triangle.

Solution : We have

$$AB = (1 - 2)i + (-3 + 1)j + (-5 - 1)k = -i - 2j - 6k$$

$$BC = (3 - 1)i + (-4 + 3)j + (-4 + 5)k = 2i - j + k$$

and $$CA = (2 - 3)i + (-1 + 4)j + (1 + 4)k = -i + 3j + 5k$$

we have $|AB|^2 = |BC|^2 + |CA|^2$.

Example 15.17 : If the points whose position vector are $3i - 2j - k$, $2i + 3j - 4k$, $-i + j + 2k$ and $4i + 5j + \lambda k$ are Co-planar, then show that $\lambda = \frac{-146}{17}$.

Solution :

$$OA = 3i - 2j - k$$

$$OB = 2i + 3j - 4k$$

$$OC = -i + j + 2k$$

$$OD = 4i + 5j + \lambda k$$

$\Rightarrow$ $$AB = -i + 5j - 3k$$

$$AC = -4i + 3j + 3k$$

$$AD = i + 7j + (\lambda + 1)k$$

If A, B, C, D are coplanar [AB AC AD] = 0

$$[AB\ .\ AC\ .\ AD] = \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix} = 0$$

$$-1(3\lambda + 3 - 21) - 5(-4\lambda - 4 - 3) - 3(-28 - 3) = 0$$

$$-3\lambda + 18 + 20\lambda + 35 + 93 = 0$$

$$17\lambda = -146$$

$$\boxed{\lambda = \frac{-146}{17}}$$

EXERCISE 15.4

1. Write the condition that $\vec{a}, \vec{b}$ and $\vec{c}$ are co-planar.
2. Determine the resultant vector $\vec{r}$ whose components along two rectangular Cartesian co-ordinate axes are 3 and 4 units respectively.
3. In the adjoining figure :

   $|OA| = 4, |OB| = 3$ and

   $|OC| = 5$. Express OP in terms of its component vectors.

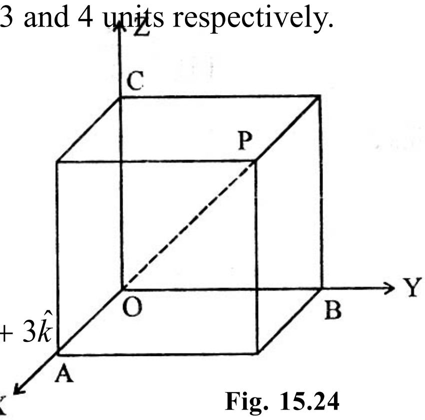


Fig. 15.24

4. If $\vec{r_1} = 4\hat{i} + \hat{j} - 4\hat{k}, \vec{r_2} = -2\hat{i} + 2\hat{j} + 3\hat{k}$

   and $\vec{r_3} = \hat{i} + 3\hat{j} - \hat{k}$

   then show that $|\vec{r_1} + \vec{r_2} + \vec{r_3}| = 7$.
5. Determine the unit vector parallel to the resultant of vectors :

   $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.
6. Find a unit vector in the direction of vector $3\vec{a} - 2\vec{b}$ where $\vec{a} = \bar{i} - \bar{j} - \bar{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$.
7. Show that the following vectors are co-planar :

   $3\vec{a} - 7\vec{b} - 4\vec{c}, \overrightarrow{3a} - 2\vec{b} + \vec{c}$ and $\vec{a} + \vec{b} + 2\vec{c}$ where $\vec{a}, \vec{b}$ and $\vec{c}$ are three non coplanar vectors.

15.10 SECTION FORMULA

Recall that the position vector of a point P is space with respect to an origin of reference O is $\vec{r} = \overrightarrow{OP}$.

In the following, we try to find the position vector of a point dividing a line segment joining two points in the ratio $m : n$ internally.

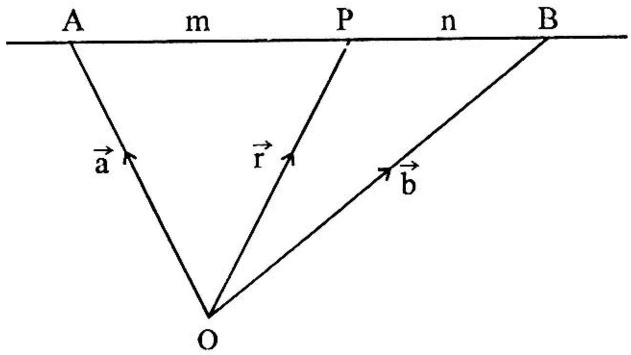


Fig. 15.25

Let A and B be two points and $\vec{a}$ and $\vec{b}$ be their position vectors w.r.t. the origin of reference O, so that $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Let P divide AB in the ratio $m : n$ so that

$$\frac{AP}{PB} = \frac{m}{n} \quad \text{(or)} \quad n\,\overrightarrow{AP} = m\,\overrightarrow{PB}$$

Since $n\,\overrightarrow{AP} = m\,\overrightarrow{PB}$, it follows that

$$n\,(\overrightarrow{OP} - \overrightarrow{OA}) = m\,(\overrightarrow{OB} - \overrightarrow{OP})$$

or $$(m+n)\overrightarrow{OP} = m\overrightarrow{OB} + n\,\overrightarrow{OA}$$

or $$\overrightarrow{OP} = \frac{m\overrightarrow{OB} + n\,\overrightarrow{OA}}{m+n}$$

or $$\vec{r} = \frac{m\vec{b} + n\,\vec{a}}{m+n}$$

where $\vec{r}$ is the position vector of P with respect to O.

Corollary 1 : If $\frac{m}{n} = 1 \Rightarrow m = n,$ then P becomes mid-point of AB

$\therefore$ The position vector of the mid-point of the join of two given points, whose position vectors are $\vec{a}$ and $\vec{b}$ is given by $\frac{1}{2}(\vec{a} + \vec{b})$.

Corollary 2 : The position vector P can also be written as

$$\vec{r} = \frac{\vec{a} + \frac{m}{n}\vec{b}}{1 + \frac{m}{n}} = \frac{\vec{a} + k\vec{b}}{1+k}$$

where $k = \frac{m}{n}$, $k \neq -1$

(ii) represents the position vector of a point which divides the join of two points with position vectors $\vec{a}$ and $\vec{b}$ in the ratio $k : 1$.

Corollary 3 : The position vector of a point P which divides AB in the ratio $m : n$ externally is

$$\vec{r} = \frac{n\vec{a} - m\vec{b}}{n-m}$$

[**Hint :** The division is in the ratio $-m : n$]

Example 15.16: Find the position vector of a point which divides the join of two points whose position vectors are given by $\vec{x}$ and $\vec{y}$ in the ratio 2 : 3 internally.

Solution: Let $\vec{r}$ be the position vector of the point.

$$\vec{r} = \frac{3\vec{x} + 2\vec{y}}{3+2} = \frac{1}{5}(3\vec{x} + 2\vec{y})$$

Example 15.17 : Find the position vector of mid-point of the line segment AB, if the position vectors of A and B are respectively, $\vec{x} + 2\vec{y}$ and $2\vec{x} - \vec{y}$.

Solution : Position vector of mid-point of AB

$$= \frac{(\vec{x} + 2\vec{y}) + (2\vec{x} - \vec{y})}{2}$$

$$= \frac{3}{2}\vec{x} + \frac{1}{2}\vec{y}$$

Example 15.18 : The position vectors of vertices A, B and C of ΔABC are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Find the position vector of the centroid of ΔABC.

Solution : Let D be the mid-point of side BC of ΔABC.

Let G be the centroid of Δ ABC.

Then G divides AD in the ratio 2 : 1

i.e. AG : GD = 2 : 1.

Now position vector of D is $\dfrac{\vec{b}+\vec{c}}{2}$

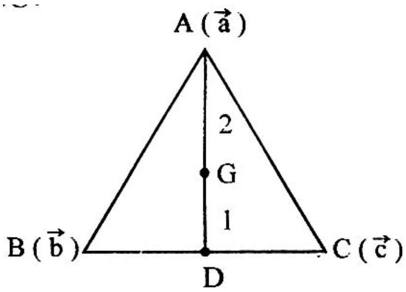


Fig. 15.26

$\therefore$ Position vector of G is $\dfrac{2\,\dfrac{\vec{b}+\vec{c}}{2}+1\,.\,\vec{a}}{2+1}$

$$= \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

EXERCISE 15.5

1. Find the position vector of the point C if it divides AB in the ratio

   (i) $\dfrac{1}{2}:\dfrac{1}{3}$ (ii) 2 : –3 given that the position vectors of A and B are $\vec{a}$ and $\vec{b}$ respectively.

2. Find the point which divides the join of $P(\vec{p})$ and $Q(\vec{q})$ internally in the ratio 3 : 4.

3. CD is trisected at points P and Q. Find the position vectors of points of trisection, if the position vectors of C and D are $\vec{c}$ and $\vec{d}$ respectively.

4. Using vectors, prove that the medians of a triangle are concurrent.

5. Using vectors, prove that the line segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.

15.11 PRODUCT OF VECTORS

In Section 15.9, you have multiplied a vector by a scalar. The product of vector with a scalar gives us a vector quantity. In this section we shall take the case when a vector is multiplied by another vector. There are two cases:

(i) When the product of two vectors is a scalar, we call it a scalar product, also dot product corresponding to the symbol '•' used for this product.

(ii) When the product of two vectors is a vector, we call it a vector product, also known as cross product corresponding to the symbol ' $\times$' used for this product.

15.12 SCALAR PRODUCT OF THE VECTORS

Let $\vec{a}$ and two vectors and θ be the angle between them. The scalar product, denoted by $\vec{a}.\vec{b}$, is defined by

$$\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}|\cos\theta$$

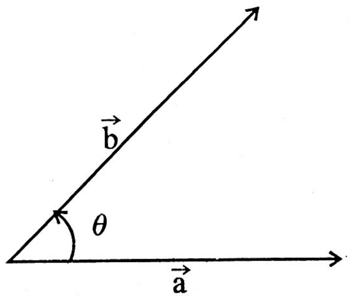


Fig. 15.27

Clearly, $\vec{a}.\vec{b}$ is a scalar as $|\vec{a}|$, $|\vec{b}|$ and $\cos\theta$ are all scalars.

Remarks

1. If $\vec{a}$ and $\vec{b}$ are like vectors, then $\vec{a}.\vec{b} = ab\cos\theta = ab$, where a and b are magnitudes of $\vec{a}$ and $\vec{b}$.
2. If $\vec{a}$ and $\vec{b}$ are unlike vectors, then $\vec{a}.\vec{b} = ab\cos\pi = -ab$.
3. Angle θ between the vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$
4. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ and $\vec{a}.(\vec{b}+\vec{c}) = (\vec{a}.\vec{b} + \vec{a}.\vec{c})$
5. $n(\vec{a}.\vec{b}) = (n\vec{a}).\vec{b} = \vec{a}.(n\vec{b})$ where n is any real number.
6. $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$ and $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$ as $\hat{i}, \hat{j}$ and $\hat{k}$ are mutually perpendicular unit vectors.

Vector equations of Line and plane

(i) The vector equation of the straight line passing through the point A($\vec{a}$) and parallel to the vector $\vec{b}$ is $r = a + t\vec{b}$ $t \in \mathbf{R}$.

(ii) Cartesian equation for the line passing through A(x_1 y_1 z_1) and parallel to the vector $\vec{b} = li + mj + nk$ is $\dfrac{x - x_i}{l} = \dfrac{y - y_i}{m} = \dfrac{z - z_i}{n}$

(iii) The vector equation of the line through the points A(a) and B(b) is $r = (1 - t)\vec{a} + t\vec{b},\ t \in \mathbf{R}$.

(iv) The equation of the plane passing through the point A(a) and parallel to the vectors b and c is $r = a + tb + sc,\ t, s \in \mathbf{R}$.

(v) The equation of the plane passing through the points A(a), B(b) and parallel to the vector C is $r = (1 - t)a + tb + sc \quad t, s \in \mathbf{R}$.

(vi) The equation of the plane passing through the points A(a), B(b) and C(c) is

$$\vec{r} = (1 - t - s)a + tb + sc \quad \text{where} \quad t, s \in \mathbf{R}.$$

Example 15.19 : If $\vec{a} = 3\hat{i} + 2\hat{j} - 6\hat{k}$ and $\vec{b} = 4\hat{i} - 3\hat{j} + \hat{k}$, find $\vec{a}, \vec{b}$.

Also find angle between $\vec{a}$ and $\vec{b}$.

Solution: $\vec{a} \cdot \vec{b} = (3\hat{i} + 2\hat{j} - 6\hat{k}) \cdot (4\hat{i} - 3\hat{j} + \hat{k})$

$$= 3 \times 4 + 2 \times (-3) + (-6) \times 1$$

$$[\because \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \text{ and } \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0]$$

$$= 12 - 6 - 6$$

$$= 0$$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

Then $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = 0$

$\therefore \quad \theta = \dfrac{\pi}{2}$.

15.13 VECTOR PRODUCT OF TWO VECTORS

Before we define vector product of two vectors, we discuss below right handed and left handed screw and associate it with corresponding vector triad.

15.13.1 Right Handed Screw

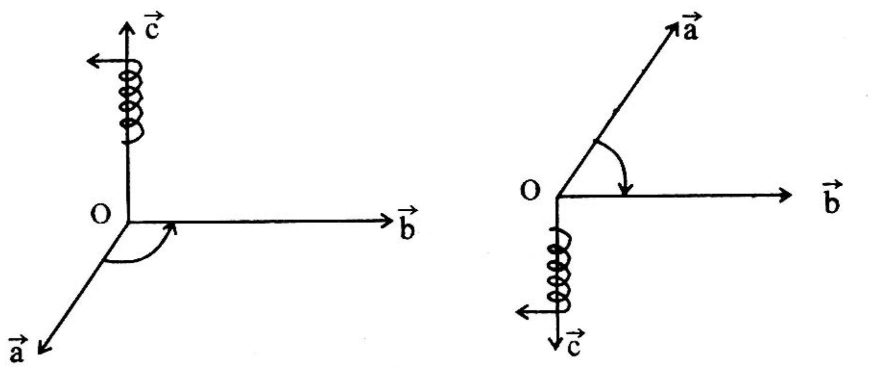
If a screw is taken and rotated in the anticlockwise direction, it translates towards the reader. It is called ***right handed screw***.

15.13.2 Left handed Screw

If a screw is taken and rotated in the clockwise direction, it translates away from the reader. It is called a left handed screw.

Now we associate a screw with given ordered vector triad

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors whose initial point is O.



(i) (ii)

Fig. 15.28

Now if a right handed screw at O is rotated from $\vec{a}$ towards $\vec{b}$ through an angle < 180°, it will undergo a translation along $\vec{c}$ [Fig. 15.28(i)]

Similarly if a left handed screw at O is rotated from $\vec{a}$ to $\vec{b}$ through an angle <180°, it will undergo a translation along $\vec{c}$ [Fig. 15.28 (ii)]. This time the direction of translation will be opposite to the first one.

Thus an ordered vector triad $\vec{a}$, $\vec{b}$, $\vec{c}$ is said to be right handed or left handed according as the right handed screw translated along $\vec{c}$ or opposite to $\vec{c}$ when it is rotated through an angle less than 180°.

15.13.3 Vector product

Let $\vec{a}$ and $\vec{b}$ be two vectors and θ be the angle between them such that $0 < \theta < \pi$.

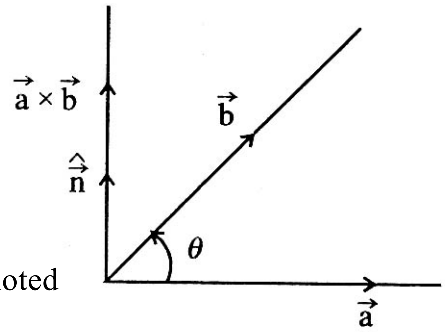


Fig. 15.29

The vector product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and is defined as the vector

$\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}|\ \sin\theta\ \hat{n}$ where $\hat{n}$ is the unit.

vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed triad of vectors.

Remark :

1. Clearly $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $\vec{a} \times \vec{b} = \vec{0}$
3. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$
4. $\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = i,\ \hat{k} \times \hat{i} = \hat{j}$ and

   $\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$
5. If $\vec{a} \times \vec{b} = \vec{0}$ then either $\vec{a} = 0$ or $\vec{b} = 0$ or $\vec{a} \| \vec{b}$.
6. θ is not defined if any or both of $\vec{a}$ and $\vec{b}$ are $\vec{0}$. As $\vec{0}$ has no direction and so $\hat{n}$ is not defined. In this case $\vec{a} \times \vec{b} = \vec{0}$.
7. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

EXERCISE 15.6

1. Find the angle between two vectors.

   (a) $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $2\hat{i} + 3\hat{j} + 4\hat{k}$

   (b) $2\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$

15.14 PROPERTIES OF DOT PRODUCT

15.14 Definition : Let a and b be two vectors. The scalar (or dot) product of a and b written a $a \,.\, b$ is defined to the scalar zero. If one of a, b is the zero vector, otherwise $a| \; |b| \cos\theta$ where θ is the angle between the vectors a and b.

Note: (i) $a \,.\, b$ is a scalar

(ii) $a \,.\, a = |a|\,|a| \cos\theta = |a|^2$ and $a.a$ is generally denoted by a^2.

15.14.1 Let *a*, *b* be two vectors. Then

i) $a \,.\, b = b \,.\, a$ (commutative law)

ii) $(la) \,.\, b = a \,.\, (lb) = l(a \,.\, b)\; l \in R$

iii) $(la) \,.\, (m.b) = lm\,(a \,.\, b)\;\; l, m \in R$

iv) $(-a) \,.\, (b) = a \,.\, (-b) = -(a \,.\, b)$

v) $(-a) \,.\, (-b) = a \,.\, b$

Note: (i) Expression for scalar (dot) product interms of i, j, k, we oberseve that, if i, j, k are mutually perpendicular unit vectors, then

$i \,.\, i = j \,.\, j = k \,.\, k = 1, \;\; i \,.\, j = 0, \;\; j \,.\, k = 0, \;\; k \,.\, i = 0.$

(ii) Let (i, j, k) be the Orthogonal Unit triad let

$a = a_1 i + a_2 j + a_3 k$, and $b = b_1 i + b_2 j + b_3 k$.

Then $a \,.\, b = a_1 b_1 + a_2 b_2 + a_3 b_3$

(iii) If θ is the angle between two non-zero vectors a and b then, from the definition of $a \,.\, b$, we have

$\theta = \cos^{-1}\left(\dfrac{a.b}{|a|\,|b|}\right)$ and in particular if $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$. then $\theta = \cos^{-1}\left(\dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\;\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$

(iv) a, b are perpendicular to each other If and only if $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$.

15.14.2 Trigonometrical Theorems - Through Vector Methods (applilcation)

15.14.2.1 Angle in a Semicircle is a right angle

Let AB be a diameter of a circle with centre O

Let OA = a so that OB = $-a$

Let P ne a point on the circle OP = r

$$PA \cdot PB = (a - r) \cdot (-a - r) = -(a^2 - r^2) = 0$$

$(|a| = |r| = \text{Radius})$

$\angle PAB = 90^0$

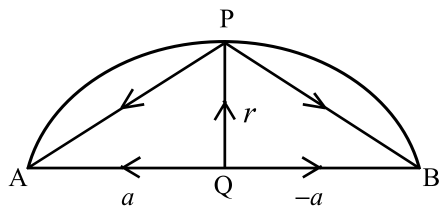


Fig. 15.30

Note : In ΔABC, let a, b, c be the sides opposite to the vertices Am B and C respectively. Then the following are valid

i) $a^2 = b^2 + c^2 - 2abc \cos A$ ii) $a = b \cos C + c \cos B$.

15.14.3 (Parallelogram Law)

In a parallelogram, the sum of the squares of the lengths of the diagonals is equal to sum of the squares of the lengths of its sides .

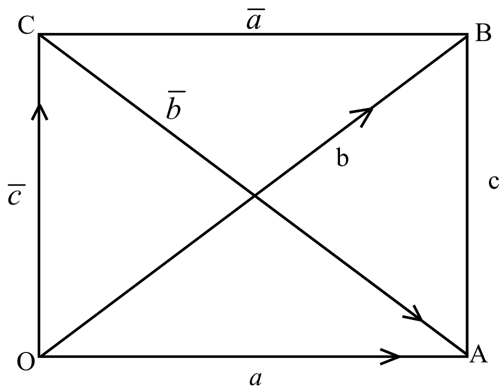


Fig. 15.31

Proof : Let OABC be a parallelogram in which OB and CA are diagonals.

Let OA = a and OC = c

$\therefore$ OB = $a + c$ and CA = $a - c$

$$OB^2 + CA^2 = |a + c|^2 + |a - c|^2 = (a^2 + 2ac + c^2) + (a^2 - 2ac + c^2)$$

$$= 2|a|^2 + 2|c|^2$$

$$= OA^2 + AB^2 + CB^2 + OC^2 \quad (OA = BC \text{ and } OC = AB).$$

15.15 VECTOR EQUATION OF A PLANE - NORMAL FORM

The equation of the plane whose perpendicular distance from the Origin is P and whose unit normal drawn from the origin towards the plane is n, is $r . n = $ P

Note :

(i) If the plane σ passes through the origin 'O' then $p = 0$ and hence the equation σ is $r . n = 0$

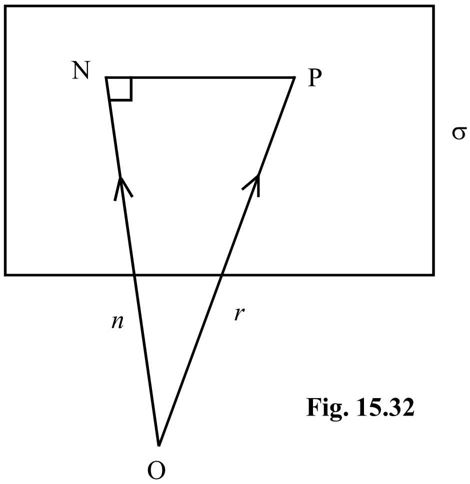


Fig. 15.32

(ii) If (l, m, n) are the direction cosines of the normal to the plane s and $p(x, y, z)$ is any point than $p \in \sigma \Leftrightarrow r.n = p$

$$\Leftrightarrow (xi + yj + zk) . (li + mj + nk) = 0$$

$\Leftrightarrow$ $lx + my + nz = $ P. This equation of the plane is called 'normal form' in cartesian coordinates.

(iii) Vector equation of the plane passing through the point A(a) and perpendicular to a vector n is $(r . a) . n = 0$.

15.16 VECTOR EQUATION OF A SPHERE AND ANGLE BETWEEN TWO PLANES

Definition : Let C be a fixed point in the space and 'a' a non-negative real number. Then, the set of all points P in the space such that the distance CP is equal to 'a' is called the sphere with centre at the point C and radius 'a'. A sphere with radius zero is called point sphere.

i) The vector equation of the sphere with centre at C whose position vector is Cand radius 'a' is $|r - c| = a$, equivalently $r^2 - 2(r \cdot c) + c^2 = a^2$

ii) If the origin of reference lies on the sphere (ie OC = a), then the equation of the sphere is $r^2 - 2r \cdot c = 0$.

iii) If the centre of the sphere is the Origin of reference (ie C = 0), then the equation of the sphere is $r^2 = a^2$ or $|r| = a$.

iv) Cartesian form of the equation of sphere is $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = a^2$

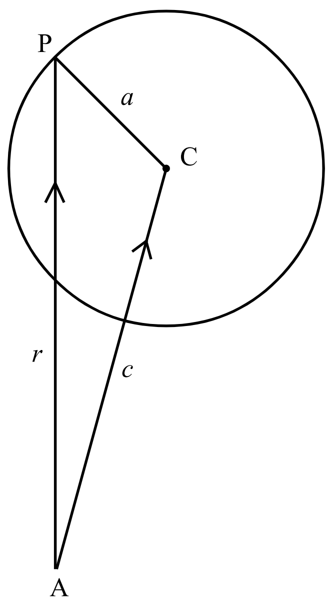


Fig. 15.33

v) The verctor equation of a sphere with A(a) and B(b) as the end points of a diameter is $(r - a) \cdot (r - b) = 0$ or equivalently $r^2 - r.(a + b) + a \cdot b = 0$

15.16.1 Angle between two planes

Let σ_1 and σ_2 be two planes and n_1, n_2 be unit normals for σ_1 and σ_2 respectively. Then the angle between σ_1 and σ_2 is defined to be the angle between their normals n_1 and n_2 as their respective unit normals, then $\cos\theta = n_1 . n_2$.

Example 15.20 : If $|a| = 11$ $|b| = 23$ and $|a - b| = 30$, then find the angle bwetween the vectors a, b and $|a+b|$.

Sol : By hypothesis $|a - b| = 30$

Let θ be the angle between a and b

$$|a - b|^2 = 900$$

$$a^2 - 2a.b \cos\theta + b^2 = 900$$

$$|\,2\,| \; -2 + \; 11.23 \cos\theta + 529 = 900$$

$$650 - 506 \cos\theta = 900$$

$$506 \cos\theta = 650 - 900 = -250$$

$$506 \cos\theta = -250$$

$$\cos\theta = -250$$

$$\cos\theta = \frac{-250}{506} = -\frac{125}{253}$$

$$\theta = \pi - \cos^{-1}\left(\frac{125}{253}\right)$$

$$|a + b|^2 = a^2 + 2a \,.\, b \cos\theta + b^2$$

$$= |2| + 2 + 11 + 23 \cos\theta + 529$$

$$= 121 + 2 \times 11 \times 23\left(-\frac{125}{253}\right) + 529$$

$$|a + b|^2 = 400$$

$$\Rightarrow |a + b| = 20.$$

Example 15.21: If $a = i + 2j + 3k, b = 3i + j + 2k$ then find i) angle between a and b ii) $a.b$ value.

Sol: $a \,.\, b = (i + 2j + 3k)\,(3i + j + 2k) = 3 + 2 + 6 = 11$

$$|a| = \sqrt{1^2 + 2^2 + 3^2}\,; \; |b| = \sqrt{3^2 + 1^2 + 2^2}$$

$$= \sqrt{14} \qquad ; \qquad = \sqrt{14}$$

$$\cos\theta = \frac{a.b}{|a||b|} = \frac{11}{\sqrt{14}\sqrt{14}} = \frac{11}{14} \Rightarrow \theta = \cos^{-1}\left(\frac{11}{14}\right).$$

Example 15.22: If the vectors $\lambda i - 3j + 5k$ and $2\lambda i - \lambda j - k$ are perpendicular to each other find λ.

Sol : $(\lambda i - 3j + 5k) \,.\, (2\lambda i - \lambda j - k) = 0$

$$\therefore\ 2\lambda^2 + 3\lambda - 5 = 0$$

$$(2\lambda + 5)(\lambda - 1) = 0 \ \therefore\ \lambda = \frac{-5}{2} \text{ or } 1.$$

Example 15.23 : Show that the points $2i - j + k,\ i - 3j - 5k,\ 3i - 4j - 4k$ are the vertices of a right angled triangle. Also find the other angles.

Sol : Let the given points be A, B and Cresoectively

Then $AB = -i - 2j - 6k,\ BC = 2i - j + k$

$CA = -i + 3j + 5k$

$\Rightarrow$ $BC.\,CA = -2 - 3 + 5 = 0$

$\Rightarrow$ $\angle C = 90^0$

$$\cos B = \frac{BC.BA}{|BC|\,|BA|} = \sqrt{\frac{6}{41}}$$

$$\cos A = \frac{AB.AC}{|AB|\,|AC|} = \sqrt{\frac{35}{41}}$$

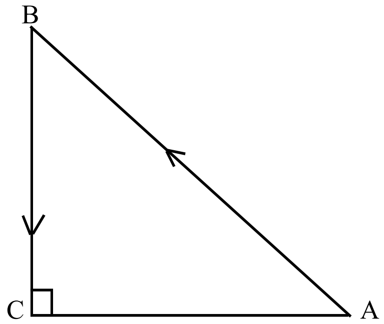


Fig. 15.34

Example 15.24 : The vectors $AB = 3i + 2j + 2k$ and $AD = i - 2k$ represent adjacent sides of a parallelogram ABCD. Find the angle between the diagonals.

Solution: From the figure Diagonal

$AC = AB + BC$

$= (3i - 2j + 2k) + (i - 2k)$

$AC = 4i - 2j$

Diagonal $BD = -2i + 2j - 4k$

Let θ be the angle between AC and BD

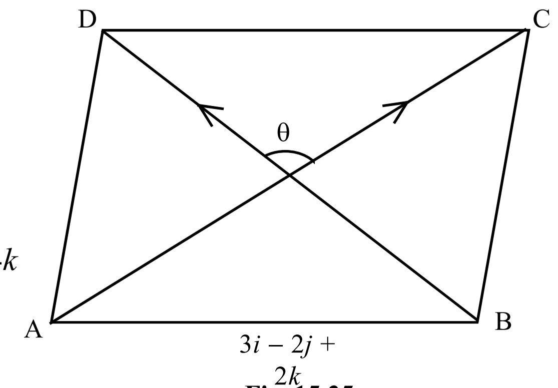


Fig. 15.35

$$\cos\theta = \frac{AC.BD}{|AC|\,|BD|} = \frac{-8-4}{\sqrt{20}\sqrt{24}} = \frac{-\sqrt{3}}{\sqrt{10}}$$

Example 15.25: Find the equation of the sphere with the line segment joining the points A$(1, -3, -1)$, and B$(2, 4, 1)$ as diameter.

Sol: Let $a = i - 3j - K$ and $b = 2i + 4j + K$ be the position vectors of A and B respectively. Then the equation of the sphere with AB as a diameter is

$$(r - a) \cdot (r - b) = 0$$

$$r^2 - r \cdot (a + b) + a \cdot b = 0$$

$$r^2 - r \cdot (3i + j) - 11 = 0$$

$\therefore$ The cartesian equation is $(x - 1)(x - 2) + (y + 3)(y - 4) + (z + 1)(z - 1) = 0$

$$x^2 + y^2 + z^2 - 3x - y - 11 = 0.$$

EXERCISE 15.7

1. If the vectors $2i + \lambda j - k$, $4i - 2j + 2k$ are perpendicular to each other then find λ value.
2. Find the angle between the planes $r \cdot (2i - j + 2k) = 3$ and $r \cdot (3i + 6j + k) = 4$.
3. Find the vector equation and its cartesian form of the sphere with the points $a = 3i + 4j - 2k$ and $b = -2i - j$ an the end points of a diameter.

15.17 VECTOR PRODUCT (CROSS PRODUCT) OF TWO VECTORS AND PROPERTIES

15.17.1 Definition

Let a and b be vectors. The cross (or vector) product of a and b, written as $a \times b$ (read as a cross b) is defined to be the null vector 0, if one of a, b is the defined to be the vector $a \times b$ is $(|a|\,|b| \sin\theta)n$ where θ is the angle between a and b and n is the unit vector perpendicular to both a and b such that (a, b, n) is a right handed system.

15.17.2 If a, b and c are vectors, then

i) $a \times (b + c) = a \times b + a \times c$

ii) $(a + b) \times c = a \times c + b \times c$ are under distributive law.

If (i, j, k) is an orthogonal triad, then from the definition of the cross product of two vectors, it is easy to see that

(i) $i \times i = j \times j = k \times k \times 0$

(ii) $i \times j = k, j \times k = i, k \times i = j$

15.18 VECTOR PRODUCT IN (*i, j, k*) SYSTEM

(i) Let $a = a_1 i + a_2 j + a_3 k$ and $b = b_1 i + b_2 j + b_3 k$ then

$a \times b = (a_2 b_3 - a_3 b_2)\, i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1) k$

(ii) The above formula for $a \times b$ can now be expressed as

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(iii) If $a = a_1 i + a_2 j + a_3 k, b = b_1 i + b_2 j + b_3 k$ and θ is the angle between 'a' and 'b' then

$$\sin\theta = \frac{\sqrt{\Sigma (a_2 b_3 - a_3 b_2)^2}}{\sqrt{\Sigma a_1^2}\,\sqrt{\Sigma b_1^2}}$$

(iv) For any two vectors a and b

$|a \times b|^2 = (a \,.\, a)(b \,.\, b) - (a \,.\, b)^2 = a^2 b^2 - (a \,.b)^2$.

(v) If a and b are non-collinear vectors, then unit vectors perpendicular to both a and b are $\dfrac{\pm(a \times b)}{|a \times b|}$.

MODULE - III
Dimensional
Geometry Vectors

Notes

15.19 VECTOR AREAS

15.19.1 Definition

Let D be a plane Region bounded by closed curve C. Let P_1, P_2, P_3 be three points on C (Taken in this order) let n be the unit vector perpendicular to the region D such that, from the side of n, the points P_1, P_2 and P_3 are in anticlock sense. If A is the are of the region D. Then an is called the vector area of D.

(i) The vector are of ΔABC is

$$\frac{1}{2}(AB\times AC) = \frac{1}{2}(BC\times BA) = \frac{1}{2}(CA\times CB)$$

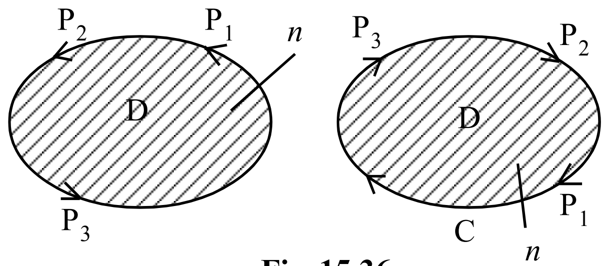


Fig. 15.36

(ii) If a, b, c are the position vectors of the vertices A, B, C (described in counter clock sense) of ΔABC, then the vector area of ΔABC is $\frac{1}{2}(b\times c + c\times a + a\times b)$ and its area is $\frac{1}{2}|b\times c + c\times a + a\times b|$.

15.19.2 (Vector area of a parallelogram)

Let ABCD be a parallelogram with vertices A, B, C and D described in counter clocksense. Then, the vector area of ABCD in terms of the diagonals AC and BD is $\frac{1}{2}(AC\times BD)$.

Example 15.26: If $a = 2i - 3j + 5k$, $b = -i + 4j + 2k$ then find $a \times b$ and unit vector perpendicular to both a and b.

Sol : $a \times b = \begin{vmatrix} i & j & k \\ 2 & -3 & 5 \\ -1 & 4 & 2 \end{vmatrix} = -26i - 9j + 5k$

the unit vector perpendicular to both a and b

$$= \pm \frac{a \times b}{|a \times b|} = \pm \frac{1}{\sqrt{782}}(-26i - 9j + 5k)$$

Example 15.27: If $a = i + 2j + 3k$ and $b = 3i + 5j - k$ are two sides of a triangle, then find its area.

Sol : Area of the triangle is equal to half of the area of the parallelogram for which a and b are adjacent sides

$$= \frac{1}{2}|a \times b| \text{ but}$$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 5 & -1 \end{vmatrix} = -17i + 10j - k$$

Area of the triangle $= \frac{1}{2}|a \times b| = \frac{\sqrt{390}}{2}$.

Example 15.28: Let $a = 2i - j + k,\ b = 3i + 4j - k$ If θ is the angle between a and b, then find 'sin θ'

Sol: $a \times b = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -3i + 5j + 11k$

and $|a| = \sqrt{6},\ |b| = \sqrt{26},\ |a \times b| = \sqrt{155}$

Now $\sin\theta = \frac{|a \times b|}{|a||b|} = \frac{\sqrt{155}}{\sqrt{6}\sqrt{26}} = \sqrt{\frac{155}{156}}$

EXERCISE 15.8

1. If $a = 2i - j + k,\ b = i - 3j - 5k$ then find $|a + b|$.
2. Find the unit vector perpendicular to both $i + j + k$ and $2i + j + 3k$.
3. Find the area of the parallelogram having $a - 2j - k$ and $b = -i + k$ as adjacent sides.
4. Find the area of the triangle having $3i + 4j$ and $-5i + 7j$ as two of its sides.
5. Let a and b be vectors, satisfying $|a| = |b| = 5$ and $(a, b) = 45^0$. Find the area of the triangle having $a - 2b$ and $3a + 2b$ as two of its sides.

15.20 SCALAR TRIPLE PRODUCT

15.20.1 Definition : Let a, b and c be three vectors. We call $(a \times b).c$, the scalar triple product of a, b and c and denebe this by $[abc]$. Usually $[a\ b\ c]$ is called box $[a\ b\ c]$.

15.20.2 Let a, b and c be three non-coplanar vector and OA = a, OB = b, OC = c. Let V be the volume of the parallelopiped with OA, OB and OC as coterminus edges. Then

i) $(a \times b).c = \text{V}$, if (a, b, c) is a right handed system

ii) $(a \times b).c = -\text{V}$, if (a, b, c) is a left handed system.

15.20.3 For any three vectors a, b and c

$(a \times b).c = (b \times c).a = (c \times a).b$ that is

$[a\ b\ c] = [b\ c\ a] = [c\ a\ b]$

15.20.4 If a, b, c are any three vectors, then $(a \times b).c = a\,.\,(b \times c)$ (That is in a scalar tripe product, the operations dot and cross can be inter angled) (dot product is commutative)

15.20.5 If a, b, c are three non zero vectors such that no two are collinear, then $[a\ b\ c] = 0$ If and only If a, b and c are coplanar.

15.20.6 For distinct points A, B, C and D are coplanar if and only if [AB AC AD] = 0

15.20.7 Let (i, j, k) be orthogonal triad of unit vectors which is a right handed system

Let $a = a_1 i + a_2 j + a_3 k,\ b = b_1 i + b_2 j + b_3 k$ and $c = c_1 i + c_2 j + c_3 k$

$$\text{Then } [abc] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

15.20.8 Let $a = a_1 i + a_2 j + a_3 k,\ b = b_1 i + b_2 j + b_3 k, c = c_1 i + c_2 j + c_3 k$. Then

a, b, c are coplanar if and if only if $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

15.20.9 The volume of a tetrahedrom with a, b and c as coterminus edges is $\frac{1}{6}\left\|[a\,b\,c]\right\|$

15.20.10 The volume of the tetrahedron whose vertices are A, B, C and D is $\frac{1}{6}\left\|[\mathrm{DA}\ \ \mathrm{DB}\ \ \mathrm{DC}]\right\|$

15.21 VECTOR EQUATION OF A PLANE IN DIFFERENT FORMS, SKEW LINES, SHORTEST DISTANCE BETWEEN TWO SKEW LINES

15.21.1 The vector equation of a plane passing through the point A(a) and parallel to two non-collinear vectors b and c is a, $[\,r\,b\,c] = [a\,b\,c]$

15.21.2 Vector equation of the plane passing through points A(a), B(b) and parallel to the vector c is

$$[\,r\,b\,c] + [\,r\,c\,a] = [a\,b\,c]$$

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15.21.3 Vector equation of the plane passing through three non-collinear points A(a), B(b) and C(c) is

$$[\, r\; b\; c\,] + [\, r\; c\; a\,] + [\, r\; a\; b\,] = [\, a\; b\; c\,]$$

15.21.4 The equation of the plane containing the line $r = a + tb$, $t \in$ R are perpendicular to the plane $r\,.\,c = q$ is $[r\; b\; c] = [\, a\; b\; c]$

15.21.5 Definition (Skew lines) :

Two lines in the space are said to be skew lines, if there is not plane containing both the lines.

15.21.6 Let L_1 and L_2 be skew lines whose equations are $r = a + tb$ and $r = c + sd$, respectively. Then the shortest distance between L_1 and L_2 is $\dfrac{|(a-c).(b\times d)|}{|b\times d|}$

15.22 VECTOR TRIPLE PRODUCT SOME RESULTS

Definition : a, b, c are three vectors, then $(a \times b)\times c$ is called the vector. Triple product or vector product of three vectors. In this section we study some properties of the vector product $(a \times b)\times c$ of three vectors a, b, c.

15.22.1 Let a, b, c be three vectors, then

(i) $(a\times b)\times c = (a\,.\,c)b - (b\,.\,c)a$

(ii) $a \times (b \times c) = (a\,.\,c)b - (a\,.\,b)c$

in general the vector product of three vectors is not associative

(iii) If a, b are non-collinear vectors a and b is perpendiculat to neither a nor to c then $(a \times b)\times c = a \times (b \times c) \Leftrightarrow$ vectors a and c are collinear.

15.22.2 If b is vector, and b is $\perp r$ to a, c then

$$(a \times b)\times c = a \times (b\times c).$$

15.23 SCALAR AND VECTOR PRODUCTS OF FOUR VECTORS

15.23.1 For any four vectors a, b, c and d

i) $(a \times b) \,.\, (c \times d) = \begin{vmatrix} a.c & a.d \\ b.c & b.d \end{vmatrix}$

ii) $(a \times b)^2 = (a \times b) \,.\, (a \times b) = a^2 b^2 - (a\,.b)^2.$

15.23.2 For any four vectors a, b, c and d

$$(a \times b) \times (c \times d) = [a\ c\ d]b - [b\ c\ d]a$$

15.23.3 If a, b, c are non-coplanar vectors and 'r' is any vector, then

$$r = \frac{[bcr]}{[abc]}a + \frac{[car]}{[abc]}b + \frac{[abr]}{[abc]}c$$

Example 15.29 : If the vectors $a = 2i - j + k$, $b = i + 2j - 3k$ and $c = 3i + pj + 5k$ are coplanar, then find p.

Sol : a, b, c are coplanar if and only if $[a\ b\ c] = 0$

$$\Rightarrow \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & p & 5 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(10 + 3p) + 1(5 + 9) + (p - 6) = 0$$

$$\Rightarrow \quad 20 + 6p + 14 + p - 6 = 0$$

$$\Rightarrow \quad p = -4$$

Example 15.30: Find the volume of the parallelo piped with coterminus edges $2i - 3j$, $i + j - k$, and $3i - k$.

Sol : $[a\ b\ c] = \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$

$$= 2(-1 - 0) + 3(-1 + 3)$$

$$= -2 + 6 = 4$$

$\therefore$ Volume $= |[a\ b\ c]| = 4$

Example 15.31 : Show that the vectors $a - 2b + 3c$, $-2a + 3b - 4c$ and $a - 3b + 5c$ are coplanar, where a, b, c are non coplanar vectors.

Sol : Let $\alpha = a - 2b + 3c$, $\beta = -2a + 3b - 4c$ and $\gamma = a - 3b + 5c$

$$[\alpha\beta\gamma] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} [abc]$$

But $\begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 1(15-12) + 2(-10+4) + 3(6-3)$

$$= 3 - 12 = 9$$
$$= 0.$$

$\therefore [\alpha\beta\gamma] = 0 \quad \therefore$ α, β, γ are coplanar vectors.

Example 15.32 : Let $a = i + j + k$, $b = 2i - j + 3k$, $c = i - j$ and $d = 6i + 2j + 3k$ Express d, interms of $b \times c$, $c \times a$ and $a \times b$.

Sol: $[abc] = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & 0 \end{vmatrix}$

$$= 1(0 + 3) - 1(0 - 3) + 1(-2 + 1) = 5$$

$$d \cdot a = (6i + 2j + 3k) \cdot (i + j + k) = 6 + 2 + 3 = 11$$

$$d \cdot b = 19; \qquad d.c = 4$$

$$d = x(b \times c) + y(c \times a) + z(a \times b)$$

$$x = \frac{d.a}{[abc]},\ y = \frac{d.b}{[abc]},\ z = \frac{a.c}{[abc]}$$

$$x = \frac{11}{5};\ y = \frac{19}{5};\ z = \frac{4}{5}$$

$$\therefore\ d = \frac{11}{5}(3i + 3j - k) + \frac{19}{5}(-i - j + 2k) + \frac{4}{5}(4i - j - 3k).$$

Example 15.33: Find the shortest distance between the skew lines $r = (6i + 2j + 2k) + t(i - 2j + 2k)$ and $r = (-4i - k) + s(3i - 2i - 2k)$.

Sol: The first line passes through the point A(6, 2, 2) and is parallel to the vector $b = (i - 2j + 2k)$. Second line passes through the point C(−4, 0, −1) and is parallel to the vector $d = 3i - 2j - 2k$.

Shortest distance = $\dfrac{|[Ac\ b\ d]|}{b \times d}$

$$[Ac\ b\ d] = \begin{vmatrix} -10 & -2 & -3 \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = -108$$

$$b \times d = \begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = 8i + 8j + 4k$$

$|b \times d| = 12$

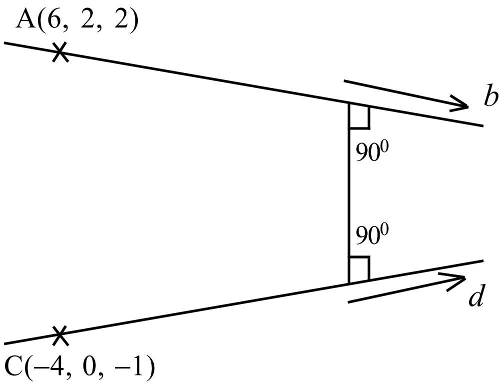

Fig. 15.37

Shortest Distance between the skew lines $= \dfrac{|[Ac\ b\ d]|}{|b \times d|} = \dfrac{108}{12} = 9$.

EXERCISE 15.9

1. Find the volume of the parallelopiped having coterminus edges $i + j + k$, $i - j$ and $i + 2j + k$.

2. Compute $[i - j\ \ j - k\ \ k - i]$.

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3. Simplify $(i - 2j + 3k) \times (2i + j - k) . (j + k)$

4. Let a, b and c be non-coplanar vector if $[2a - b + 3c,\ a + b - 2c,\ a + b - 3c] = \lambda[abc]$ then find λ.

5. If a, b and c are non-coplanar vectors, then prove that the four points with position vectors $2a + 3b - c$, $a - 2b + 3c$, $3a + 4b - 2c$ and $a - 6b + 6c$ are coplanar.

6. If A = (1, –2, –1) B = (4, 0, –3), C = (1, 2, –1) and D = (2, –4, –5) Find the distance between AB and CD.

7. If $a = i - 2j - 3k$, $b = 2i + j - k$ and $c = i + 3j - 2k$ verify that $a \times (b \times c) \neq (a \times b) \times c$

8. If $a = i - 2j + 3k$, $b = 2i + j + k$, $c = i + j + 2k$ then find $|(a \times b) \times c|$, $|a \times (b \times c)|$.

KEY WORDS

- A physical quantity which can be represented by a number only is called a scalar.
- A quantity which has both magnitude and direction is called a vector.
- A vector whose magnitude is 'a' and direction from A to B can be represented by $\overrightarrow{AB}$ and its magnitude is denoted by $\overrightarrow{AB} = a$.
- A vector whose magnitude is equal to the magnitude of another vector $\vec{a}$ but of opposite direction is called negative of the given vector and is denoted by $-\vec{a}$.
- A unit vector is of magnitude unity. Thus, a unit vector parallel to $\vec{a}$ is denoted by $\hat{a}$ `and is equal to $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

- A zero vector, denoted by $\vec{0}$ is of magnitude 0 while it has no definite direction
- Unlike addition of scalars, vectors are added in accordance with triangle law of addition of vectors and therefore, the magnitude of sum of two vectors is always less than or equal to sum of their magnitudes.
- Two or more vectors are said to be collinear if their supports are the same or parallel.
- Three or more vectors are said to be coplanar if their supports are parallel to the same plane or lie on the same plane.
- If $\vec{a}$ is a vector and x is a scalar, then $x\vec{a}$ is a vector whose magnitude is $|x|$ times the magnitude of and whose direction is the same or opposite to that of $\vec{a}$ depending upon $x > 0$ or $x < 0$.
- Any vector co-planar with two given non-collinear vectors is expressible as their linear combination.
- Any vector in space is expressible as a linear combination of three given non-coplanar vectors.
- The position vector of a point that divides the line segment joining the points with position vectors $\vec{a}$ and $\vec{b}$ in the ratio of $m : n$ internally/ externally are given by $\dfrac{n\vec{a} + m\vec{b}}{m+n}, \dfrac{n\vec{a} - m\vec{b}}{m-n}$ respectively.
- The position vector of mid-point of the line segment joining the points with position vectors $\vec{a}$ and $\vec{b}$ given by $\dfrac{\vec{a} + \vec{b}}{2}$.
- The scalar product of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$.

- The vector product of two vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}| \sin\theta\ \hat{n}$, where θ,is the angle between $\vec{a}$, $\vec{b}$ and $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$

- If a, b an two vectors, then $a\,.\,b = b\,.\,a$

- $i\,.\,i = j\,.\,j = k\,.\,k = 1;\ \ i.j = j.k = k.i = 0$

- $a = a_1 i + a_2 j + a_3 k;\ \ b = b_1 i + b_2 j + b_3 k$ then $a.b = a_1 a_2 + b_1 b_2 + c_1 c_2$.

- The vectors a and b are perpendicular to each other then $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

- $\theta = \cos^{-1}\left(\dfrac{a.b}{|a||b|}\right)$ then $\theta = \cos^{-1}\left(\dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1 + a_2 + a_3}\sqrt{b_1 + b_2 + b_3}}\right)$

- The vector equation of plane $(r - a)\,.\,n = 0$

- Angle between the planes $\cos\theta = n_1.n_2$.

- Vector product of two vectors $a \times b = (|a|\ |b| \sin\theta)n$

- i, j, k are orthogonal trial of unit vectors then $i \times i = j \times j = k \times k = 0$ $i \times j = k,\ j \times k = i,\ k \times i = j.$

- If 'θ' is Angle between the two vectors then $\sin\theta = \dfrac{\sqrt{\Sigma(a_2 b_3 - a_2 b_2)^2}}{\sqrt{z a_1^2}\ \sqrt{z b_1^2}}$

- If a and b are non-collinear vectors, then, unit vectors perpendicular to both a and b are $= \dfrac{\pm\,(a \times b)}{|a \times b|}$

- Vector area of a parallelogram (diagonals) $= \dfrac{1}{2}(\mathrm{AC} \times \mathrm{BD})$

- $r = a+tb,\ r = c + sd\ \ t, s \in$ R then the shortest distance $= \dfrac{|(a-c).(b\times d)|}{|b\times d|}$.
- Let a, b, c be three vectors. Then

  $(a \times b)\times c = (a\ .\ c)\ b - (b\ .\ c)a$

  $a \times (b \times c) = (a\ .\ c)b - (a\ .\ b)c$

- For any Four vectors a, b, c and d

  i) $(a\times b)\ .\ (c\times d) = \begin{vmatrix} a.c & a.d \\ b.c & b.d \end{vmatrix}$

  ii) $(a\times b) \times (c\times d) = [acd]a - [bcd]c$

SUPPORTIVE WEBSITES

- http://www.wikipedia.org
- http://mathworld.wolfram.com

PRACTICE EXERCISE

1. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that any two of them are non-collinear. Find their sum if the vector $\overline{a} + \overline{b}$ is collinear with the vector $\vec{c}$ and if the vector $\vec{b} + \vec{c}$ is collinear with $\vec{a}$.
2. Prove that any two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exist numbers x and y, both not zero simultaneously, such that $x\ \vec{a} + y\vec{b} = \vec{0}$.
3. ABCD is a parallelogram in which M is the mid-point of side CD. Express the vectors $\overrightarrow{BD}$ and $\overrightarrow{AM}$ in terms of vectors $\overrightarrow{BM}$ and $\overrightarrow{MC}$.
4. Can the length of the vector $\vec{a} - \vec{b}$ ‹be (i) less than, (ii) equal to or (iii) larger than the sum of the lengths of vectors $\vec{a}$ and $\vec{b}$?

5. Let $\vec{a}$ and $\vec{b}$ be two non-collinear vectors. Find the number x and y, if the vector $(2-x)\,\vec{a}+\vec{b}$ and $y\,\vec{a}+(x-3)\vec{b}$ are equal.

6. The vectors $\vec{a}$ and $\vec{b}$ are non-collinear. Find the number x if the vector $3\vec{a}+x\vec{b}$ and $(1-x)\,\vec{a}-\frac{2}{3}\,\vec{b}$ are parallel.

7. Determine x and y such that the vector $\vec{a}=-2i+3j+yk$ is collinear with the vector $\vec{b}=xi-6j+2k$. Find also the magnitudes of $\vec{a}$ and $\vec{b}$.

8. Determine the magnitudes of the vectors $\vec{a}+\vec{b}$ if $\vec{a}=3i-5j+8k$ and $\vec{b}=-i+j-4k$.

9. Find a unit vector in the direction of $\vec{a}$ where $\vec{a}=-6i+3j-2k$.

10. Find a unit vector parallel to the resultant of vectors $3i-2j+k$ and $-2i+4j+k$.

11. The following forces act on a particle P :

    $\overrightarrow{F1}=2i+j-3k$, $\overrightarrow{F2}=-3i+2j+2k$ and $\overrightarrow{F3}=3i-2j+k$ measured in Newtons. Find (a) the resultant of the forces, (b) the magnitude of the resultant.

12. Show that the following vectors are co-planar :

    $(\vec{a}-2\,\vec{b}+\vec{c})$, $(2\vec{a}+\vec{b}-3\vec{c})$ and $(-3\vec{a}+\vec{b}+2\vec{c})$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three non-coplanar vectors.

ANSWERS

EXERCISE 15.1

1. (d) 2. (b)

3.

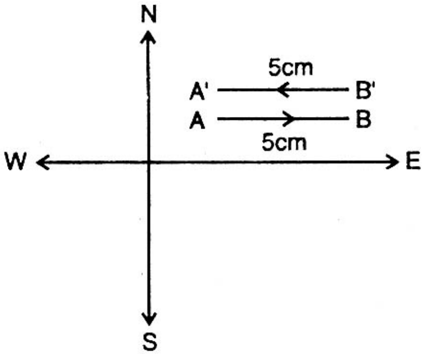

Fig. 15.38

4. Two vectors are said to be like if they have same direction what ever be their magnitudes. But in case of equal vectors magnitudes and directions both must be same.

5.

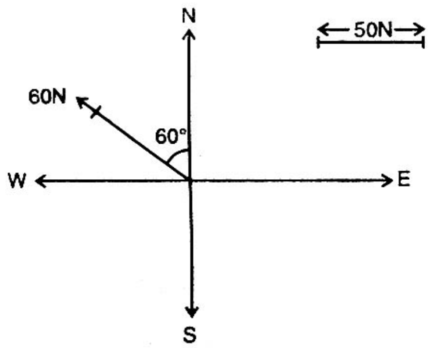

Fig. 15.39

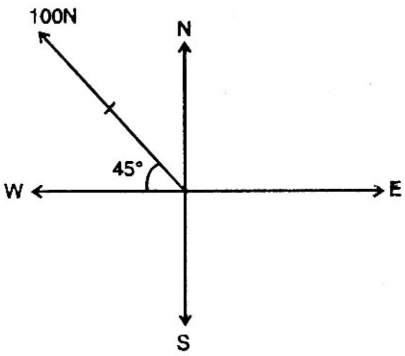

Fig. 15.40

EXERCISE 15.2

1. $\vec{O}$

2. $\vec{O}$

EXERCISE 15.3

1. $\vec{b} - \vec{a}$

2. (i)It is a vector in the direction of $\vec{a}$ and whose magnitudes is 3 times that of $\vec{a}$.

   (ii) It is a vector in the direction opposite to that of $\vec{b}$ and with magnitude 5 times that of $\vec{b}$.

3. $\overrightarrow{DB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = 2\vec{a} + 3\vec{b}$

4. $|y\vec{n}| = y\,|\vec{n}|$ if $y > 0$

   $= -y\,|\vec{n}|$ if $y < 0$

   $= 0$ if $y = 0$

5. vector

6. $\vec{p} = x\vec{q}$, x is a non zero scalar.

EXERCISE 15.4

1. If there exist scalars x and y such that $\vec{c} = x\,\vec{a} + y\vec{b}$

2. $\vec{r} = 3\hat{i} + 4\hat{j}$

3. $\overrightarrow{OP} = 4\hat{i} + 3\hat{j} + 5\hat{k}$

5. $\frac{1}{7}\,(3\hat{i} + 6\hat{j} - 2\hat{k})$

6. $\frac{1}{\sqrt{51}}\,\hat{i} - \frac{5}{\sqrt{51}}\,\hat{j} - \frac{5}{\sqrt{51}}\,\hat{k}$

EXERCISE 15.5

1. (i) $\frac{1}{5}(2\vec{a}+3\vec{b})$ (ii) $3\vec{a}-2\vec{b}$
2. $\frac{1}{7}(4\vec{p}+3\vec{q})$
3. $\frac{1}{3}(2\vec{c}+\vec{d})$, $\frac{1}{3}(\vec{c}+2\vec{d})$

EXERCISE 15.6

1. (a) $\frac{\pi}{2}$ (b) $\cos^{-1}\left(\frac{1}{14}\right)$

EXERCISE 15.7

1. $\lambda = 3$
2. $\cos^{-1}\left(\frac{2}{3\sqrt{46}}\right)$
3. $r^2 - (i + 3j - 2k) \,.\, r - 10 = 0.$

EXERCISE 15.8

1. $\sqrt{210}$
2. $\neq \frac{1}{\sqrt{6}}(2i - j - k)$
3. 3
4. $\frac{41}{2}$
5. $50\sqrt{2}$

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EXERCISE 15.9

1. $\frac{1}{6}$
2. 0
3. 12
4. $\lambda = -3$
6. $\frac{4}{3}$
8. $5\sqrt{14}, \sqrt{54}$

PRACTICE EXERCISE

1. $\vec{a} + \vec{b} + \vec{c} = \vec{0}$
2. $\overrightarrow{BD} = \overrightarrow{BM} - \overrightarrow{MC}, = \overrightarrow{AM} = \overrightarrow{BM} + 2\overrightarrow{MC}$
4. (i) Yes, $\vec{a}$ and $\vec{b}$ are either any non-collinear vectors or non-zero vectors of same direction.

   (ii) Yes, $\vec{a}$ and $\vec{b}$ are either in the opposite directions or at least one of them is a zero vector.

   (iii) Yes, $\vec{a}$ and $\vec{b}$ have opposite directions.
5. $x = 4, y = -2$
6. $x = 2, -1$
7. $x = 4, y = -1, |\vec{a}| = \sqrt{14}, |\vec{b}| = 2\sqrt{14}$
8. $|\vec{a} + \vec{b}| = 6, |\vec{a} - \vec{b}| = 14$
9. $-\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{2}{7}\hat{k}$
10. $\pm\frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$
11. $2\hat{i} + \hat{j}\ ;\ \sqrt{5}$